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THE MAXIMAL ADDITIVE FAMILY FOR THE CLASS OF GRAPH CONTINUOUS REAL FUNCTIONS

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ABSTRACT. In 1977 Z. Grande introduced the notion of graph continuity for functions from $[0,1]$ to $\mathbb{R}$. The aim of this note is to deal with the maximal additive family for the class of graph continuous functions from topological space X to real numbers.

1. Basic notations and some results

In what follows, X, Y denote topological spaces. For a subset A of a topological space X symbols $\overline{A}$ and $\operatorname{Int} A$ denote the closure and the interior of A in a topological space X, respectively. If $G(f)$ denotes the graph of a function $f\colon X \to Y$, then $\overline{G(f)}$ denotes the closure of the graph of f in a topological space $X \times Y$.

The letter $\mathbb{R}$ stands for the set of real numbers.

Now let us recall several definitions and basic facts which will be used throughout this note.

DEFINITION 1. ([Gr]). Let X, Y be topological spaces. A function f is *graph continuous* if there exists a continuous function $g\colon X \to Y$ such that $G(g) \subset \overline{G(f)}$ (the closure of the graph of f contains the graph of g).

Remark 1. We shall often use this characterization of graph continuity: Let $f\colon X \to Y$ be a function, such that X is a topological space and Y is a metric space with a metric d. Then f is graph continuous if and only if there exists a continuous function $g\colon X \to Y$ such that for any $\varepsilon > 0$ the set $A_\varepsilon = \bigl\{x\colon d\bigl(f(x), g(x)\bigr) < \varepsilon\bigr\}$ is dense in X.

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Definition 2. ([BD]). Let X be a topological space and Y be a metric space with a metric d. A function $f\colon X \to Y$ is *cliquish at a point* $x \in X$ if for each $\varepsilon > 0$ and each neighbourhood U of x there is a nonempty open set $G \subset U$ such that $d\big(f(y), f(z)\big) < \varepsilon$ for each $y, z \in G$. If f is cliquish at every point x of X, then it is called *cliquish*.

We shall use the following proposition.

Proposition 1. ([S2]). *Let X be a topological space and (Y,d) be a metric space. Let $f\colon X \to Y$ be a graph continuous and cliquish function. Then there exists the unique continuous function $g\colon X \to Y$ such that $G(g) \subset \overline{G(f)}$.*

An useful characterization of graph continuity and cliquishness is Proposition 2:

Proposition 2. ([S2]). *Let X be a topological space and (Y,d) be a metric space. Then $f\colon X \to Y$ is graph continuous and cliquish if and only if there exists a continuous function $g\colon X \to Y$ such that for any $\varepsilon > 0$ the set $B_\varepsilon = \{x\colon d\big(f(x), g(x)\big) > \varepsilon\}$ is nowhere dense in X.*

And still we can introduce some more notation.

Definition 3. If X is a topological space then $\mathscr{G}_R$ is the system of all graph continuous functions from X to $\mathbb{R}$ and $\mathscr{G}_{RC}$ is the system of all graph continuous and cliquish functions from X to $\mathbb{R}$.

2. The maximal additive family for the class of graph continuous real functions

Theorem 1. *Let $f_1, f_2 \in \mathscr{G}_R$ be such that one from the functions f_1, f_2 is cliquish. Then:*

$$f_1 + f_2 \in \mathscr{G}_R .$$

Proof. To show that $f_1 + f_2$ is graph continuous we can assume that f_1 is cliquish. So from Proposition 1 we have the existence of continuous function g_1 such that for every $\varepsilon > 0$ the set $B_1 = \{x\colon |f_1(x) - g_1(x)| > \frac{\varepsilon}{2}\}$ is nowhere dense in X. Graph continuity of f_2 implies the existence of a continuous function g_2 such that for every $\varepsilon > 0$ the set $A_2 = \{x\colon |f_2(x) - g_2(x)| < \frac{\varepsilon}{2}\}$ is dense in X. Then the set $B_1 \cup (X \setminus A_2)$ is co-dense in X. Let

$$H = \big\{x\colon \big|(f_1 + f_2)(x) - (g_1 + g_2)(x)\big| < \varepsilon\big\} .$$

To prove that the set H is dense in X it suffices to show that $X \setminus H \subset B_1 \cup (X \setminus A_2)$. So let $x \notin X \setminus H \subset B_1 \cup (X \setminus A_2)$, i.e., $x \notin B_1$ and $x \in A_2$. From the definitions of A_2 and B_1 the following inequalities hold for x: $\big|f_1(x) - g_1(x)\big| \le \frac{\varepsilon}{2}$

and $\left|f_2(x)-g_2(x)\right|<\frac{\varepsilon}{2}$. So

$$\left|(f_1+f_2)(x)-(g_1+g_2)(x)\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

and this implies that $x\in H$. Thus g_1+g_2 is a continuous function such that for every $\varepsilon>0$ the set H is dense in X and therefore by virtue of Remark 1 the function f_1+f_2 is graph continuous. □

Theorem 1 motivates the introduction of this well known definition.

Definition 4. ([Bru]). The set $\mathcal{A}(\mathscr{G}_R)$ is the set of all function $h\in\mathscr{G}_R$ such that for any $f\in\mathscr{G}_R\colon f+h\in\mathscr{G}_R$. System

$$\mathcal{A}(\mathscr{G}_R)=\left\{h\colon X\to\mathbb{R}: h+f\in\mathscr{G}_R \text{ for every } f\in\mathscr{G}_R\right\}$$

is called the *maximal additive class* for $\mathscr{G}_R$.

Then an immediate consequence of Theorem 1 is the following corollary.

Corollary 1. *We have*

$$\mathscr{G}_{RC}\subset\mathcal{A}(\mathscr{G}_R).$$

We can reverse the inclusion relation from Corollary 1 in the following way.

Theorem 2. *Let X be a connected T_2 topological space. Then:*

$$\mathcal{A}(\mathscr{G}_R)=\mathscr{G}_{RC}.$$

P r o o f . Because of Corollary 1, we have to show only the inclusion $\mathcal{A}(\mathscr{G}_R)\subset\mathscr{G}_{RC}$. So we have to show that if the sum of two arbitrary graph continuous functions f_1, f_2 is graph continuous, then one (e.g., f_1) from these two functions is cliquish. Equivalently we can prove this: for arbitrary graph continuous function (f_1) which is not cliquish we can find a graph continuous function f_2 such that the function f_1+f_2 is not graph continuous.

So let f_1 be graph continuous, i.e., there exists a continuous function g_1 such that for every $\varepsilon>0$ the set $A=\{x\colon|f_1(x)-g_1(x)|<\varepsilon\}$ is dense in X. While f_1 is not cliquish, there is $\varepsilon_1>0$ for which the set $B_1=\{x\colon|f_1(x)-g_1(x)|>\varepsilon_1\}$ is not nowhere dense in X. So we have found the continuous function g_1 and $\varepsilon_1>0$ such that A is dense and B_1 is not nowhere dense. The fact that B_1 is not nowhere dense implies the existence of a set U open in X such that for every open $G\subset U\colon G\cap B_1\neq\emptyset$ ($B_1\cap U$ is dense in U).

Because X is connected and T_2, there exists an open set $W\subset U$ for which $X\setminus W$ is not open and $\operatorname{Int}(X\setminus W)\neq\emptyset$. Evidently $B_1\cap W$ is dense in W.

Then we define the function f_2 as follows:

$$f_2(x)=\begin{cases}-f_1(x), & x\notin W,\\ -g_1(x), & x\in W\cap B_1,\\ -f_1(x)+\varepsilon_1, & x\in W\setminus B_1.\end{cases}$$

Obviously for the continuous function $g_2 = -g_1$ holds $G(g_2) \subset \overline{G(f_2)}$, in other words f_2 is graph continuous.

Hence $f_1 + f_2$ is defined as follows:

$$(f_1 + f_2)(x) = \begin{cases} 0\,, & x \notin W\,, \\ f_1(x) - g_1(x)\,, & x \in W \cap B_1\,, \\ \varepsilon_1\,, & x \in W \setminus B_1\,. \end{cases}$$

One can easy prove that the distance between $(f_1 + f_2)(x)$ and 0 is greater or equal to ε_1, i.e., the function $f_1 + f_2$ is not graph continuous. This completes the proof. □

The following example shows that the assumption that X is T_2 cannot be replaced by the assumption that X is T_1.

EXAMPLE 1. Let X be an infinite set and let τ be the smallest topology such that (X, τ) is a T_1 space (i.e., $A \in \tau$ iff $A = \emptyset$ or A has the finite complement). Then (X, τ) is a T_1 connected space. Let A be an infinite subset of X with the infinite complement. Then the characteristic function of the set A is graph continuous but not cliquish and its sum with every graph continuous function on X is graph continuous.

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