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ON THE STRONG QUASI-CONTINUITY OF MULTIVALUED MAPS OF TWO VARIABLES

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ABSTRACT. Various properties describing the strong quasi-continuity of real functions of two variables were considered in [5]. The results contained here generalize some of them onto the case of multivalued maps.

Let $\mathbb{R}$ denote the set of real numbers and let E denote $\mathbb{R}$ or $\mathbb{R}\times\mathbb{R}$. For $y \in E$ and $\varepsilon > 0$ we denote $K(y,\varepsilon) = \{x \in E\colon |x-y| < \varepsilon\}$. Moreover, for a nonempty set $A \subset E$ we have $K(A,\varepsilon) = \bigcup\{K(y,\varepsilon)\colon y \in A\}$. Let $\mathscr{L}(E)$ be the family of Lebesgue measurable sets on E and let $\mu^*(\mu)$ be the outer Lebesgue measure (the Lebesgue measure) in E. Denote by

$$d_u(A,p) = \limsup_{h\to 0} \mu^*\big(A \cap K(p,h)\big)/\mu\big(K(p,h)\big)$$

$$\Big(d_l(A,p) = \liminf_{h\to 0} \mu^*\big(A \cap K(p,h)\big)/\mu\big(K(p,h)\big)\Big)$$

the upper (lower) outer density of a set $A \subset E$ at p. If there exists a measurable set $B \subset A$ such that $d_l(B,p) = 1$ then p is called a density point of A. The family of sets

$$\mathscr{T}_d(E) = \{A \in \mathscr{L}(E)\colon \text{ every point } p \in A \text{ is a density point of } A\}$$

forms a topology which is called the density topology [1, 3, 8, 9].

We consider the multivalued maps $F\colon E \to \mathbb{R}$ (i.e., $F(x)$ is a non-empty subset of $\mathbb{R}$). For any set $A \subset E$ we denote

$$F^+(A) = \{x \in E\colon F(x) \subset A\},$$
$$F^-(A) = \{x \in E\colon F(x) \cap A \neq \emptyset\}.$$

The following definitions and theorems generalize those of Z. Grande [5]. Some of them were considered in [10] in the case of multivalued maps of one variable.

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A multivalued map $F\colon E \to \mathbb{R}$ is said to be *strongly lower* (*upper*) *quasi-continuous at* $p \in E$ if for every open set $G \subset \mathbb{R}$ such that $p \in F^-(G)$ ($p \in F^+(G)$) and for any set $U \in \mathcal{T}_d(E)$ containing p there exists a nonempty open set $W \subset E$ such that $W \cap U \neq \emptyset$ and $W \cap U \subset F^-(G)$ ($W \cap U \subset F^+(G)$).

F is said to be strongly lower (upper) quasi-continuous if it is strongly lower (upper) quasi-continuous at each point $p \in E$.

A multivalued map $F\colon E \to \mathbb{R}$ is said to be *strongly cliquish at* $p \in E$ if for every $\varepsilon > 0$ and $U \in \mathcal{T}_d(E)$ containing p there is a nonempty open set $W \subset E$ such that $W \cap U \neq \emptyset$ and

$$\bigcap_{x \in W \cap U} K\big(F(x), \varepsilon\big) \neq \emptyset .$$

F is said to be cliquish if it is cliquish at every $p \in E$.

A multivalued map $F\colon E \to \mathbb{R}$ has *the property* $B^-(p)$ ($B^+(p)$) at $p \in E$ if for any open set $G \subset \mathbb{R}$ such that $p \in F^-(G)$ ($p \in F^+(G)$) we have

$$d_u\big(\operatorname{Int} F^-(G), p\big) > 0$$
$$\big(d_u(\operatorname{Int} F^+(G), p\big) > 0\big) .$$

In short we write it $F \in B^-(p)$ ($F \in B^+(p)$).

Theorem 1. ([10], Theorem 1) *Any multivalued map $F\colon E \to \mathbb{R}$ which has the property $B^-(p)$ ($B^+(p)$) at $p \in E$ is strongly lower (upper) quasi-continuous at this point.*

The converse is not true, as the following example shows.

Example. ([10], Example 1) Let $C \subset [0,1]$ be the Cantor's set such that $\mu(C) > 0$. Denote by P a set of all centres of components of the set $[0,1] \backslash C$. Define a multivalued map $F\colon [0,1] \to \mathbb{R}$ as follows:

$$F(x) = \begin{cases} [0,1] & \text{if } x \in P\,, \\ \{0\} & \text{if } x \in [0,1] \setminus P\,. \end{cases}$$

Then F is strongly upper quasi-continuous at each point $p \in [0,1]$, but $F \notin B^+(p)$ at a point $p \in C$ which is a density point of the set C.

Theorem 2. *Let $F\colon E \to \mathbb{R}$ be a multivalued map with compact values. If F is both strongly lower and upper quasi-continuous, then $F \in B^-(p) \cap B^+(p)$ for every $p \in E$.*

For the proof we refer the reader to [10].

Now let $E = \mathbb{R} \times \mathbb{R}$. We consider multivalued maps $F\colon E \to \mathbb{R}$. For a given $x \in \mathbb{R}$ we call an F_x section the multifunction $F_x(y) = F(x,y)$. The F^y section is defined analogously.

The classical result of K e m p i s t y concerning the quasi-continuity of functions of two variables states that the quasi-continuity of f_x sections and f^y sections implies the quasi-continuity of f. For multivalued maps we have another result given by N e u b r u n n [6, 7]. Roughly speaking, the upper (lower) quasi-continuity of F_x and both upper and lower quasi-continuity of F^y imply the upper (lower) quasi-continuity of F. The situation is different for the strong quasi-continuity. There are functions $f\colon E \to \mathbb{R}$ such that all sections f_x and f^y are continuous and f are not strongly quasi-continuous (see [4, 5]). But we can consider the following property.

Definition 1. A multivalued map $F\colon E \to \mathbb{R}$ has the property $H^-(x,y)$ ($H^+(x,y)$) at $(x,y) \in E$ if for every open set $G \subset \mathbb{R}$ such that $(x,y) \in F^-(G)$ ($(x,y) \in F^+(G)$) and for any sets $U, V \in \mathscr{T}_d(\mathbb{R})$ such that $x \in U$ and $y \in V$ there exists a nonempty open set $W \subset E$ such that $W \cap (U \times V) \neq \emptyset$ and $W \cap (U \times V) \subset F^-(G)$ ($W \cap (U \times V) \subset F^+(G)$).

In short we write it $F \in H^-(x,y)$ ($H^+(x,y)$).

Lemma 1. *For a multivalued map $F\colon E \to \mathbb{R}$ with compact values the following conditions are equivalent:*

(i) $F \in H^+(x,y)$,

(ii) *for any $\varepsilon > 0$ and any sets $U, V \in \mathscr{T}_d(\mathbb{R})$ such that $x \in U$, $y \in V$ there exists a nonempty open set $W \subset E$ such that $W \cap (U \times V) \neq \emptyset$ and $W \cap (U \times V) \subset F^+\big(K(F(x,y),\varepsilon)\big)$.*

Lemma 2. *For any multivalued map $F\colon E \to \mathbb{R}$ the following conditions are equivalent:*

(i) $F \in H^-(x,y)$,

(ii) *for any $\varepsilon > 0$, $z \in F(x,y)$ and $U, V \in \mathscr{T}_d(\mathbb{R})$ such that $x \in U$, $y \in V$ there exists a nonempty open set $W \subset E$ such that $W \cap (U \times V) \neq \emptyset$ and $W \cap (U \times V) \subset F^-\big(K(z,\varepsilon)\big)$.*

P r o o f . We give only the proof of Lemma 1; that of 2 is similar. The implication (i)$\Rightarrow$(ii) is obvious, so we will prove (ii)$\Rightarrow$(i). Let us take $U, V \in \mathscr{T}_d(\mathbb{R})$, such that $x \in U$, and $y \in V$. Let $G \subset \mathbb{R}$ be an arbitrary open set such that $(x,y) \in F^+(G)$. Then $F(x,y) \subset G$ and for every $z \in F(x,y)$ there exists a real $\varepsilon_z > 0$ such that $K(z, 2\varepsilon_z) \subset G$. $F(x,y)$ is compact, and so there are $z_1, z_2, \dots, z_n \in F(x,y)$ such that $F(x,y) \subset \bigcup_{i=1}^{n} K(z_i, \varepsilon_{z_i}) \subset G$. By assumption, for $\varepsilon = \min\{\varepsilon_{z_i} : i = 1, 2, \dots, n\}$ there is a nonempty open set $W \subset E$ such that $\emptyset \neq W \cap (U \times V) \subset F^+\big(K(F(x,y),\varepsilon)\big)$. It is easy to check that $K(F(x,y),\varepsilon) \subset G$. Consequently, $W \cap (U \times V) \subset F^+(G)$ and (i) is proved. $\square$

Then the following theorem holds.

THEOREM 3. *Let $F\colon E \to \mathbb{R}$ be a multivalued map with compact values such that F_x are strongly upper (lower) quasi-continuous for every $x \in \mathbb{R}$ and F^y are both strongly lower and upper quasi-continuous for every $y \in \mathbb{R}$. Then $F \in H^+(x,y)$ $(F \in H^-(x,y))$ for every $(x,y) \in E$.*

Proof. We give the proof only for an "upper" case. The "lower" case may be proved in much the same way. Let $(x,y) \in E$ be an arbitrary point. Fix $\varepsilon > 0$ and $U, V \in \mathscr{T}_d(\mathbb{R})$ such that $x \in U$ and $y \in V$. Since F^y is both strongly lower and upper quasi-continuous, $F^y \in B^+(x)$, by Theorem 2. This means that $d_u\big(\mathrm{Int}(F^y)^+\big(K\big(F(x,y),\varepsilon/4\big)\big),\, x\big) > 0$. So we can choose an open interval I such that $I \cap U \neq \emptyset$ and for every $t \in I$

$$F(t,y) \subset K\big(F(x,y),\varepsilon/4\big)\,. \tag{1}$$

Let $P = \mathrm{cl}(I \cap U)$ and $t \in P$. As F_t is strongly upper quasi-continuous at y, there is an open interval J_t with rational endpoints such that $J_t \cap V \neq \emptyset$ and for every $v \in J_t \cap V$

$$F(t,v) \subset K\big(F(t,y),\varepsilon/4\big)\,. \tag{2}$$

There is an open interval J such that a set $K = \{t \in P\colon J_t = J\}$ is of the second category in P. Therefore, we can choose an open interval $I_1 \subset I$ such that $I_1 \cap P \neq \emptyset$ and $I_1 \cap K$ is dense in $I_1 \cap P$. Putting $W = I_1 \cap J$ we have $W \cap (U \times V) = (I_1 \cap U) \times (J \cap V) \neq \emptyset$. Take $(u,v) \in W \cap (U \times V)$ and an arbitrary point $z_0 \in F(u,v)$. As $F^v \in B^-(u)$, we have $d_u\big(\mathrm{Int}(F^v)^-\big(K(z_0,\varepsilon/2)\big),\, u\big) > 0$. Consequently, there is an open interval $I_2 \subset I_1$ such that $I_2 \cap P \neq \emptyset$ and for every $t \in I_2$, $F(t,v) \cap K(z_0,\varepsilon/2) \neq \emptyset$. Take $s \in I_2 \cap K$. Then $F(s,v) \cap K(z_0,\varepsilon/2) \neq \emptyset$. Let $p \in F(s,v) \cap K(z_0,\varepsilon/2) \neq \emptyset$. From (2) we obtain $F(s,v) \subset K\big(F(s,y),\varepsilon/4\big)$. Hence $|p - z_0| < \varepsilon/2$ and there is $p_1 \in F(s,y)$ such that $|p - p_1| < \varepsilon/4$. Since $p_1 \in F(s,y)$ and $s \in I_2 \subset I$, we conclude from (1) that there is $p_2 \in F(x,y)$ such that $|p_1 - p_2| < \varepsilon/4$ and, in consequence, $|z_0 - p_2| \le |z_0 - p| + |p - p_2| + |p_2 - p_1| < \varepsilon/2 + \varepsilon/4 + \varepsilon/4 = \varepsilon$. Thus $F(u,v) \subset K\big(F(x,y),\varepsilon\big)$ and, by Lemma 1, $F \in H^+(x,y)$. $\square$

One of the Fudali's results (see [2]) concerning the cliquishness of functions on product spaces states that the cliquishness of f_x sections and the quasi-continuity of f^y sections imply the cliquishness of f. One question still unanswered is whether the above is true if we will replace the cliquishness and the quasi-continuity by the strong cliquishness and the strong quasi-continuity respectively. However, we can consider the following property, which is a generalization of a property introduced in [5].

DEFINITION 2. A multivalued map $F\colon E \to \mathbb{R}$ has the property $K(x,y)$ at $(x,y) \in E$ if for every $\varepsilon > 0$ and $U, V \in \mathscr{T}_d(\mathbb{R})$ such that $x \in U$, $y \in V$ there is a nonempty open set $W \subset E$ such that $W \cap (U \times V) \neq \emptyset$ and

$$\bigcap_{(u,v) \in W \cap (U \times V)} K\big(F(u,v),\varepsilon\big) \neq \emptyset\,.$$

THEOREM 4. *Let $F\colon E \to \mathbb{R}$ be a multivalued map with compact values such that all sections $F^y \in B^+(x)$ at each $x \in \mathbb{R}$ and F_x are strongly cliquish for every $x \in \mathbb{R}$. Then $F \in K(x,y)$ at each $(x,y) \in E$.*

Proof. Fix $(x,y) \in E$. Let $\varepsilon > 0$ be an arbitrary real number and $U, V \in \mathscr{T}_d(\mathbb{R})$ such that $x \in U$, $y \in V$. Let $L = \mathrm{cl}(U)$ and $t \in L$. Since F_t is strongly cliquish at y, there is an open interval J_t with rational endpoints such that $J_t \cap V \neq \emptyset$ and

$$\bigcap_{s \in J_t \cap V} K\big(F(t,s), \varepsilon/2\big) \neq \emptyset .$$

Then there is a closed interval P_t with rational endpoints such that $\mu(P_t) < \varepsilon$ and for every $s \in J_t \cap V$ we have $F(t,s) \cap P_t \neq \emptyset$. Thus there are open intervals I, J and a closed interval P such that $I \cap U \neq \emptyset$ and a set $A = \{t \in L \cap I \colon J_t = J,\ P_t = P\}$ is dense in $I \cap U$. Let $W = I \times J$. Then $W \cap (U \times V) \neq \emptyset$. Take $(u,v) \in W \cap (U \times V)$. Suppose $F(u,v) \cap P = \emptyset$. Since $F(u,v)$ is compact, we have $\delta = \mathrm{dist}\big(P, F(u,v)\big) > 0$. As $F^v \in B^+(u)$, we have $d_u\big(\mathrm{Int}\big\{t \in \mathbb{R} \colon F(t,v) \subset K\big(F(u,v),\delta\big)\big\}, u\big) > 0$. Therefore, there is an open interval $I_1 \subset I$ such that $I_1 \cap U \neq \emptyset$ and for every $t \in I_1$ $F(t,v) \subset K\big(F(u,v),\delta\big)$. Since A is dense in $I \cap U$, there is $p \in I_1 \cap A$. This means that $F(p,v) \subset K\big(F(u,v),\delta\big)$ and, consequently, $F(p,v) \cap P = \emptyset$, contrary to the definition of A. It follows that for every $(u,v) \in W \cap (U \times V)$ $F(u,v) \cap P \neq \emptyset$. As $\mu(P) < \varepsilon$, the property $K(x,y)$ is proved. ☐

From Theorems 2 and 4 we have

THEOREM 5. *If $F\colon E \to \mathbb{R}$ is a multivalued map with compact values such that all sections F^y are both strongly lower and upper quasi-continuous and all sections F_x are strongly cliquish, then $F \in K(x,y)$ at every $(x,y) \in E$.*

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