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APPLICATIONS OF THE HAUSDORFF APPROACH TO SEMICONTINUOUS FUNCTIONS

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ABSTRACT. Let $(T, \mathscr{T})$ be a measurable space, X a metric space, D a subset of $T \times X$, and f a real-valued function on D. We give sufficient conditions for f to be the limit of an increasing sequence of Carathéodory functions. The proof of the main theorem is based on the parametrization of a Hausdorff formula for l.s.c. functions.

1. Preliminaries

In this section we introduce the notation and formulate the problem.

Throughout this paper $(T, \mathscr{T})$ is a measurable space, X and Y metric spaces, and D a nonempty subset of $T \times X$. The set D is always considered with the trace σ-field $\mathscr{D} = \{D \cap A \colon A \in \mathscr{T} \otimes \mathscr{B}(X)\}$, where $\mathscr{B}(X)$ stands for the Borel σ-field on X. By D_t and D^x we denote t and x-sections of D, respectively, i.e., $D_t = \{x \in X \colon (t,x) \in D\}$ and $D^x = \{t \in T \colon (t,x) \in D\}$. For $x \in X$ and $r > 0$, $B(x,r)$ stands for the open ball with center x and radius r.

A function $f \colon D \to Y$ is said to be *Carathéodory* if it is measurable (with respect to the σ-field $\mathscr{D}$), and $f(t,\cdot)$ is continuous on D_t for all $t \in T$. Denote by $F(D)$ the family of all real-valued functions on D which are pointwise limits of increasing sequences of Carathéodory functions. If $f \in F(D)$ then it is measurable, and $f(t,\cdot)$ is lower semicontinuous (abbreviated to l.s.c.) on D_t for all $t \in T$. The class $F(D)$ appears in the optimization theory, e.g., in the study of optimal measurable selections (see [12], [13]).

The main result of this paper, Theorem 3.2, gives sufficient conditions for f to be in the class $F(D)$. We have already studied this problem in [7] and [8], but unlike those papers, we do not assume that f is measurable. Hence, our results can serve as criteria of the measurability of a function of two variables. In [9] we gave a characterization of functions belonging to $F(D)$.

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By a multifunction from T to X we mean a mapping $\varphi\colon T \to 2^X$ with nonempty values. A multifunction φ is *measurable* if $\{t \in T\colon \varphi(t) \cap G \neq \emptyset\} \in \mathscr{T}$ for each open $G \subset X$. Note that Himmelberg [4] calls such a multifunction weakly measurable. If X is separable, then φ is measurable iff the real-valued function $t \to \operatorname{dist}(x, \varphi(t))$ is measurable for each $x \in X$ (see, e.g., [4], Theorem 3.3). We conclude this paper with some corollaries on the measurability of a multifunction of two variables.

We shall assume that D has nonempty t-sections. Hence, D may be treated as a graph of the multifunction $t \to D_t$, $t \in T$. A countable family U of measurable functions $u\colon T \to X$ is called a *Castaing representation of* D if for each $t \in T$ the set $\{u(t)\colon u \in U\}$ is dense in D_t. Different measurable selection theorems give sufficient conditions for the existence of a Castaing representation (see, e.g., [4], [14]).

2. Auxiliary results

It is well known that a l.s.c. function defined on a metric space is the pointwise limit of an increasing sequence of continuous functions. In 1919 Hausdorff gave a formula for these approximating maps.

Theorem 2.1. ([3]). *Let (X, d) be a metric space and $f\colon X \to \mathbb{R}$. If f is l.s.c. and bounded from below, then the functions*

$$f_n(x) = \inf\{f(y) + n \cdot d(x, y)\colon y \in X\}, \qquad x \in X, \ n \in \mathbb{N},$$

are Lipschitz continuous, and the sequence (f_n) is increasing and converges to f.

Let E be a fixed dense subset of X, and $f\colon X \to \mathbb{R}$. Remind that

$$\liminf_{y \in E,\, y \to x} f(y) = \lim_{\delta \to 0} \bigl(\inf f(E \cap B(x, \delta))\bigr).$$

For $E = X$ it reduces to the standard lower limit $\liminf_{t \to x} f(y)$.

We shall consider the following two conditions:

(i) For each $x \in X$

$$\liminf_{y \in E,\, y \to x} f(y) = f(x).$$

(ii) For each open $G \subset X$ the set $f(E \cap G)$ is dense in $f(G)$.

The next theorem relates these two conditions.

Theorem 2.2. *The function f satisfies* (i) *iff it satisfies* (ii) *and is l.s.c.*

Proof. Assume f satisfies (i). Fix open $G \subset X$ and $r \in f(G)$. Let $x \in G$ be such that $r = f(x)$. Denote $r_n = \inf f(E \cap B(x, 1/n))$. Because of (i),

$f(x) = \lim_{n\to\infty} r_n$. For $\varepsilon > 0$ and n sufficiently large we have $f(x)-\varepsilon \le r_n \le f(x)$. By the definition of r_n, there exists some $y \in E\cap G$ such that $r_n \le f(y) < r_n+\varepsilon$. Consequently, $f(x) - \varepsilon \le f(y) \le f(x) + \varepsilon$. Thus $f(E \cap G)$ is dense in $f(G)$.

Now we prove the lower semicontinuity of f. Fix $x \in X$. By the first part of the proof, $f\big(E \cap B(x,1/n)\big)$ is dense in $f\big(B(x,1/n)\big)$ for all $n \in \mathbb{N}$. Hence, $r_n = \inf f\big(B(x,1/n)\big)$, and $f(x) = \liminf_{y\to x} f(y)$.

Assume that f is l.s.c. and satisfies (ii). We have already proved that under condition (ii),

$$\liminf_{y\in E,\, y\to x} f(y) = \liminf_{y\to x} f(y)\,.$$

Since f is l.s.c., the last equality implies (i). It completes the proof. $\square$

We shall use the following lemma with a straightforward proof:

Lemma 2.3. *Let E be a dense subset of X, and $f, g\colon X \to \mathbb{R}$. If f satisfies* (i) *and g is continuous, then $f + g$ also satisfies* (i).

For convex and l.s.c. functions we have the following result:

Theorem 2.4. *Let X be a convex subset of a Banach space, such that $\operatorname{Int} X \ne \emptyset$, and $f\colon X \to \mathbb{R}$ a convex and l.s.c. function. Then for each dense subset $E \subset X$, f satisfies condition (i).*

Proof. Let E be a dense subset of X, and $A \subset X$ convex with nonempty interior. We are going to prove that

$$\inf\{f(x)\colon x \in A\} = \inf\{f(x)\colon x \in E \cap A\}\,.$$

Denote by l and r, respectively, the left and the right hand side of the required equality. Of course, $l \le r$. Since $f|\operatorname{Int} A$ is continuous (see, e.g., [11], Prop. 3.3), $f(x) \ge r$ for all $x \in \operatorname{Int} A$. Now fix $x \in A \cap \operatorname{Fr} A$ and $y \in \operatorname{Int} A$. The segment $S = \{\lambda x + (1-\lambda)y\colon 0 \le \lambda \le 1\}$ is contained in $\operatorname{Int} A$ except the point x. Being convex on the segment, $f|S$ is upper semicontinuous. Hence, the set $\{z \in S\colon f(z) \ge r\}$ is closed in S and, consequently, contains x. Thus $f(x) \ge r$ for all $x \in A$ which implies $l = r$.

Now fix $x \in X$. Since for each $n \in \mathbb{N}$ the set $B(x,1/n) \cap X$ is convex and has nonempty interior,

$$\inf f\big(E \cap B(x,1/n) \cap X\big) = \inf f\big(B(x,1/n) \cap X\big)\,.$$

Hence,

$$\liminf_{y\in E,\, y\to x} f(y) = \liminf_{y\to x} f(y)\,.$$

Since f is l.s.c., it means that f satisfies condition (i). $\square$

Remark 1. The proof of Theorem 2.4 is a slight generalization of the argumentation of Dynkin ([2], Lemma 3.1), who considered the case $X \subset \mathbb{R}^n$. Such X has a nonempty relative interior, and the additional assumption $\operatorname{Int} X \neq \emptyset$ is superfluous.

Remark 2. If we strengthen (ii) by the requirement, that it is satisfied for each dense $E \subset X$, then this modified condition characterizes the quasi-continuity of f (see [10]). Thus we can reformulate Theorem 2.4 in the following way: If f is a convex and l.s.c. function defined on a convex subset of a Banach space with a nonempty interior, then f is quasi-continuous.

3. Main results

Remind that $(T, \mathscr{T})$ is a measurable space, X and Y metric spaces, and D a subset of $T \times X$ with nonempty t-sections. It is well known that if X is separable, then each function $f\colon T \times X \to Y$ being measurable in t and continuous in x is product-measurable. It is not the case when f is defined on $D \subset T \times X$. Consider the following example:

Example. Let $T = X = Y = [0,1]$, $\mathscr{T} = \mathscr{L}$, the Lebesgue σ-field on [0,1], $D = \{(t,x) \in T \times X : t = x\}$, $A \subset D$, and $f = \chi_A | D$ the characteristic function of A. Such f, considered as a function from D to Y, is continuous in each variable separately, but if $A \notin \mathscr{L} \otimes \mathscr{B}([0,1])$ then f is not measurable.

We shall use the following lemma:

Lemma 3.1. ([6], Lemma 1). *Let $f\colon D \to Y$. Suppose X is separable, and D has a Castaing representation U such that for each $u \in U$ the function $t \to f(t, u(t))$ is measurable, and $f(t, \cdot)$ is continuous on D_t for each $t \in T$. Then f is measurable. If we additionally assume that D has closed t-sections, then $D \in \mathscr{T} \otimes \mathscr{B}(X)$.*

The following theorem is the main result of this paper:

Theorem 3.2. *Let $f\colon D \to \mathbb{R}$. Suppose X is separable and D has a Castaing representation U such that*

a) for each $u \in U$ the function $t \to f(t, u(t))$ is measurable;

b) for each $t \in T$, $x \in D_t$,

$$\liminf_{y \in U(t),\, y \to x} f(t,y) = f(t,x),$$

where $U(t) = \{u(t) : u \in U\}$.

Then $f \in F(D)$ and, consequently, f is measurable.

P r o o f . Assume that f is bounded from below and define $f_n \colon D \to \mathbb{R}$ by the formula

$$f_n(t,x) = \inf\{f(t,y) + n \cdot d(x,y) \colon y \in D_t\}, \qquad n \in \mathbb{N}, \tag{3.1}$$

where d is the metric of X. Because of Theorem 2.1, the sequence (f_n) is increasing and pointwise convergent to f, and for each $t \in T$, $f_n(t,\cdot)$ is continuous on D_t. We are going to prove that each f_n is measurable. In order to apply Lemma 3.1, for $u \in U$ we consider

$$f_n\big(t,u(t)\big) = \inf\{f(t,y) + n \cdot d\big(u(t),y\big) : y \in D_t\}. \tag{3.2}$$

Assumption b) means that for fixed $t \in T$, the function $f(t,\cdot)$ satisfies condition (i) with the countable dense subset $U(t)$. Consequently, $g(y) = f(t,y) + n \cdot d\big(u(t),y\big)$ also satisfies (i). Now Theorem 2.2 implies that in (3.2) the infimum of g over D_t is equal to the infimum over $U(t)$, i.e.,

$$f_n\big(t,u(t)\big) = \inf\{f\big(t,v(t)\big) + n \cdot d\big(u(t),v(t)\big) : v \in U\}.$$

Under assumption a), the function $t \to f\big(t,v(t)\big) + n\cdot d\big(u(t),v(t)\big)$ is measurable. Hence, $t \to f_n\big(t,u(t)\big)$ is also measurable. Now Lemma 3.1 implies that all f_n are measurable. Thus $f \in F(D)$.

If f is not bounded from below, then we consider $\hat{f} = h \circ f$, where h is an increasing homeomorphism of $\mathbb{R}$ and $(0,\infty)$. It not difficult to see that $\hat{f}$ satisfies conditions a) and b). By the first part of the proof, $\hat{f}$ is the limit of an increasing sequence of Carathéodory functions $\hat{f}_n \colon D \to \mathbb{R}$. It follows from (3.1) and the lower semicontinuity of $\hat{f}$ that $\hat{f}_n(t,x) > 0$ for all $n \in \mathbb{N}$, $(t,x) \in D$. If we define $f_n = h^{-1} \circ \hat{f}_n$, then (f_n) is an increasing sequence of Carathéodory functions, pointwise convergent to f. It completes the proof. □

Remark 1. Note that if $u \colon T \to X$ and $f \colon D \to Y$ are measurable, then the superposition $t \to f\big(t,u(t)\big)$ is also measurable. Hence, the superpositional measurability of f, assumed in Lemma 3.1 and Theorem 3.2, is a necessary condition for the measurability of f.

We say that D is *separable* (with separating set E) if X contains a countable dense subset E such that $E \cap D_t$ is dense in D_t for $t \in T$, and $D^x \in \mathscr{T}$ for $x \in E$ (cf. [12], [13]).

Corollary 3.3. (cf. [13], Prop. 11.6). *Let $f \colon D \to \mathbb{R}$. Suppose D is separable with the separating set E, for each $x \in E$ the function $f(\cdot,x)$ is measurable on D^x, and*

$$\liminf_{y \in E\cap D_t,\, y \to x} f(t,y) = f(t,x)$$

for all $t \in T$, $x \in D_t$. *Then* $f \in F(D)$.

Proof. We show that D has a Castaing representation U such that all assumptions of Theorem 3.2 are satisfied, and $U(t) = E \cap D_t$. Since $D^e \in \mathcal{T}$, for each $e \in E$ there exists a measurable function $p_e : T \to E$ such that $p_e(t) \in D_t$ for $t \in T$, and $p_e(t) = e$ iff $e \in D_t$. Hence, $\{p_e(t) : e \in E\} = E \cap D_t$ (cf. [12]). Each p_e has a countable range and, consequently, the function $t \to f(t, p_e(t))$ is measurable. Thus $U = \{p_e : e \in E\}$ satisfies all required conditions. □

Corollary 3.4. *Let* $f : T \times X \to \mathbb{R}$. *Suppose there exists a countable dense subset* $E \subset X$ *such that for each* $x \in E$ *the function* $f(\cdot, x)$ *is measurable, and*

$$\liminf_{y \in E,\, y \to x} f(t, y) = f(t, x)$$

for all $t \in T$, $x \in X$. *Then* $f \in F(T \times X)$.

In the next corollaries we require some convexity assumptions.

Corollary 3.5. *Let* X *be a separable Banach space and* $f : D \to \mathbb{R}$. *Suppose* D_t *is convex and has a nonempty interior for* $t \in T$, $D^x \in \mathcal{T}$ *for* $x \in X$, $f(t, \cdot)$ *is convex and l.s.c. for* $t \in T$, *and* $f(\cdot, x)$ *is measurable for* $x \in X$. *Then* $f \in F(D)$.

Proof. Let E be a countable dense subset of X. Since D_t is convex with a nonempty interior, $E \cap D_t$ is dense in D_t. Thus D is separable with the separating set E. It follows from Theorem 2.4 that

$$\liminf_{y \in E \cap D_t,\, y \to x} f(t, y) = f(t, x)$$

for all $t \in T$, $x \in D_t$. Now an application of Corollary 3.3 completes the proof. □

Corollary 3.6. (cf. [2], Lemma 3.2). *Let* X *be a convex subset of a Banach space with a nonempty interior, and* $f : T \times X \to \mathbb{R}$ *a function measurable in* t, *and convex and l.s.c. in* x. *Then* $f \in F(T \times X)$ *and, consequently,* f *is product-measurable.*

If we assume that T is a topological space, then we have the following theorem:

Theorem 3.7. (cf. [1]). *Let* T *be a topological space,* $\mathcal{T} = \mathcal{B}(T)$, X *a separable metric space, and* $f : T \times X \to \mathbb{R}$. *If* f *is upper semicontinuous in* t *and l.s.c. in* x, *then* $f \in F(T \times X)$.

Proof. Assume f is bounded from below. Then $f_n : T \times X \to \mathbb{R}$ defined by the parametrization of the Hausdorff formula

$$f_n(t, x) = \inf\{f(t, y) + n \cdot d(x, y) : y \in X\}$$

is continuous in x, and the sequence (f_n) is increasing and convergent to f. Since for each fixed x and y the function $t \to f(t,y) + n \cdot d(x,y)$ is upper semicontinuous (abbreviated by u. s. c.), f_n is also u. s. c. in t. Hence, each f_n is $\mathscr{B}(T) \otimes \mathscr{B}(X)$-measurable. It means that $f \in F(T \times X)$.

If f is not bounded from below, then we argue in the same way as in the proof of Theorem 3.2. □

Remark 2. It would be interesting to generalize Theorem 3.7 for the case, when f is defined on $D \subset T \times X$ instead of the whole product $T \times X$.

Remark 3. Theorem 3.7 is related to the classical result of Kempisty [5]: If a real-valued function f, defined on the product of two metric spaces T and X, is u.s.c. in one and l.s.c. in the second variable, then it is in the first class of Baire (i.e., it is the limit of a sequence of continuous functions). It is not difficult to see that the Kempisty proof holds in the case, when f is defined on $D \subset T \times X$ with open t and x-sections.

Now we give some corollaries on the measurability of a multifunction $\varphi\colon D \to 2^Y$.

Corollary 3.8. *Let X, Y be separable metric spaces, $D \subset T \times X$, and φ a multifunction from D to Y. Suppose for each $t \in T$ and $y \in Y$ the function $x \to \operatorname{dist}\bigl(y, \varphi(t,x)\bigr)$ is continuous on D_t, and D has a Castaing representation U such that the multifunction $t \to \varphi\bigl(t, u(t)\bigr)$ is measurable for all $u \in U$. Then φ is measurable.*

Proof. For each $y \in Y$ the function $f\colon D \to \mathbb{R}$ defined by $f(t,x) = \operatorname{dist}\bigl(y, \varphi(t,x)\bigr)$ satisfies all assumptions of Lemma 3.1. Thus f is measurable. □

Let φ be a multifunction from a convex set A to a linear space Z. We say that φ is *convex*, if $\lambda\varphi(a) + (1-\lambda)\varphi(b) \subset \varphi\bigl(\lambda a + (1-\lambda)b\bigr)$ for all $a, b \in A$ and $\lambda \in [0,1]$. Note that φ is convex iff its graph $\operatorname{Gr}\varphi = \{(a,z)\colon z \in \varphi(a)\}$ is a convex subset of $A \times Z$.

Let ψ be a multifunction from a topological space X to a metric space Y. Such ψ is *H-upper semicontinuous* if for each $x_o \in X$ and each $\varepsilon > 0$ there is a neighbourhood V of x_o such that $\psi(x) \subset B\bigl(\psi(x_0), \varepsilon\bigr)$ for all $x \in V$. A multifunction ψ is *lower semicontinuous* if for each open $G \subset Y$ the preimage $\{x \in X\colon \psi(x) \cap G \neq \emptyset\}$ is open in X.

Corollary 3.9. *Let X be a separable Banach space, Y a separable normed space, D a subset of $T \times X$, and φ a multifunction from D to Y. We assume that for each $t \in T$, D_t is convex with a nonempty interior and the multifunction $\varphi(t,\cdot)$ is convex and H-upper semicontinuous, and for each $x \in \operatorname{proj}_T D$, $D^x \in \mathscr{T}$ and $\varphi(\cdot, x)$ is measurable. Then φ is measurable.*

P r o o f . For each $y \in Y$ the function $(t,x) \to \operatorname{dist}\big(y, \varphi(t,x)\big)$ satisfies all assumptions of Corollary 3.5. Thus it is measurable. □

COROLLARY 3.10. *Let T be a topological space, $\mathscr{T} = \mathscr{B}(T)$, X and Y separable metric spaces, and $\varphi\colon T \times X \to 2^Y$ a multifunction. If φ is lower semicontinuous in t and H-upper semicontinuous in x, then φ is $\mathscr{B}(T) \otimes \mathscr{B}(X)$-measurable.*

P r o o f . For each $y \in Y$ the real-valued function $f(t,x) = \operatorname{dist}\big(y, \varphi(t,x)\big)$ is u.s.c. in t and l.s.c. in x. It follows from Theorem 3.7 that f is product-measurable. □

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