Tatra Mt. Math. Publ. **19** (2000), 257–261

TATRA MOUNTAINS
Mathematical Publications

SECOND NOTE ON THE THEOREM OF ALEKSANDROV

Oľga Kulcsárová

ABSTRACT. The paper deals with the modification of the Aleksandrov Theorem stating that a monotone, compact and subadditive small system is continuous for a small system of fuzzy sets.

Many problems of measure theory can be formulated and proved by the help of the small systems ([1–9]).

Recall that the classical Aleksandrov Theorem states that an additive, regular set function is σ-additive. In weaker form: a subadditive, regular set function is upper continuous in $\emptyset$. Instead of regular measures, we shall consider compact measures. Instead of compact measures, we shall consider compact small systems. One version of the Aleksandrov Theorem for small system defined on a ring is formulated in [9].

The main result of this paper is that this version can be used and proved for a small system defined on a DF-σ-algebra (by [9]). There are only gentle distinctions in the proof. Thanks for this idea to D. Butnariu and B. Riečan.

Preliminaries; main result

Let $X \neq \emptyset$. A set $\mathscr{F} \subset [0,1]^X$, $\mathscr{F} \neq \emptyset$, is said to be a DF-$\sigma$-algebra, if the following is true:

1° $f \in \mathscr{F} \Longrightarrow f^{\perp} \overset{\text{def.}}{=} 1 - f \in \mathscr{F}$.

2° $f_n \in \mathscr{F}$, $n = 1, 2, \dots \Longrightarrow \bigvee f_n \overset{\text{def.}}{=} \sup f_n \in \mathscr{F}$.

3° $f, g \in \mathscr{F}$, $g \leqq f \Longrightarrow f - g \in \mathscr{F}$.

2000 Mathematics Subject Classification: 28B10.
Keywords: small system of sets, fuzzy sets.

Supported by grant VEGA 95/5305/471 and 2/5124/98.

As an example of a DF-σ-algebra, the set of fuzzy subsets of X can be considered. Also the set of all fuzzy events (fuzzy sets from X to $[0,1]$, measurable with respect to a given σ-algebra) forms a DF-σ-algebra. Since the set of all characteristic functions of sets belonging to a σ-algebra is a DF-σ-algebra, too, the classical results of the Aleksandrov type are immediate consequences of ours.

Let $\mathscr{F}$ denote a DF-σ-algebra and $\mathbb{N}$ denote the set of all positive integers.

Let f, g, f_i ($i \in \mathbb{N}$) be the elements of $\mathscr{F}$. We have $f \wedge g = 1-(1-f) \vee (1-g) \in \mathscr{F}$, and $\bigwedge\limits_{i=1}^{\infty} f_i = 1 - \bigvee\limits_{i=1}^{\infty} (1-f_i) \in \mathscr{F}$.

A sequence $\{\mathscr{N}_n\}_{n=1}^{\infty}$ of subsets of $\mathscr{F}$ is called a small system on $\mathscr{F}$, if $\mathscr{N}_{n+1} \subset \mathscr{N}_n$ for every $n \in \mathbb{N}$.

The notions as monotonicity, subadditivity, compactness and continuity can be (for DF-σ-algebra we apply the modification of terminology from [4], [9]) formulated abstractly by the help of a small system $\{\mathscr{N}_n\}_{n=1}^{\infty}$.

A small system $\{\mathscr{N}_n\}_{n=1}^{\infty}$ is said to be monotone if

$$f \in \mathscr{F},\ f \leqq g,\ g \in \mathscr{N}_n \implies f \in \mathscr{N}_n. \tag{m}$$

A small system $\{\mathscr{N}_n\}_{n=1}^{\infty}$ has the property (a) if the following is true:

$$\forall n \in \mathbb{N}\ \ \exists\, k_1, k_2 \in \mathbb{N}: \quad f \in \mathscr{N}_{k_1},\ g \in \mathscr{N}_{k_2} \implies f + g \in \mathscr{N}_n\,. \tag{a}$$

A small system $\{\mathscr{N}_n\}_{n=1}^{\infty}$ is said to be subadditive if

$$\forall n \in \mathbb{N}\ \ \exists \{k_i\}_{i=1}^{\infty} \subset \mathbb{N}: \quad f_i \in \mathscr{N}_{k_i},\ p \in \mathbb{N} \implies \bigvee_{i=1}^{p} f_i \in \mathscr{N}_n\,. \tag{sa}$$

A subsystem $\mathscr{C}$ of $\mathscr{F}$ is said to be compact if for every sequence $\{f_i\}_{i=1}^{\infty} \subset \mathscr{C}$ the following implication is true

$$\left(\forall k \in \mathbb{N},\ \bigwedge_{i=1}^{k} f_i \neq 0\right) \implies \bigwedge_{i=1}^{\infty} f_i \neq 0\,. \tag{cp}$$

A small system $\{\mathscr{N}_n\}_{n=1}^{\infty}$ of subsets of $\mathscr{F}$ is said to be compact if there exists a compact system $\mathscr{C}$, $\mathscr{C} \subset \mathscr{F}$, which satisfies the property

$$\forall n \in \mathbb{N}\ \ \forall f \in \mathscr{F}\ \ \exists\, g \in \mathscr{C}: \quad g \leqq f \quad \text{and} \quad f - g \in \mathscr{N}_n\,. \tag{co}$$

A small system $\{\mathscr{N}_n\}_{n=1}^{\infty}$ of subsets of $\mathscr{F}$ is said to be continuous if for every sequence $\{f_i\}_{i=1}^{\infty} \subset \mathscr{F}$, such that $f_i \searrow 0$, $i \to \infty$ (i.e., $f_{i+1} \leqq f_i$, $i = 1, 2, \dots$, $\bigwedge\limits_{i=1}^{\infty} f_i = 0$) the following is true:

$$\forall n \in \mathbb{N}\ \ \exists\, i_0 \in \mathbb{N}\ \ \forall i,\ i \geq i_0: \quad f_i \in \mathscr{N}_n\,. \tag{c}$$

THEOREM 1 (Aleksandrov for small systems)**.** *Every monotone, subadditive and compact small system $\{\mathcal{N}_n\}_{n=1}^{\infty}$ of subsets of a DF-σ-algebra is continuous.*

P r o o f . Let (likewise as in [9])

$$\{f_i\}_{i=1}^{\infty} \subset \mathscr{F}, \qquad f_{i+1} \leqq f_i\,, \quad i = 1, 2, \dots, \qquad \text{and} \qquad \bigwedge_{i=1}^{\infty} f_i = 0\,. \tag{i}$$

We want to prove (c). Contrary, assume that for some $n_0 \in \mathbb{N}$ and for every $i \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$, $k_i \geq i$, such that $f_{k_i} \notin \mathcal{N}_{n_0}$. Put $g_i = f_{k_i}$. By (sa) for n_0 there exists a sequence $\{m_i\}_{i=1}^{\infty}$ of integers such that for $d_i \in \mathcal{N}_{m_i}$ and $m \in \mathbb{N}$

$$\bigvee_{i=1}^{m} d_i \in \mathcal{N}_{n_0}\,. \tag{ii}$$

The small system $\{\mathcal{N}_n\}_{n=1}^{\infty}$ is compact (there exists the system $\mathscr{C}$ in $\mathscr{F}$ with (co)), so

$$\forall m_i \in \mathbb{N} \ \ \forall g_i \in \mathscr{F} \ \ \exists c_i \in \mathscr{C}\colon \quad c_i \leqq g_i \quad \text{and} \quad g_i - c_i \in \mathcal{N}_{m_i}\,.$$

We can prove that for every $k \in \mathbb{N}$ there holds $\bigwedge\limits_{i=1}^{k} c_i \neq 0$.

Let, indirectly, there exists a number $k_0 \in \mathbb{N}$ such that $\bigwedge\limits_{i=1}^{k_0} c_i = 0$. Then

$$g_{k_0} = g_{k_0} - \bigwedge_{i=1}^{k_0} c_i \leqq \bigvee_{i=1}^{k_0} (g_{k_0} - c_i) \leqq \bigvee_{i=1}^{k_0} (g_i - c_i) \in \mathcal{N}_{n_0}\,,$$

by (ii) because $g_i - c_i \in \mathcal{N}_{m_i}$, $i = 1, 2, \dots, k_0$. By help of monotonicity (m) $g_{k_0} \in \mathcal{N}_{n_0}$. This is a contradiction with the assumption $g_{k_0} = f_{k_{k_0}} \notin \mathcal{N}_{n_0}$. So for every $k \in \mathbb{N}$ there holds $\bigwedge\limits_{i=1}^{k} c_i \neq 0$.

Now we can complete the proof of the theorem.

From the compactness of $\mathscr{C}$ we have $\bigwedge\limits_{i=1}^{\infty} c_i \neq 0$. But $c_i \leqq g_i = f_{k_i}$ and $\bigwedge\limits_{i=1}^{\infty} c_i \leqq \bigwedge\limits_{i=1}^{\infty} f_{k_i}$, therefore $\bigwedge\limits_{i=1}^{\infty} f_{k_i} \neq 0$, too. This is a contradiction with (i). □

A small system $\{\mathcal{N}_n\}_{n=1}^{\infty}$ of subsets of $\mathscr{F}$ is said to be σ-subadditive if there holds:

$$\forall n \in \mathbb{N} \ \ \exists \{k_i\}_{i=1}^{\infty} \subset N\colon \quad f_i \in \mathcal{N}_{k_i}\,, \ i = 1, 2, \cdots \implies \bigvee_{i=1}^{\infty} f_i \in \mathcal{N}_n\,. \tag{ssa}$$

THEOREM 2. *Every monotone, subadditive and compact small system $\{\mathcal{N}_n\}_{n=1}^{\infty}$ of a DF-σ-algebra $\mathcal{F}$ with the property* (a) *is σ-subadditive.*

P r o o f. Let $n \in \mathbb{N}$. By the property (a) are $k_1, k_2 \in \mathbb{N}$ such that

$$f \in \mathcal{N}_{k_1},\ g \in \mathcal{N}_{k_2} \implies f + g \in \mathcal{N}_n . \tag{$*$}$$

By (sa) there exist $m_1, m_2, m_3, \dots$ such that

$$h_t \in \mathcal{N}_{m_t},\ t = 1, 2, 3, \dots, p \implies \bigvee_{t=1}^{p} h_t \in \mathcal{N}_{k_1} . \tag{$**$}$$

Hence we have constructed a sequence $\{m_i\}_{i=1}^{\infty}$. Let $f_i \in \mathcal{N}_{m_i}$. Since $\bigvee_{m=1}^{\infty} f_m - \bigvee_{m=1}^{i} f_m \searrow 0$ $(i \to \infty)$, by (c) there exists i_0 such that for any $i \geq i_0$ there holds

$$\bigvee_{m=1}^{\infty} f_m - \bigvee_{m=1}^{i} f_m \in \mathcal{N}_{k_2} . \tag{$***$}$$

By $(**)$ we obtain

$$\bigvee_{m=1}^{i} f_t \in \mathcal{N}_{k_1} . \tag{$****$}$$

By $(***)$, $(****)$ and $(*)$ we get

$$\bigvee_{m=1}^{\infty} f_m = \bigvee_{m=1}^{i} f_m + \left(\bigvee_{m=1}^{\infty} f_m - \bigvee_{m=1}^{i} f_m \right) \in \mathcal{N}_n .$$

The proof of the Theorem 2 is complete. □

REFERENCES

[1] RIEČAN, B.: *Abstract formulation of some theorems of measure theory*, Mat.–fyz. časop. SAV **16** (1966), 286–273.

[2] RIEČAN, B.: *On fuzzy set of sets of small measure*, Ann. Univ. Sci. Budapest., Sect. Comput. **12** (1991), 209–212.

[3] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure and Ordering*, Kluwer, Dordrecht, 1997.

[4] RIEČAN, B.: *A note on measures in fuzzy topological spaces*, Fuzzy Sets and Systems **54** (1993), 341–345.

[5] JOHNSON, R. A.—NIEWIAROVSKI, J.—SWIATKOWSKI, T.: *Small systems convergence and metrizability*, Proc. Amer. Math. Soc. **103** (1988), 105–112.

[6] WAJCH, E.: *On small systems and compact families of Borel functions*, Math. Slovaca **40** (1990), 63–69.

[7] WAJCH, E.: *Small systems — on approximation of compact sets of measurable functions to compact subsets of* $C_C - 0(x)$, Czechoslovak Math. J. **41** (1991), 619–633.
[8] HEJDUK, J.—WAJCH, E.: *A characterization of compactness in the sense of convergence with respect to* σ*-ideal*, Rad. Mat. **7** (1991), 5–9.
[9] KULCSÁROVÁ, O.: *A note on the theorem of Aleksandrov*, in: Proc. 7th IFSA, Prague, 1997, 13–14.

Received November 1, 1998

Department of Mathematical Analysis
University of P. J. Šafárik
Jesenná 5
SK–041-54 Košice
SLOVAKIA

E-mail: kulcsarova@duro.fac_sci.upjs.sk