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EXTENSION CRITERION FOR CONTINUOUS CONVEXITY PRESERVING MAPS

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ABSTRACT. We present a necessary and sufficient condition for extending continuous convexity preserving maps defined on some subsets of a topological convexity space and with values in a compact topological median space. This result can be applied in topology to the theory of superextensions as well as to the lattice theory.

1. Introduction

A classical theorem of Taĭmanow [6] states that a map f defined on a dense subset of a topological space X with values in a compact Hausdorff space Y can be extended to a continuous map $F\colon X \to Y$ if and only if f satisfies the following condition: if $A, B \subset Y$ are closed and disjoint then $\operatorname{cl} f^{-1}(A) \cap \operatorname{cl} f^{-1}(B) = \emptyset$, where "cl" denotes the closure in X.

We present an analogue of Taĭmanow's Theorem for maps of topological convexity spaces. Let us start with definitions.

By *a convexity space* we mean a pair $(X, \mathscr{G})$ where $\mathscr{G} \subset \mathscr{P}(X)$ is stable under intersections and the unions of chains and $\emptyset, X \in \mathscr{G}$ (cf. [8, p. 3]). Elements of $\mathscr{G}$ are called *convex sets*, their complements are called *concave* and $\mathscr{G}$ itself is called *a convexity* on X. By *a topological convexity space* we understand any triple $(X, \mathscr{T}, \mathscr{G})$ where $\mathscr{T}$ is a topology and $\mathscr{G}$ is a convexity on X (we do not assert any compatibility conditions on $\mathscr{T}$ and $\mathscr{G}$). We will use the following notation:

$$\operatorname{conv} A = \bigcap \{G \in \mathscr{G} \colon A \subset G\} \quad \text{(the convex hull of } A\text{)},$$

$$\operatorname{clco} A = \bigcap \{G \in \mathscr{G} \colon A \subset G \text{ and } G \text{ is closed}\} \quad \text{(the closed convex hull of } A\text{)}.$$

The convex hull of a two-element subset $\{a, b\}$ is called *the segment joining* a, b and denoted by $[a, b]$. Every convexity space can be viewed in as a topological convexity space with discrete topology and every topological space is a

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topological convexity space with discrete convexity (consisting of all subsets). A topological convexity space X is *normal* provided one-point subsets are closed and convex and for any two disjoint closed convex sets $A, B \subset X$ there exist closed convex sets A', B' with $A \cap B' = \emptyset = A' \cap B$ and $A' \cup B' = X$. For convexity spaces (with discrete topology) normality is called *the Kakutani separation property* S_4 and is equivalent to the fact that two disjoint convex sets can be separated by a halfspace (a convex set with the convex complement), see [8, Theorem I.3.8]. Let X and Y be two convexity spaces. We say that a map $f: X \to Y$ is *convexity preserving* (cp for short) provided $f^{-1}(G)$ is convex in X for every convex set $G \subset Y$. This is equivalent to the condition $f(\operatorname{conv} S) \subset \operatorname{conv} f(S)$ for every finite $S \subset X$ (cf. [8, p. 15]).

For a study of axiomatic convexity theory we refer to the monograph of van de Vel [8].

An important example of a convexity space is a lattice $(L, \wedge, \vee)$ with the convexity consisting of all order-convex sublattices, i.e., a subset $G \subset L$ is convex iff for every $a, b \in G$ we have $I(a, b) \subset G$, where $I(a, b) = \{x \in L: a \wedge b \leqslant x \leqslant a \vee b\}$ (cf. [7]). It is easy to check that $I(a, b)$ equals the segment joining a, b. A lattice is a normal convexity space iff it is distributive (cf. [7] or [8, Proposition I.3.12.3]).

2. Median spaces

A median space is a topological convexity space X with the following properties:

(M1) If $a, b \in X$ are distinct then there exist closed convex sets A, B such that $a \notin B$, $b \notin A$ and $A \cup B = X$.

(M2) If $\mathscr{A}$ is a finite collection of convex subsets of X and $\bigcap \mathscr{A} = \emptyset$ then there exist $A, B \in \mathscr{A}$ with $A \cap B = \emptyset$.

These conditions imply that for each $a, b, c \in X$ there exists a unique element in $[a, b] \cap [a, c] \cap [b, c]$, called *the median of* a, b, c (this explains the name "median space"). Indeed, if $x_1, x_2 \in [a, b] \cap [a, c] \cap [b, c]$ then x_1, x_2 cannot be separated by two convex sets as in (M1) above. Moreover, condition (M2) is equivalent to $[a, b] \cap [a, c] \cap [b, c] \neq \emptyset$ for each $a, b, c \in X$, see [8]. Note that, by (M1), every median space is a Hausdorff topological space (condition (M1) is a "convex" analogue of the topological separation axiom T_2). Our definition of a median space is more restrictive than the one in [8] or [10].

Every distributive lattice is a median space (with discrete topology), see [8]. In a distributive lattice, the median of its three points a, b, c is equal to $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$. An important example of a median space is a Hilbert cube $[0, 1]^\kappa$

with the product topology and with the convexity of a lattice (with coordinatewise order).

Proposition 2.1. *Every compact median space is normal.*

Proof. Let X be a compact median space. Using (M1) we see that one-point subsets are closed and convex. Fix two disjoint closed convex sets $A, B \subset X$.

Suppose first that $A = \{a\}$. Using (M1) we can find for each $b \in B$ two closed convex sets G_b, F_b with $a \notin F_b$, $b \notin G_b$ and $F_b \cup G_b = X$. Now the collection of closed convex sets $\{B\} \cup \{G_b : b \in B\}$ has empty intersection. By the compactness of X and (M2) there exists a $b_0 \in B$ such that $B \cap G_{b_0} = \emptyset$, since $a \in G_b$ for every $b \in B$. Setting $F = F_{b_0}$, $G = G_{b_0}$ we obtain two closed convex sets such that $B \cap G = \emptyset = \{a\} \cap F$ and $F \cup G = X$.

Now let A be an arbitrary closed convex set. Using the first part of our proof we can find for each $a \in A$ two closed convex sets C_a, D_a such that $a \notin C_a$, $B \cap D_a = \emptyset$ and $C_a \cup D_a = X$. The same argument as above gives us an $a_0 \in A$ with $A \cap C_{a_0} = \emptyset$ and setting $C = C_{a_0}$, $D = D_{a_0}$ we get $A \cap C = \emptyset = B \cap D$ and $C \cup D = X$. □

Using the same arguments as in [8, Proposition III.4.13.3] one can prove that every median space with compact segments is normal. For our purpose only the normality of compact median spaces will be needed. Let us also mention that every median space has the Kakutani separation property (as a convexity space), see [10, Thm. I.2.14].

Lemma 2.2. *Let X be a topological convexity space and let Y be a compact median space. Then a map $f: X \to Y$ is continuous and convexity preserving if and only if for every $x \in X$ and for every open concave set $U \subset Y$ with $f(x) \in U$ there exists an open concave set $W \subset X$ such that $x \in W$ and $f(W) \subset U$.*

Proof. The "only if" part is trivial. Suppose that f satisfies the above condition. Then the pre-image under f of every convex closed set is convex and closed in X. Consider the collection $\mathscr{B}$ of all closed convex subsets of Y. By (M1) $\mathscr{B}$ generates a Hausdorff topology on Y which is weaker than its original one. Hence, by the compactness of Y, $\mathscr{B}$ is a closed subbase of the topology of Y and consequently f is continuous. For the proof that f is cp we need only to check that $f^{-1}(\operatorname{conv} S)$ is convex for every finite $S \subset Y$ (see [8, Proposition I.1.12]). For this purpose, we show that $\operatorname{conv} S$ is closed for arbitrary finite $S \subset Y$.

Fix a $y \in Y \setminus \operatorname{conv} S$, where $S = \{s_1, \ldots, s_n\}$. By (M2), there exists a point

$$p \in \operatorname{conv} S \cap \bigcap_{i=1}^{i=n} [y, s_i].$$

Now, using (M1), we get two closed convex sets A, B with $p \notin B$, $y \notin A$ and $A \cup B = Y$. As $p \in [y, s_i]$, we see that $s_i \in A$. Hence $\operatorname{conv} S \subset A$. It follows that $\operatorname{conv} S$ is closed. □

3. Main result

We first describe a condition for maps, an analogue to what has appeared in Taĭmanow's Theorem.

Proposition 3.1. *Let X, Y be two topological convexity spaces, $M \subset X$ and let $f: M \to Y$ be a map. The following conditions are equivalent:*

(a) $\operatorname{clco} f(A) \cap \operatorname{clco} f(B) \neq \emptyset$ *holds for each* $A, B \subset M$ *with* $\operatorname{clco} A \cap \operatorname{clco} B \neq \emptyset$.

(b) *If* $C, D \subset Y$ *are closed convex and disjoint then* $\operatorname{clco} f^{-1}(C) \cap \operatorname{clco} f^{-1}(D) = \emptyset$.

P r o o f. (a)⇒(b) If $\operatorname{clco} f^{-1}(C) \cap \operatorname{clco} f^{-1}(D) \neq \emptyset$ then by (a) we have $\emptyset \neq \operatorname{clco} f\big(f^{-1}(C)\big) \cap \operatorname{clco} f\big(f^{-1}(D)\big) \subset C \cap D$.

(b)⇒(a) If $\operatorname{clco} f(A) \cap \operatorname{clco} f(B) = \emptyset$ then by (b) we get $\emptyset = \operatorname{clco} f^{-1}\big(\operatorname{clco} f(A)\big) \cap \operatorname{clco} f^{-1}\big(\operatorname{clco} f(B)\big) \supset \operatorname{clco} A \cap \operatorname{clco} B$. □

We say that a map f *satisfies condition* (T) if f fulfills (a) (or (b)) above. Observe that if X and Y are topological spaces considered with discrete convexity then condition (T) is exactly condition of Taĭmanow.

Let X be a topological convexity space (or, in particular, a median space). We say that a subset $M \subset X$ is *median-stable* provided for each $a, b, c \in M$, $[a, b] \cap [a, c] \cap [b, c] \subset M$ (cf. [8, p. 130] for the case of discrete convexity spaces). We define *the median stabilization* of $A \subset X$ as

$$\operatorname{med} A = \bigcap \{G \subset X : A \subset G \text{ and } G \text{ is closed median-stable}\}.$$

We say that a subset $M \subset X$ is *geometrically dense* if $\operatorname{med} M = X$. Observe that if M is geometrically dense then for each two open concave sets $U, V \subset X$ the following implication holds true:

$$U \cap V \neq \emptyset \implies U \cap V \cap M \neq \emptyset. \tag{$*$}$$

Indeed, if $U \cap V \cap M = \emptyset$ then $M \subset (X \setminus U) \cup (X \setminus V)$ and the union of two convex sets is median-stable. Notice that every topologically dense set is geometrically dense (see also [8, Lemma I.6.20] for other examples).

We now can state our main result. The proof is quite similar to the proof of Taĭmanow's Theorem in [1, p. 164].

Theorem 3.2. *Let M be a geometrically dense subset of a topological convexity space X, let Y be a compact median space and let $f: M \to Y$ be a map satisfying condition* (T). *Then there exists a unique continuous convexity preserving map $F: X \to Y$ such that $F|M = f$.*

Proof. (i) For $x \in X$ denote by $\operatorname{nbd}(x)$ the collection of all open concave sets containing x and set

$$\mathscr{F}(x) = \{\operatorname{clco} f(M \cap U) \colon U \in \operatorname{nbd}(x)\}.$$

Since M is geometrically dense, if $U_1, U_2 \in \operatorname{nbd}(x)$ then, by condition $(*)$, $U_1 \cap U_2 \cap M \neq \emptyset$. Hence $\operatorname{clco} f(U_1 \cap M) \cap \operatorname{clco} f(U_2 \cap M) \supset \operatorname{clco} f(U_1 \cap U_2 \cap M) \neq \emptyset$. The compactness of Y together with condition (M2) imply that $\bigcap \mathscr{F}(x) \neq \emptyset$.

(ii) Suppose that there exist distinct $y_1, y_2 \in \bigcap \mathscr{F}(x)$. Since Y is normal (Proposition 2.1), there exist two open concave sets $U_1, U_2 \subset Y$ with $y_i \in U_i$ and $\operatorname{clco} U_1 \cap \operatorname{clco} U_2 = \emptyset$. Condition (T) implies that $\operatorname{clco} f^{-1}(U_1) \cap \operatorname{clco} f^{-1}(U_2) = \emptyset$. Assume that $x \notin \operatorname{clco} f^{-1}(U_1)$. Setting $W = X \setminus \operatorname{clco} f^{-1}(U_1)$ we have $W \in \operatorname{nbd}(x)$ and hence $\operatorname{clco} f(M \cap W) \in \mathscr{F}(x)$. On the other hand

$$\begin{aligned} \operatorname{clco} f(M \cap W) &= \operatorname{clco} f\big(M \setminus \operatorname{clco} f^{-1}(U_1)\big) \\ &\subset \operatorname{clco} f\big(M \setminus f^{-1}(U_1)\big) \subset \operatorname{clco}(Y \setminus U_1) = Y \setminus U_1, \end{aligned}$$

which gives a contradiction, since $y_1 \in \operatorname{clco} f(M \cap W)$.

(iii) Thus we have proved that $\left|\bigcap \mathscr{F}(x)\right| = 1$ for every $x \in X$. Define $F: X \to Y$ by letting $F(x) \in \bigcap \mathscr{F}(x)$. If $x \in M$ then $f(x) \in \bigcap \mathscr{F}(x)$, consequently $F(x) = f(x)$. It remains to check that F is continuous and cp.

(iv) Let $U \in \operatorname{nbd}\big(F(x)\big)$. As $\bigcap \mathscr{F}(x) = \{F(x)\}$, we have

$$(Y \setminus U) \cap \bigcap \mathscr{F}(x) = \emptyset.$$

Now (M2) and the compactness of Y give a $W \in \operatorname{nbd}(x)$ with $\operatorname{clco} f(M \cap W) \cap (Y \setminus U) = \emptyset$. It follows that for each $x' \in W$ we have $F(x') \in \operatorname{clco} f(M \cap W) \subset U$. Hence $F(W) \subset U$. In view of Lemma 2.2, F is continuous and convexity preserving.

(v) If $F_1, F_2: X \to Y$ are two continuous cp extensions of f then the set

$$G = \{x \in X \colon F_1(x) = F_2(x)\}$$

is closed median-stable and contains M; hence $G = X$ and $F_1 = F_2$. This completes the proof. □

Remark. Condition (T) is necessary for the existence of a continuous cp extension. Indeed, if f can be extended to a continuous cp map $F: X \to Y$ then for two disjoint closed convex sets $C, D \subset Y$ we have $\operatorname{clco} f^{-1}(C) \cap \operatorname{clco} f^{-1}(D) \subset F^{-1}(C) \cap F^{-1}(D) = \emptyset$.

4. Applications

Proof of Taĭmanow's Theorem. Let $f\colon M \to Y$ be a map satisfying the condition of Taĭmanow, where M is a (topologically) dense subset of a topological space X and Y is a compact Hausdorff space. Embed Y into the Hilbert cube $H = [0,1]^\kappa$ and consider X as a topological convexity space with discrete convexity. Now H is a compact median space and $f\colon M \to H$ satisfies (T). Applying Theorem 3.2 we obtain the unique continuous map $F\colon X \to H$ which extends f. Finally, $F(X) = F(\operatorname{cl} M) \subset \operatorname{cl} F(M) = \operatorname{cl} f(M) \subset Y$, since Y is closed in H. □

We now give an application of Theorem 3.2 to the theory of superextensions.

A collection $\mathscr{P}$ is called a *T_1-subbase* for a topological convexity space $(X, \mathscr{T}, \mathscr{G})$ provided:

(i) $\mathscr{P}$ is a closed subbase for the topology $\mathscr{T}$ and $\mathscr{P}$ generates the convexity $\mathscr{G}$ (i.e., $\mathscr{G}$ is the smallest convexity containing $\mathscr{P}$);

(ii) for every $x \in X$ there exists a $P \in \mathscr{P}$ with $x \notin P$;

(iii) if $x \notin P \in \mathscr{P}$ then there exists a $Q \in \mathscr{P}$ with $x \in Q$ and $P \cap Q = \emptyset$.

Let $\mathscr{P}$ be a T_1-subbase of a space X. Then there exists a topological convexity space $\lambda(X, \mathscr{P})$, called *the superextension of X with respect to $\mathscr{P}$*, with the following properties:

(1) X is continuously cp embedded into $\lambda(X, \mathscr{P})$ and X is geometrically dense in $\lambda(X, \mathscr{P})$.

(2) If $P, Q \in \mathscr{P}$ are disjoint then their closed convex hulls in $\lambda(X, \mathscr{P})$ are disjoint as well.

(3) If $\mathscr{A}$ is any collection of closed convex subsets of $\lambda(X, \mathscr{P})$ with $\bigcap \mathscr{A} = \emptyset$ then there exist $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$.

The details one can find in [8, pp. 13, 279] or [3] (in a different language). Condition (3) says that the collection of all closed convex subsets of $\lambda(X, \mathscr{P})$ is *binary*. Topological spaces having a binary closed subbase are called *supercompact*, see [3]. Notice that, by Alexander Subbase Lemma, every supercompact space is compact. If a T_1-subbase $\mathscr{P}$ is *normal*, that is, for each two disjoint $P, Q \in \mathscr{P}$ there exist $P', Q' \in \mathscr{P}$ with $P \cap Q' = \emptyset = P' \cap Q$ and $P' \cup Q' = X$, then $\lambda(X, \mathscr{P})$ satisfies (M1) and consequently it is a compact median space. In [3] a topological space with normal binary T_1-subbase is called *normally supercompact*; in our language a topological space $(X, \mathscr{T})$ is normally supercompact iff there exists a convexity $\mathscr{G}$ on X such that $(X, \mathscr{T}, \mathscr{G})$ is a compact median space.

Applying Theorem 3.2 and condition (2) above, we obtain the following result due to Verbeek [9] and van Mill, van de Vel [4] (see also [8, Corollary III.4.17]).

Theorem 4.1. *Let $\mathscr{P}$ be a T_1-subbase of a topological convexity space X, let Y be a compact median space and let $f: X \to Y$ be such a map that $f^{-1}(G) \in \mathscr{P}$ whenever $G \subset Y$ is closed convex. Then there exists a unique continuous convexity preserving map $F: \lambda(X, \mathscr{P}) \to Y$ such that $F|X = f$.*

Remark. Let Y be a topological convexity space satisfying condition (M1) and suppose that Y fulfills the statement of Theorem 3.2. Then Y is a compact median space.

Indeed, taking the collection $\mathscr{P}(Y)$ of all subsets of Y we see that $\mathscr{P}(Y)$ is a normal T_1-subbase for the discrete topology and the discrete convexity on Y; therefore, by Theorem 4.1, the identity map $\mathrm{id}_Y: Y \to Y$ can be extended to a continuous cp map $F: \lambda\big(Y, \mathscr{P}(Y)\big) \to Y$ which is onto. As $\lambda\big(Y, \mathscr{P}(Y)\big)$ is a compact median space and condition (M2) is preserved by images under cp maps, it follows that Y is a compact median space.

Let us now present a discrete version of Theorem 3.2. We need an auxiliary result on median stabilization.

Lemma 4.2. *In every median space, the median stabilization of a finite set is finite.*

P r o o f . Let Y be a median space. As we have already mentioned, Y has the Kakutani separation property S_4, i.e., two disjoint convex sets can be separated by a halfspace. Hence, there exists a cp embedding $j: Y \to \mathscr{P}(\mathscr{H})$, given by the formula $j(y) = \{H \in \mathscr{H} : y \in H\}$, where $\mathscr{H}$ is the collection of all halfspaces in Y and $\mathscr{P}(\mathscr{H})$ is the power set of $\mathscr{H}$ considered with the lattice convexity (see [8, Lemma I.3.16] for the details). Thus, assuming that Y is a subspace of $\mathscr{P}(\mathscr{H})$, we see that the median of $a, b, c \in Y$ is precisely equal to $(a \cap b) \cup (a \cap c) \cup (b \cap c)$. It follows that the median stabilization of a finite set $S \subset Y$ is contained in the lattice of sets generated by S and therefore is finite. □

Theorem 4.3. (cf. [2]). *Let Y be a median convexity space and let $f: M \to Y$ be a map defined on a geometrically dense subset of a convexity space X. If f satisfies the condition*

(I) $\operatorname{conv} f(S) \cap \operatorname{conv} f(T) \neq \emptyset$ *whenever* $S, T \subset M$ *are finite and* $\operatorname{conv} S \cap \operatorname{conv} T \neq \emptyset$,

then there exists a unique convexity preserving map $F: X \to Y$ such that $F|M = f$.

P r o o f . Denote by $\mathscr{F}$ the collection of all finite subsets of M. If $S \in \mathscr{F}$ then, according to Lemma 4.2, $\operatorname{med} f(S)$ is finite and can be viewed as a compact median space (with discrete topology). Moreover, $f|S$ satisfies condition (T). Applying Theorem 3.2 we obtain the unique cp extension $F_S: \operatorname{med} S \to$

Y of $f|S$. By uniqueness we have $F_S \subset F_T$ whenever $S \subset T$. Then setting $F = \bigcup_{S \in \mathscr{F}} F_S$ we obtain a convexity preserving map with the domain $\bigcup_{S \in \mathscr{F}} \operatorname{med} S = \operatorname{med} M = X$ and $F|M = f$. □

Finally, we apply the last result to obtain an extension criterion for maps of lattices which, in the case of Boolean algebras, is known as Sikorski Extension Criterion [5]; see also [2].

Theorem 4.4. *Let L be a distributive lattice and let K be a lattice generated by its subset M. If $f\colon M \to L$ is a map satisfying the implication*

$$a_1 \wedge \cdots \wedge a_n \leqslant b_1 \vee \cdots \vee b_m \implies f(a_1) \wedge \cdots \wedge f(a_n) \leqslant f(b_1) \vee \cdots \vee f(b_m) \quad \text{(S)}$$

for all $a_1, \ldots, a_n, b_1, \ldots, b_m \in M$, then f can be uniquely extended to a lattice homomorphism $F\colon K \to L$.

Proof. First, add to K two elements $0_K, 1_K$ in such a way that $0_K < x < 1_K$ for all $x \in K$ and set $K' = K \cup \{0_K, 1_K\}$. Let us make the same operation for L and set $L' = L \cup \{0_L, 1_L\}$. Then K' is a lattice and L' is a distributive lattice. Now set $M' = M \cup \{0_K, 1_K\}$ and extend f to a map $f'\colon M' \to L'$ by letting $f'(0_K) = 0_L$, $f'(1_K) = 1_L$. It is easy to see that f' satisfies the condition (S) above. If G is a median-stable subset of K' containing M' then G is a sublattice, since for $x, y \in G$ we have $x \wedge y \in [x, y] \cap [x, 0_K] \cap [y, 0_K]$ and $x \vee y \in [x, y] \cap [x, 1_K] \cap [y, 1_K]$. It follows that M' is geometrically dense in K'.

Observe that, in any lattice, the convex hull of a finite set P is equal to the segment $[\inf P, \sup P]$. Let S, T be two finite subsets of M'. If there exists a point $x \in \operatorname{conv} S \cap \operatorname{conv} T$ then $\inf S \leqslant x \leqslant \sup T$ and $\inf T \leqslant x \leqslant \sup S$. By the condition (S) we get $\inf f'(S) \leqslant \sup f'(T)$ and $\inf f'(T) \leqslant \sup f'(S)$. Hence, setting $y = \inf f'(S) \vee \inf f'(T)$, we have $y \in \operatorname{conv} f'(S) \cap \operatorname{conv} f'(T)$. It follows that f' satisfies condition (I) of Theorem 4.3.

Since every distributive lattice is a median convexity space, we can apply Theorem 4.3 to obtain a unique cp map $F\colon K' \to L'$ with $F|M' = f'$. As we have observed, for every $x, y \in K'$ the point $x \wedge y$ belongs to $[x, y] \cap [x, 0_K] \cap [y, 0_K]$ and consequently $F(x \wedge y)$ is the median of $F(x), F(y), F(0_K)$. Since $F(0_K) = 0_L$, it follows that $F(x \wedge y) = F(x) \wedge F(y)$. Similarly $F(x \vee y) = F(x) \vee F(y)$. Hence F is a lattice homomorphism. Finally, $F(K)$ is a sublattice generated by $f(M)$; hence $F(K) \subset L$. This completes the proof. □

Let us finally mention that the last theorem is no longer true when we drop the assumption of the distributivity of the lattice L. Indeed, if L satisfies the above statement then taking K equal to the free distributive lattice generated by the set L we see that L is a homomorphic image of a distributive lattice; consequently L is distributive.

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