Tatra Mt. Math. Publ. **19** (2000), 155–165

TATRA MOUNTAINS
Mathematical Publications

ON DARBOUX FUNCTIONS $f\colon \mathbb{R}^n \to \mathbb{R}^m$ WITH FINITE VARIATION

Andrzej Rychlewicz

ABSTRACT. In the present paper the relations between subclasses of the class of all bounded Darboux functions in the sense of Pawlak $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with finite variation are studied. The porosity of the class of all quasi-continuous functions in the class of all cliquish functions is proved (for $m \geq 2$).

Let $f\colon X \to Y$, where X and Y are arbitrary topological spaces. We say that f is a Darboux function in the sense of Pawlak (briefly P-Darboux) if $f(\mathcal{L})$ is a connected set for any arc $\mathcal{L}$ contained in X (see [4], [6]).

A map f is quasi-continuous at a point x if for any neighbourhood U of the point x and any neighbourhood V of the point $f(x)$ there exists an open non-empty set $W \subset U$ such that $f(W) \subset V$.

Let us additionally assume that Y is a metric space. A map f is cliquish at a point x if for any neighbourhood U of the point x and for any $\varepsilon > 0$ there exists an open non-empty set $W \subset U$ such that $d\big(f(x_1), f(x_2)\big) < \varepsilon$, for any $x_1, x_2 \in W$.

The function f is said to be quasi-continuous (cliquish) if f is quasi-continuous (cliquish) at any point $x \in X$.

Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$. A function $\mathscr{N}_f\colon \mathbb{R}^m \to [0, +\infty]$ defined in the following way: $\mathscr{N}_f(p)$ is equal to the number of elements of the set $f^{-1}(p)$ when the last set is finite and equal to $+\infty$ when $f^{-1}(p)$ is not finite, is called the Banach indicatrix of the function f.

We say that f has finite variation (in Banach sense) if the Banach indicatrix of the function f is measurable and $\int\limits_{\mathbb{R}^m} \mathcal{N}_f(p)\, dp < +\infty$.

Denote by $C(f)$, $QC(f)$, $CL(f)$ the sets of all points at which function f is continuous, quasi-continuous and cliquish, respectively.

Let $\mathscr{X}(n,m)$ be the space of all bounded P-Darboux functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$ with finite variation, with metric of uniform convergence, where n, m are positive

2000 Mathematics Subject Classification: 26A21.
Keywords: Darboux functions, quasi-continuous functions, cliquish functions, porosity.

integers and $m \geq 2$. Denote by

$\mathscr{C}(n,m)$ – the class of all continuous functions $f \in \mathscr{X}(n,m)$;
$\mathcal{Q}\mathscr{C}(n,m)$ – the class of all quasi-continuous functions $f \in \mathscr{X}(n,m)$;
$\mathcal{D}\mathscr{C}(n,m)$ – the class of all functions $f \in \mathscr{X}(n,m)$ for which $C(f)$ is a dense set;
$\mathcal{D}\mathcal{Q}\mathscr{C}(n,m)$ – the class of all functions $f \in \mathscr{X}(n,m)$ for which $QC(f)$ is a dense set;
$\mathscr{C}\ell(n,m)$ – the class of all cliquish functions $f \in \mathscr{X}(n,m)$.

We know that $C(f) \subset QC(f) \subset CL(f)$, and $CL(f)$ is a closed set (see [3]) and $CL(f)\setminus C(f)$ is the set of the first category (see [5]). Above facts imply that $\mathcal{D}\mathscr{C}(n,m) = \mathcal{D}\mathcal{Q}\mathscr{C}(n,m) = \mathscr{C}\ell(n,m)$. Of course $\mathscr{C}(n,m) \subset \mathcal{Q}\mathscr{C}(n,m) \subset \mathcal{D}\mathscr{C}(n,m)$.

Moreover for $m = 1$ we have the following chain of equalities

$$\mathscr{C}(n,1) = \mathcal{Q}\mathscr{C}(n,1) = \mathcal{D}\mathscr{C}(n,1) = \mathcal{D}\mathcal{Q}\mathscr{C}(n,1) = \mathscr{C}\ell(n,1).$$

The above equalities are a consequence of the known fact that for any Darboux, not continuous function f mapping a Hausdorff metric space with countable basis into a one dimensional Euclidian space, f does not have a finite variation (see [8]).

Let X be a metric space $x \in M \subset X$ and $R > 0$. Then we denote the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $K(z,r) \subset K(x,R)\setminus M$ by $\gamma(x,R,M)$. The number $p(M,x) = 2\cdot\limsup_{R\to 0^+} \frac{\gamma(x,R,M)}{R}$ is called the porosity of M at x. If $p(M,x) > 0$, then M is said to be porous at x. A set M is said to be porous, if it is porous at each of its points.

For $\mathscr{A}, \mathscr{B} \in \mathscr{X}(n,m)$ we write $\mathscr{A} \rightsquigarrow \mathscr{B}$ if $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}$ is a porous set in $\mathscr{B}$.

Theorem 1 in [7] implies that $\mathscr{C}(2,2) \rightsquigarrow \mathscr{X}(2,2)$, $\mathcal{Q}\mathscr{C}(2,2) \rightsquigarrow \mathscr{X}(2,2)$ and $\mathcal{D}\mathscr{C}(2,2) \rightsquigarrow \mathscr{X}(2,2)$. Theorem 2 in [7] implies that $\mathscr{C}(2,2) \rightsquigarrow \mathcal{Q}\mathscr{C}(2,2)$. The question is: do the following relations

1. $\mathscr{C}(2,2) \rightsquigarrow \mathcal{D}\mathscr{C}(2,2)$,
2. $\mathcal{Q}\mathscr{C}(2,2) \rightsquigarrow \mathcal{D}\mathscr{C}(2,2)$

hold?

Positive answer to Question 1 is a corollary of the proof of the Theorem 2 in [7], but the same answer to Question 2 is given by Theorem 1 from this paper.

Throughout the paper, we adopt the standard symbols, definitions and notations. The distance is denoted by d. The symbols $K^k(x,r)$, $\overline{K}^k(x,r)$, $S^k(x,r)$

is used to write down an open ball, closed ball, and sphere in the space $\mathbb{R}^k$ with the centre $x \in \mathbb{R}^k$ and radius r, respectively. An open ball with the centre y_0 and radius r in the space $\mathscr{L}(n,m)$ is denoted by $K(y_0, r)$. The symbol $[a,b]$ denotes an interval with end points a, b (the case $a > b$ is also accepted). $\mathrm{Cl}_X(A)$ ($\mathrm{Int}_X(A)$, $\mathrm{Fr}_X(A)$) means a closure (interior, boundary) of the subset A in the space X.

Let $x_0 \in \mathbb{R}^k$.

If $A \subset \mathbb{R}^k$ we denote

$$d_{x_0}(A) = \{y \geq 0\colon d(x_0, x) = y \text{ for some } x \in A\}.$$

If $k \geq 2$ and $r > 0$ we denote

$$K^k_{\mathbb{Q}}(x_0, r) = \Big(\big((\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q})\big) \times \mathbb{R}^{k-2}\Big) \cap K^k(x_0, r),$$

where $\mathbb{Q}$ denotes all rationals.

Let $x_0 \in \mathbb{R}^k$ $(k \geq 2)$. The following two lemmas are true.

Lemma 1. *The set $K^k_{\mathbb{Q}}(x_0, r)$ is connected, $\overline{K^k_{\mathbb{Q}}(x_0, r)} = \overline{K}^k(x_0, r)$ and for any $\beta \in (0, r)$ there exists $x \in K^k_{\mathbb{Q}}(x_0, r)$ such that $d(x_0, x) = \beta$.*

Lemma 2. *Let C be the Cantor set in $[0,b]$ and C_0 be the set of all end points of components of the complement of the C. Then there exists a function $\phi_{[0,b]}\colon C \setminus C_0 \to K^k_{\mathbb{Q}}(x_0, b)$ possessing the following property*

$$\phi_{[0,b]}\big([y_1, y_2] \cap (C \setminus C_0)\big) = K^k_{\mathbb{Q}}(x_0, b)$$

for any $y_1, y_2 \in [0,b]$ $(y_1 \neq y_2)$ such that $[y_1, y_2] \cap (C \setminus C_0) \neq \emptyset$.

Theorem 1. *In the space of all bounded P-Darboux and cliquish functions $f\colon \mathbb{R}^n \to \mathbb{R}^m$ $(m \geq 2)$ with finite variation, with the metric of uniform convergence, quasi-continuous functions constitute a porous set.*

P r o o f. Let f_0 be a bounded quasi-continuous P-Darboux function with finite variation. Let $x_0 \in \mathbb{R}^n$, $f_0(x_0) = \alpha_0$.

Let $\tau > 0$. There exists $z_0 \in \mathbb{R}^n$ and $\delta > 0$ such that

$$f_0\big(\overline{K}^n(z_0, \delta)\big) \subset K^m(\alpha_0, \tau). \tag{1}$$

Denote the Cantor set in $[0, \delta]$ and the set of all end points of components of the complement of this Cantor set in $[0, \delta]$ by C^0 and C^0_0, respectively. By the Lemma 2 there exists the function $\phi_0 = \phi_{[0,\delta]}\colon C^0 \setminus C^0_0 \to K^m_{\mathbb{Q}}(\alpha_0, \tau)$ possessing the following property

$$\phi_0\big([y_1, y_2] \cap (C^0 \setminus C^0_0)\big) = K^m_{\mathbb{Q}}(\alpha_0, \tau)$$

for any $y_1, y_2 \in [0,\delta]$ such that $[y_1, y_2] \cap (C^0 \setminus C_0^0) \neq \emptyset$.

Let us denote by g_0 the map of $\mathbb{R}^n$ into $\mathbb{R}^m$ given by the formula

$$g_0(x) = \begin{cases} \phi_0\big(d(z_0, x)\big), & d(z_0, x) \in (C^0 \setminus C_0^0) \cap [0, \delta), \\ \alpha_0, & d(z_0, x) \in [0, \delta), \setminus (C^0 \setminus C_0^0), \\ f_0(x), & d(z_0, x) \geq \delta. \end{cases}$$

Let us denote the Cantor set in $[0, \tau]$ and the set of all end points of components of the complement of this Cantor set in $[0, \tau]$ by C^1 and C_0^1, respectively. By the Lemma 2 there exists the function $\phi_1 = \phi_{[0,\tau]} \colon C^1 \setminus C_0^1 \to K_{\mathbb{Q}}^m(\alpha_0, \tau)$ possessing the following property

$$\phi_1\big([y_1, y_2] \cap (C^1 \setminus C_0^1)\big) = K_{\mathbb{Q}}^m(\alpha_0, \tau)$$

for any $y_1, y_2 \in [0, \tau]$ ($y_1 \neq y_2$) such that $[y_1, y_2] \cap (C^1 \setminus C_0^1) \neq \emptyset$.

Let now define the function $g_1{}' \colon [0, \tau) \to \mathbb{R}^m$ in the following way

$$g_1{}'(y) = \begin{cases} \phi_1(y), & y \in (0, \tau) \cap (C^1 \setminus C_0^1), \\ \alpha_0, & y \in \big[[0, \tau) \setminus (C^1 \setminus C_0^1)\big] \cup \{0\}, \end{cases}$$

and the function $g_1 \colon \mathbb{R}^m \to \mathbb{R}^m$ by the formula

$$g_1(x) = \begin{cases} g_1{}'\big(d(\alpha_0, x)\big), & x \in K^m(\alpha_0, \tau), \\ x, & x \in \mathbb{R}^m \setminus K^m(\alpha_0, \tau). \end{cases}$$

Consider the function $h_0 = g_1 \circ g_0 \colon \mathbb{R}^n \to \mathbb{R}^m$.

Now we shall demonstrate that h_0 is a P-Darboux function. We shall first show that g_0 is a P-Darboux function.

Let $\mathcal{L}$ be an arbitrary arc contained in $\mathbb{R}^n$.

If $\mathcal{L} \subset \mathbb{R}^n \setminus K^n(z_0, \delta)$ then the set $g_0(\mathcal{L}) = f_0(\mathcal{L})$ is connected because f_0 possesses a P-Darboux property.

Let $\mathcal{L} \subset K^n(z_0, \delta)$. Then $d_{z_0}(\mathcal{L}) \subset [0, \delta)$ is connected. If $d_{z_0}(\mathcal{L})$ is a one-point set then $g_0(\mathcal{L})$ is a one-point set, too.

Let $d_{z_0}(\mathcal{L})$ be a non-degenerate interval. There are two cases possible.

(i) $[y_1, y_2] \cap (C^0 \setminus C_0^0) = \emptyset$ for any $y_1, y_2 \in d_{z_0}(\mathcal{L})$. Then $d_{z_0}(\mathcal{L}) \subset [0, \delta) \setminus (C^0 \setminus C_0^0)$ and the set $d_{z_0}(\mathcal{L}) = \{\alpha_0\}$ is connected.

(ii) $[y_1, y_2] \cap (C^0 \setminus C_0^0) \neq \emptyset$ for some $y_1, y_2 \in d_{z_0}(\mathcal{L})$ ($y_1 \neq y_2$). Then $g_0(\mathcal{L}) \supset \phi_0\big([y_1, y_2] \cap (C^0 \setminus C_0^0)\big) = K_{\mathbb{Q}}^m(\alpha_0, \tau)$. Since

$$g_0(\mathcal{L}) = K_{\mathbb{Q}}^m(\alpha_0, \tau) \cup \{\alpha_0\}, \qquad K_{\mathbb{Q}}^m(\alpha_0, \tau) \subset g_0(\mathcal{L}) \subset \overline{K_{\mathbb{Q}}^m(\alpha_0, \tau)},$$

so $g_0(\mathcal{L})$ is a connected set.

Thus, in order to finish the proof of the P-Darboux property of the function g_0, it suffices to consider the case when

$$\mathcal{L} \cap K^n(z_0, \delta) \neq \emptyset \qquad \text{and} \qquad \mathcal{L} \setminus K^n(z_0, \delta) \neq \emptyset\,.$$

The connectedness of $d_{z_0}(\mathcal{L})$ and the Cantor set properties imply that $[y_1, y_2] \cap (C^0 \setminus C^0_0) \neq \emptyset$ for some $y_1, y_2 \in d_{z_0}\big(\mathcal{L} \cap K^n(z_0, \delta)\big)$. Moreover

$$g_0\big(\mathcal{L} \cap K^n(z_0, \delta)\big) \supset \phi_0\big([y_1, y_2] \cap (C^0 \setminus C^0_0)\big) = K^m_{\mathbb{Q}}(\alpha_0, \tau)\,.$$

It follows by condition (1) and definition of the function g_0 that

$$K^m_{\mathbb{Q}}(\alpha_0, \tau) \subset g_0\big(\mathcal{L} \cap \overline{K}^n(z_0, \delta)\big) \subset K^m(\alpha_0, \tau) \subset \overline{K}^m(\alpha_0, \tau) = \overline{K^m_{\mathbb{Q}}(\alpha_0, \tau)}\,.$$

The above inclusions allow one to deduce that

$$\text{the set } g_0\big(\mathcal{L} \cap \overline{K}^n(z_0, \delta)\big) \text{ is connected}. \tag{2}$$

If $\mathcal{L} \subset \overline{K}^n(z_0, \delta)$ then (2) follows connectedness of $g_0(\mathcal{L})$. So, let us assume that

$$\mathcal{L} \setminus \overline{K}^n(z_0, \delta) \neq \emptyset\,. \tag{3}$$

We shall show the following condition

$$\text{if } K \text{ is a component of the set } \mathcal{L} \setminus K^n(z_0, \delta) \text{ then } K \cap S^n(z_0, \delta) \neq \emptyset\,. \tag{4}$$

In fact, it is clear that $\mathcal{L} \setminus K^n(z_0, \delta)$ is a non-empty and closed set in $\mathcal{L}$. Because of $\operatorname{Int}_{\mathcal{L}}\big(\mathcal{L} \setminus K^n(z_0, \delta)\big) \supset \mathcal{L} \setminus \overline{K}^n(z_0, \delta) \neq \emptyset$ then, in view of closeness of $\mathcal{L} \setminus K^n(z_0, \delta)$ in $\mathcal{L}$, we infer that the inclusion $\operatorname{Fr}_{\mathcal{L}}\big(\mathcal{L} \setminus K^n(z_0, \delta)\big) \subset \mathcal{L} \cap S^n(z_0, \delta)$ holds. So, the inequality $K \cap \operatorname{Fr}_{\mathcal{L}}\big(\mathcal{L} \setminus K^n(z_0, \delta)\big) \neq \emptyset$ (see [1], Part I, Th. 13, p. 239) implies that condition (4) is true.

Let further $\{K_s\}_{s \in S}$ denote the family of all components of the set $\mathcal{L} \setminus K^n(z_0, \delta)$. It is easy to notice that K_s is an arc or a one-point set for any $s \in S$. Since the P-Darboux property of the function f_0, we infer that

$$g_0(K_s) = f_0(K_s) \text{ is connected}. \tag{5}$$

By the (4) we can write down

$$g_0(K_s) \cap g_0\big(\mathcal{L} \cap \overline{K}^n(z_0, \delta)\big) \neq \emptyset\,. \tag{6}$$

Since (5) and (6) we can deduce that $g_0(K_s) \cup g_0\big(\mathcal{L} \cap \overline{K}^n(z_0, \delta)\big)$ is a connected set, for any $s \in S$. It follows that the set

$$g_0(\mathcal{L}) = g_0\big(\mathcal{L} \cap \overline{K}^n(z_0, \delta)\big) \cup \bigcup_{s \in S} g_0(K_s)$$

is connected. The proof of the P-Darboux property of the function g_0 is complete.

In order to finish the proof of P-Darboux property of the function h_0 it suffices to show that

$$g_1(K) \text{ is a connected set for any connected set } K \subset \mathbb{R}^m . \tag{7}$$

Indeed, if $K \subset \mathbb{R}^m \setminus K^m(\alpha_0, \tau)$ then $g_1(K) = K$. Let us assume that $K \subset K^m(\alpha_0, \tau)$. Then $d_{\alpha_0}(K) \subset [0, \tau)$ is a connected set. If $d_{\alpha_0}(K)$ is a singleton then $g_1(K)$ is a one-point set. Let now $d_{\alpha_0}(K)$ be a non-degenerate interval. There are two cases, that are possible:

(i) $[y_1, y_2] \cap (C^1 \setminus C_0^1) = \emptyset$ for any $y_1, y_2 \in d_{\alpha_0}(K)$. Then $g_1(K) = \{\alpha_0\}$.

(ii) $[y_1, y_2] \cap (C^1 \setminus C_0^1) \neq \emptyset$ for some $y_1, y_2 \in d_{\alpha_0}(K)$. The properties of the Cantor set imply that, without loss of generality, we can assume that $y_1, y_2 \neq 0$ ($y_1 \neq y_2$). In view of the definition of the function g_1, under this additional assumption, we can infer that

$$g_1(K) = K_{\mathbb{Q}}^m(\alpha_0, \tau) \cup \{\alpha_0\}, \qquad K_{\mathbb{Q}}^m(\alpha_0, \tau) \subset g_1(K) \subset \overline{K_{\mathbb{Q}}^m(\alpha_0, \tau)},$$

and consequently $g_1(K)$ is a connected set.

Thus, in order to finish the proof of the condition (7), it suffices to consider the case when

$$K \cap K^m(\alpha_0, \tau) \neq \emptyset \qquad \text{and} \qquad K \setminus K^m(\alpha_0, \tau) \neq \emptyset .$$

The set $J_1 = d_{\alpha_0}(K)$ is connected and $[y_1, y_2] \cap (C^1 \setminus C_0^1) \neq \emptyset$ for some $y_1, y_2 \in J_1 \cap [0, \tau)$. Hence it appears that

$$g_1\left(K \cap K^m(\alpha_0, \tau)\right) \supset K_{\mathbb{Q}}^m(\alpha_0, \tau) \cup \{\alpha_0\}$$

and the following equality is also true:

$$K_{\mathbb{Q}}^m(\alpha_0, \tau) \subset g_1\left(K \cap \overline{K}^m(\alpha_0, \tau)\right) \subset \overline{K}^m(\alpha_0, \tau)$$

and, consequently the set $g_1\left(K \cap \overline{K}^m(\alpha_0, \tau)\right)$ is a connected one. Because of $K \cap S^m(\alpha_0, \tau) \neq \emptyset$ we can observe that the union

$$K \cup g_1\left(K \cap \overline{K}^m(\alpha_0, \tau)\right) \text{ is a connected set} .$$

The following equality is also true

$$K \cup g_1\left(K \cap \overline{K}^m(\alpha_0, \tau)\right) \setminus g_1\left(K \cap \overline{K}^m(\alpha_0, \tau)\right) = A \cup B ,$$

where $A = \left[K \cap K^m(\alpha_0, \tau)\right] \setminus g_1\left(K \cap K^m(\alpha_0, \tau)\right)$ and $B = K \setminus \overline{K}^m(\alpha_0, \tau)$ are separated sets. On the basis of Theorem 8, p. 228 included in [2], we deduce that equality

$$g_1(K) = g_1\left(K \cap \overline{K}^m(\alpha_0, \tau)\right) \cup B$$

holds and $g_1(K)$ is a connected set. The P-Darboux property of the function h_0 has finally been concluded.

We shall show that h_0 possesses a dense set of points of quasi-continuity. Let $x \in \mathbb{R}^n$, $U = K^n(x, \delta_1)$ and V be an open neighbourhood of $h_0(x)$.

I. Let $d(z_0, x) > \delta$. Then $x \in \mathbb{R}^n \setminus \overline{K}^n(z_0, \delta)$. One can assume (without loss of generality) that

$$U \subset \mathbb{R}^n \setminus \overline{K}^n(z_0, \delta). \tag{8}$$

Let us observe that

$$h_0(x) = g_1\bigl(g_0(x)\bigr) = g_1\bigl(f_0(x)\bigr). \tag{9}$$

Denote $\eta = d\bigl(\alpha_0, f_0(x)\bigr)$. Let us consider the following three cases:

1. $\eta > \tau$. Then $g_1\bigl(f_0(x)\bigr) = f_0(x)$ and $h_0(x) = f_0(x) \in \mathbb{R}^m \setminus \overline{K}^m(\alpha_0, \tau)$. One can assume that

$$V \subset \mathbb{R}^m \setminus \overline{K}^m(\alpha_0, \tau). \tag{10}$$

The function f_0 is quasi-continuous. It follows that $f_0(W) \subset V$ for some open non-empty set $W \subset U$. Since (8) one can deduce that $g_0(W) = f_0(W) \subset V$. On account of (10) and the definition of g_1 we can ascertain that $h_0(W) = g_1\bigl(g_0(W)\bigr) = g_0(W) = f_0(W) \subset V$. Of course, x is a point of quasi-continuity of the function h_0.

2. Let $\eta \in [0, \tau] \setminus C^1 \subset \bigl([0, \tau) \setminus (C^1 \setminus C_0^1)\bigr) \cup \{0\}$. Then

$$g_1\bigl(f_0(x)\bigr) = {g_1}'(\eta) = \alpha_0. \tag{11}$$

From closeness of the Cantor set we can deduce that there exists $\tilde{\delta} > 0$ such that $(\eta - \tilde{\delta}, \eta + \tilde{\delta}) \subset [0, \tau] \setminus C^1$. Let $U_1 = K^m(\alpha_0, \eta + \tilde{\delta}) \setminus \overline{K}^m(\alpha_0, \eta - \tilde{\delta})$. It is easy to notice that U_1 is an open neighbourhood of $f_0(x)$.

The definition of the function g_1 implies that

$$g_1(U_1) = {g_1}'(\eta - \tilde{\delta}, \eta + \tilde{\delta}) = \{\alpha_0\}. \tag{12}$$

Because f_0 is quasi-continuous then there exists open non-empty set $W \subset U$ such that $f_0(W) \subset U_1$. The last condition and (8), (9), (11) allow us to write down

$$h_0(W) = g_1\bigl(g_0(W)\bigr) \subset g_1(U_1) = \{\alpha_0\} = \{h_0(x)\} \subset V$$

and h_0 is quasi-continuous at a point x.

3. Let $\eta \in C^1 \cap [0, \tau]$.
Assume that

$$\begin{gathered}\text{there is not any quasi-continuous point of the function } h_0\\ \text{belonging to } U' = K^n(x, \delta_2) \text{ for some } \delta_2 \in (0, \delta_1).\end{gathered} \tag{13}$$

Because of the inclusion $U' \subset U$, definition of g_0 and the condition (8) we have

$$g_0(U') = f_0(U'). \tag{14}$$

The point x is not a continuity point of the function h_0. It implies that the function $f_0|U'$ is not a constant function. There is $x' \in U'$ such that $f_0(x) \neq f_0(x')$. Let now x' be an arbitrary point satisfying the above condition. From the P-Darboux property of the function f_0 the set $f_0([x, x'])$ is a connected one. Of course $\eta \in J = d_{\alpha_0}(f_0([x, x']))$ and J is a connected subset of $[0, +\infty)$.

If $J \cap (\tau, +\infty) \neq \emptyset$ then $d(\alpha_0, f_0(\tilde{x})) > \tau$ for some $\tilde{x} \in f_0([x, x'])$. As can be easy observed, the assumptions of the **1** case are satisfied and $x' \in U'$ is a quasi-continuity point of the function h_0. It contradicts to (13).

If $J \subset [0, \tau]$ is a non-degenerate interval then the Cantor set properties imply that there exists $y \in J \cap ([0, \tau] \setminus C^1)$. It follows that $d(\alpha_0, f_0(x_y)) \in [0, \tau] \setminus C^1$ for some $x_y \in [x, x'] \subset U'$. The assumptions of the **2** case are satisfied. Since x_y is a quasi-continuity point of the function h_0, we obtain a contradiction of the condition (13).

Finally $J = \{\eta\}$. This, in view of arbitrariness of the choice of the point $x' \in U'$, implies the following condition

$$d_{\alpha_0}(f_0(U')) = \{\eta\}. \tag{15}$$

Two cases are possible.

(i) Let $\eta = \tau$. From (15) one can deduce that

$$f_0(x) \in f_0(U') \subset S^m(\alpha_0, \tau). \tag{16}$$

By the definition of the mapping g_1, in view of (9) one can write down

$$h_0(x) = g_1(f_0(x)) = f_0(x). \tag{17}$$

It follows that V is a neighbourhood of $f_0(x)$. From quasi-continuity we can deduce that

$$f_0(W) \subset V \tag{18}$$

for some open non-empty set $W \subset U' \subset U$. By (8) we have $g_0(W) = f_0(W)$. It implies the condition

$$h_0(W) = g_1(g_0(W)) = g_1(f_0(W)). \tag{19}$$

But (16) implies $f_0(W) \subset S^m(\alpha_0, \tau)$. Since the definition of the transformation g_1, in view of (19) we can infer that $h_0(W) = f_0(W)$. The last equality and (18) allow us to deduce that h_0 is quasi-continuous at x. It contradicts to (13).

(ii) Let $\eta \neq \tau$. Then by (14) and (15)

$$\begin{aligned}&h_0(x) \in h_0(U') = g_1\big(g_0(U')\big) = g_1\big(f_0(U')\big) = g_1\big(\{\eta\}\big)\\ &\text{and } \ g_1\big(\{\eta\}\big) \ \text{ is a singleton}.\end{aligned}$$

By the condition $h_0(U') = \{h_0(x)\} \subset V$, h_0 is quasi-continuous at x. It contradicts to (13).

It has just been shown that if x is not a quasi-continuous point of the function h_0 then there is another quasi-continuous point of this function in any neighbourhood of the point x.

II. Let $d(z_0,x) \in \big([0,\delta]\setminus C^0\big) \subset \big([0,\delta)\setminus(C^0\setminus C^0_0)\big)$. The closeness of the set C^0 allow us to deduce that δ_1 can be chosen in a such way that $d_{z_0}(U) \subset [0,\delta]\setminus C^0$. It follows that

$$h_0(U) = g_1\big(g_0(U)\big) = g_1\big(\{\alpha_0\}\big) = \{\alpha_0\}.$$

It implies that $h_0(U) = \{h_0(x)\} \subset V$ and h_0 is quasi-continuous at x.

III. Let $d(z_0,x) \in [0,\delta] \cap C^0$. Then there exists $z \in \big(K^n(x,\delta_1)\setminus\{x\}\big) \cap K^n(z_0,\delta)$. Consider the interval $[z,x]$. Then $d_{z_0}\big([z,x]\big) = \big[d(z_0,z), d(z_0,x)\big]$ is a non-degenerate interval contained in $[0,\delta]$. By the Cantor set properties we can infer that there exists $y \in d_{z_0}\big([z,x]\big)\setminus C^0$. Moreover $d(z_0,x_y) = y \in [0,\delta]\setminus C^0$ for some $x_y \in [z,x] \subset U$. So, the point $x_y \in U$ satisfies the assumptions of the case **II.** and x_y is a quasi-continuity point of the function h_0.

We have just proved that h_0 possesses a dense set of points of quasi-continuity.

Since (1) and the definition of the transformation h_0 we can observe that the Banach indicatrix satisfies the following condition

$$\mathscr{N}_{h_0|\mathbb{R}^m \setminus K^m_{\mathbb{Q}}(\alpha_0,\tau)\cup\{\alpha_0\}}(p) = \begin{cases} \mathscr{N}_{f_0}(p) & \text{if } p \in \mathbb{R}^m \setminus K^m(\alpha_0,\tau),\\ 0 & \text{if } p \in K^m(\alpha_0,\tau).\end{cases}$$

Of course m-dimensional Lebesgue measure of the set $K^m_{\mathbb{Q}}(\alpha_0,\tau) \cup \{\alpha_0\}$ is equal to zero. Therefore a measurability of the function $\mathscr{N}_{f_0}$ implies a measurability of the function $\mathscr{N}_{h_0}$. In order to finish the proof of the finite variation of the function h_0 it suffices to observe that an inequality $\int\limits_{\mathbb{R}^m} \mathcal{N}_{h_0}(p)\,dp < +\infty$ is a consequence of the condition $\int\limits_{\mathbb{R}^m} \mathcal{N}_{f_0}(p)\,dp < +\infty$.

It is easy to verify that $h_0 \in K(f_0, 4\tau)$. We shall show that there is no quasi-continuous function contained in $K\big(h_0, \frac{\tau}{4}\big)$.

Suppose that

$$\text{there is a quasi-continuous function } i_0 \in K\Big(h_0, \frac{\tau}{4}\Big). \tag{20}$$

By definition of the function h_0 we can write down

$$h_0\big(K^n(z_0,\delta)\big) = g_1\big(g_0(K^n(z_0,\delta))\big) = g_1\big(K^m_{\mathbb{Q}}(\alpha_0,\tau)\cup\{\alpha_0\}\big)$$
$$= K^m_{\mathbb{Q}}(\alpha_0,\tau)\cup\{\alpha_0\}\,.$$

The above equality and Lemma 1 imply that

$$d\Big(\alpha_0, h_0\big(z_{\frac{\tau}{2}}\big)\Big) = \frac{\tau}{2}\,. \tag{21}$$

for some $z_{\frac{\tau}{2}} \in K^n(z_0,\delta)$. The condition (20) implies a following relation

$$i_0\big(z_{\frac{\tau}{2}}\big) \in K^m\Big(h_0\big(z_{\frac{\tau}{2}}\big), \frac{\tau}{4}\Big)\,. \tag{22}$$

By quasi-continuity of the function i_0 we can infer that there exists $W_1 = K^n\big(\tilde{z},\tilde{\delta}\big) \subset K^n(z_0,\delta)$ such that

$$i_0(W_1) \subset K^m\Big(h_0\big(z_{\frac{\tau}{2}}\big), \frac{\tau}{4}\Big)\,. \tag{23}$$

Obviously the set $K = d_{z_0}(W_1)$ is a non-degenerate open interval. There two cases are possible:

(i) $K\cap(C^0\setminus C^0_0) = \emptyset$. Then

$$h_0(W_1) = g_1\big(g_0(W_1)\big) = g_1\big(\{\alpha_0\}\big) = \{\alpha_0\}\,.$$

By the above and (20) we can write down

$$i_0(W_1) \subset K^m\Big(\alpha_0, \frac{\tau}{4}\Big)\,. \tag{24}$$

From (23) and (24) we have

$$K^m\Big(\alpha_0,\frac{\tau}{4}\Big)\cap K^m\Big(h_0\big(z_{\frac{\tau}{2}}\big),\frac{\tau}{4}\Big) \neq \emptyset\,. \tag{25}$$

It contradicts to (21).

(ii) $K\cap(C^0\setminus C^0_0) \neq \emptyset$. Then $[y_1,y_2]\cap(C^0\setminus C^0_0)\neq\emptyset$ for some $y_1,y_2\in K$ ($y_1\neq y_2$). By definition of h_0 we have

$$h_0(W_1) = g_1\big(g_0(W_1)\big) \supset g_1\big(\phi_0(K)\big) \supset g_1\big(\phi_0\big([y_1,y_2]\cap(C^0\setminus C^0_0)\big)\big)$$
$$= g_1\big(K^m_{\mathbb{Q}}(\alpha_0,\tau)\big) = {g_1}'\big(d_{\alpha_0}\big(K^m_{\mathbb{Q}}(\alpha_0,\tau)\big)\big) \supset {g_1}'\big((0,\tau)\big)$$
$$= K^m_{\mathbb{Q}}(\alpha_0,\tau)\cup\{\alpha_0\}\,.$$

The last condition and (20) implies

$$K^m\Big(\alpha_0,\frac{\tau}{4}\Big)\cap i_0(W_1)\neq\emptyset\,.$$

By (23) and the above we can see that (25) is true. It contradicts to (21).

We showed that (20) does not hold.

The results of the above considerations allow us to ascertain that the porosity at the point f_0 is not less than the number

$$2\cdot\lim_{\tau\to 0^+}\frac{\frac{\tau}{4}}{4\cdot\tau} = \frac{1}{8} > 0\,.$$

The proof of the theorem has thus been completed. □

REFERENCES

[1] DUDA, R.: *Wprowadzenie do topologii*, PWN, Warszawa, 1986.

[2] KURATOWSKI, K.: *Introduction to Set Theory and Topology*, PWN, Warszawa, 1972.

[3] LIPIŃSKI, J. S.—ŠALÁT, T.: *On the points of quasi-continuity and cliquishness of functions*, Czechoslovak Math. J. **21** (1971), 484–489.

[4] NATKANIEC, T.: *Almost Continuity*, Thesis, Bydgoszcz, 1992.

[5] NEUBRUNNOVÁ, A.: *On quasi-continuous and cliquish functions*, Čas. Pěst. Mat. **99** (1974), 109–114.

[6] PAWLAK, R. J.: *Darboux Transformations*, Thesis, Łódź, 1985.

[7] RYCHLEWICZ, A.: *On continuous and quasi-continuous functions*, Real Anal. Exchange **19** (1993/94), 547–563.

[8] ŠALÁT, T.: *Generalization of the notion of the Banach indicatrix*, Fund. Math. **LXXIII** (1971), 29–36.

Received September 22, 1998

The Faculty of Mathematics
of Łódź University
Banacha 22
PL–90–238 Łódź
POLAND
E-mail: anrychle@imul.math.uni.lodz.pl