Tatra Mt. Math. Publ. **19** (2000), 229–238

TATRA MOUNTAINS
Mathematical Publications

MEASURABILITY AND APPROXIMATE CONTINUITY OF MULTIVALUED FUNCTIONS

Grażyna Kwiecińska

ABSTRACT. The notion of approximate continuity was generalized into the case of multivalued functions. In this paper we discuss approximate continuity as continuity relative to the density topology. We investigate also some connections between measurability and approximate continuity of multivalued functions.

1. Introduction

An interesting class of real functions of real variable, the approximately continuous functions, was introduced by Denjoy in his work [2] on derivatives.

It was observed in [5] that the real valued functions continuous relative to the density topology in the domain and usual topology in the range, turn out to be exactly the approximately continuous functions. A set E is open in this topology and is called d-open, if it is measurable and if the density of E exists and is equal to 1 at every point of E. This topology is called d-topology and is denoted by $\mathscr{T}_d$ (see [3] or [4]). In this paper we discuss the above property in the case of multivalued functions — Section 3. The basis for this discussion are multivalued functions $F\colon T \to Z$, where $\big(T, d, \mathscr{M}(T), \mu\big)$ is a metric measure space and $\big(Z, \mathscr{U}(Z)\big)$ a uniform one. Section 2 presents some preliminaries. Section 4 establishes the relation between approximate continuity and measurability of multivalued functions.

2. Preliminaries

Let T and Z be non-empty sets and let $F\colon T \to Z$ be a multivalued function, i.e., a function whose value $F(t)$ is a non-empty subset of Z for each $t \in T$.

2000 Mathematics Subject Classification: 54C60, 28A20, 26B35.

Keywords: set-valued maps; measurability of set-valued maps, approximate continuity.

This work was supported by the University of Gdańsk, grant BW Nr 5100–5–0188–8.

Then two counter images of a set $B \subset Z$ are defined:

$$F^{+}(B) = \{t \in T \colon F(t) \subset B\}$$

and

$$F^{-}(B) = \{t \in T \colon F(t) \cap B \neq \emptyset\}.$$

Let $\bigl(T, \mathscr{M}(T)\bigr)$ be a measurable space and let $\bigl(Z, \mathscr{T}(Z)\bigr)$ be a topological one. The symbols $\mathscr{C}(Z)$ and $\mathscr{K}(Z)$ are used to denote the classes of all non-empty closed and non-empty compact subsets of Z, respectively.

A multivalued function $F \colon T \to Z$ is called *upper* (resp. *lower*) *$\mathscr{M}(T)$-measurable* if for each $G \in \mathscr{T}(Z)$, $F^{+}(G) \in \mathscr{M}(T)$ (resp. $F^{-}(G) \in \mathscr{M}(T)$).

It is known that

(1) If $\bigl(Z, \mathscr{T}(Z)\bigr)$ is a perfect space and F is upper $\mathscr{M}(T)$-measurable, then it is lower $\mathscr{M}(T)$-measurable. If $\bigl(Z, \mathscr{T}(Z)\bigr)$ is a perfectly normal space and F is a multivalued function with compact values, then lower $\mathscr{M}(T)$-measurability of F implies its upper $\mathscr{M}(T)$-measurability ([6, Proposition 1]).

Let $\bigl(T, \mathscr{T}(T)\bigr)$ and $\bigl(Z, \mathscr{T}(Z)\bigr)$ be topological spaces. A multivalued function F is called *upper* (resp. *lower*) *semicontinuous* at a point $t \in T$ if the following condition is valid:

(2) $\forall G \in \mathscr{T}(Z)\ \bigl(F(t) \subset G \Longrightarrow t \in \operatorname{Int} F^{+}(G)\bigr)$
(resp. $\forall G \in \mathscr{T}(Z)\ \bigl(F(t) \cap G \neq \emptyset \Longrightarrow t \in \operatorname{Int} F^{-}(G)\bigr)$).

F is called *continuous* at a point $t \in T$ if it is simultaneously upper and lower semicontinuous at this point.

Let $\bigl(Z, \mathscr{U}(Z)\bigr)$ be a uniform space and let $\mathscr{P}$ be the family of pseudometrics ϱ on Z such that $\bigl\{\{(y,z) \in Z \times Z \colon \varrho(y,z) < 2^{-n}\} \colon \varrho \in \mathscr{P},\ n \in \mathbb{N}\bigr\}$ ($\mathbb{N}$ denotes the set of positive integers) is a base for the uniformity $\mathscr{U}(Z)$ ([8]).

For each $\varrho \in \mathscr{P}$, a number $r > 0$, each $z \in Z$ and $A \subset Z$ we will write

$$B(z, \varrho, r) = \{y \in Z \colon \varrho(y,z) < r\}$$

and

$$B(A, \varrho, r) = \bigcup \{B(z, \varrho, r) \colon z \in A\}.$$

A multivalued function $F \colon T \to Z$ is called *h-upper* (resp. *h-lower*) *semicontinuous* at a point $t_0 \in T$ if the following condition holds:

(3) For each $\varepsilon > 0$ and $\varrho \in \mathscr{P}$ there exists a neighbourhood $\mathscr{U}(t_0)$ of t_0 such that $F(t) \subset B\bigl(F(t_0), \varrho, \varepsilon\bigr)$ (resp. $F(t_0) \subset B\bigl(F(t), \varrho, \varepsilon\bigr)$) for each $t \in \mathscr{U}(t_0)$.

It is known that

(4) If F is upper (resp. h-lower) semicontinuous at a point $t_0 \in T$, then it is h-upper (resp. lower) semicontinuous at the point t ([7, Theorem 1.15

and 1.12]). If moreover $F(t) \in \mathscr{K}(Z)$, then conditions (2) and (3) are equivalent ([7, Theorem 1.17 and 1.14]).

Let $\big(T, d, \mathscr{M}(T), \mu\big)$ be a measure metric space with a metric d and with a σ-finite and G_δ-regular (i.e., for each $A \subset T$ there exists a G_δ set H such that $A \subset H$ and $\mu(H) = \mu^*(A)$) complete measure μ defined on a σ-field $\mathscr{M}(T)$ of subsets of T containing Borel sets (μ^* denotes here the outer measure corresponding to μ). Let $\mathscr{N}(T) \subset \mathscr{M}(T)$ be a net, i.e., $\mathscr{N}(T)$ is a denumerable disjoint family of $\mathscr{M}(T)$-measurable sets (called cells) covering T with a positive and finite measure μ, the boundaries of which are of μ-measure zero. Let $\triangle\big(\mathscr{N}(T)\big)$ denote upper bound of the diameters $\delta(I)$ of all cells I of the net $\mathscr{N}(T)$.

Let $\big(\mathscr{N}_n(T)\big)_{n\in\mathbb{N}}$ be a regular sequence of nets, i.e., each cell of the net $\mathscr{N}_{n+1}(T)$ is a subset of some cell of $\mathscr{N}_n(T)$ and $\triangle\big(\mathscr{N}_n(T)\big) \to 0$ as $n \to \infty$.

Finally let $\mathscr{F}(T) = \bigcup\limits_{n\in\mathbb{N}} \mathscr{N}_n(T)$. In this way constructed family $\mathscr{F}(T)$ is called a *net structure* of $\big(T, d, \mathscr{M}(T), \mu\big)$ ([1, 6.3, p. 35]).

If $t \in T$, then for each $n \in \mathbb{N}$ there exists exactly one cell I_n of $\mathscr{N}_n(T)$ containing t. We take $I_n \to t$ to mean that $t \in I_n \in \mathscr{N}_n(T)$ and $n \to \infty$. It is clear that for each $t \in T$ there exists a decreasing sequence (I_n) of cells of the net structure $\mathscr{F}(T)$ such that $I_n \to t$.

For example if T is m-dimensional euclidean space $\mathbb{R}_m$ and $\mathscr{N}_n$ is for every $n \in \mathbb{N}$ the family of all m-dimensional intervals of the form $\big[\frac{i_1-1}{2^n}, \frac{i_1}{2^n}\big) \times \big[\frac{i_2-1}{2^n}, \frac{i_2}{2^n}\big) \times \cdots \times \big[\frac{i_m-1}{2^n}, \frac{i_m}{2^n}\big)$, where $i_1, i_2, \dots, i_m$ are integers, then $\mathscr{F}(T) = \bigcup\limits_{n\in\mathbb{N}} \mathscr{N}_n$ forms a net structure of $(\mathbb{R}_m, \mathscr{L}_m, \lambda_m)$, where $\mathscr{L}_m$ and λ_m denote the family of Lebesgue measurable sets and Lebesgue measure in $\mathbb{R}_m$, respectively.

Let $A \subset T$ and $t \in T$. The upper outer density of A at the point t with respect to $\mathscr{F}(T)$ is

$$\limsup_{I_n \to t} \frac{\mu^*(A \cap I_n)}{\mu(I_n)}\,.$$

Replacing lim sup by lim inf we obtain the lower outer density of A at t with respect to $\mathscr{F}(T)$. These densities we will denote by $D^*_u(t, A)$ and $D^*_l(t, A)$, respectively. If both these densities are equal, then their common value is called the *outer density of A at t with respect to $\mathscr{F}(T)$* and is denoted by $D^*(t, A)$.

If $A \in \mathscr{M}(T)$, then the outer density of the set A at the point $t \in T$ with respect to $\mathscr{F}(T)$ is called *density of A at t with respect to $\mathscr{F}(T)$*.

A point $t \in T$ is called *density point of a set $A \subset T$ with respect to $\mathscr{F}(T)$* if there exists a set $B \in \mathscr{M}(T)$ such that $B \subset A$ and the density of B at t with respect to $\mathscr{F}(T)$ is equal to 1. We will write $D(t, A) = 1$.

In the remainder of the paper we assume that

(5) $\mathscr{F}(T)$ has the density property, i.e., every set $A \subset T$ satisfies the condition $\mu(\{t \in A\colon D_l^*(t, A) < 1\}) = 0$.

By the density property of $\mathscr{F}(T)$ there holds

(6) A set $M \subset T$, whose almost every point is its density point with respect to $\mathscr{F}(T)$ is a $\mathscr{M}(T)$-measurable set.

A measurable set in T will be called *homogeneous with respect to* $\mathscr{F}(T)$ if its density with respect to $\mathscr{F}(T)$ is one at each of its point.

Let $\phi(A)$ denote the set of all density points of $A \in \mathscr{M}(T)$ with respect to $\mathscr{F}(T)$. Then for $A \in \mathscr{M}(T)$, inclusion $A \subset \phi(A)$ means that A is a homogeneous set with respect to $\mathscr{F}$. It is easy to see that $\phi(A \cap B) = \phi(A) \cap \phi(B)$ for each $A \in \mathscr{M}(T)$ and $B \in \mathscr{M}(T)$. Therefore the finite intersection of homogeneous sets with respect to $\mathscr{F}(T)$ is homogeneous with respect to $\mathscr{F}(T)$.

Let $\{A_s\}_{s \in S}$ be the family of homogeneous sets with respect to $\mathscr{F}(T)$. Measure μ is σ-finite, so we can suppose that $\mu(T) < \infty$. Let b denote the upper bound of all measures of finite sums of the sets of family $\{A_s\}_{s \in S}$. Then there exists $\{s_n\} \subset S$ such that $\mu\left(\bigcup_{n \in \mathbb{N}} A_{s_n}\right) = b$. Let $A = \bigcup_{n \in \mathbb{N}} A_{s_n}$. Then $A \in \mathscr{M}(T)$ and $\mu(A_s \setminus A) = 0$ for each $s \in S$. Moreover $\phi(A_s) \subset \phi(A)$ because $A_s \setminus (A_s \setminus A) \subset A$ for each $s \in S$. The sets A_s are homogeneous with respect to $\mathscr{F}(T)$, so $A_s \subset \phi(A_s)$ for each $s \in S$. Therefore $A \subset \bigcup_{s \in S} A_s \subset \bigcup_{s \in S} \phi(A_s) \subset \phi(A)$ and $\bigcup_{s \in S} A_s \in \mathscr{M}(T)$ because $\mu(A \setminus \phi(A)) = 0$. Finally $\bigcup_{s \in S} A_s \subset \phi\left(\bigcup_{s \in S} A_s\right)$, i.e., the union of homogeneous sets with respect to $\mathscr{F}(T)$ is homogeneous with respect to $\mathscr{F}(T)$. Since the empty set and the whole space are homogeneous sets with respect to $\mathscr{F}(T)$, the space T can be topologized by taking the homogeneous sets with respect to $\mathscr{F}(T)$ as open sets. This topology we will denote as usual by $\mathscr{T}_d(T)$ (cp. [3]).

Let $(T, d, \mathscr{M}(T), \mu)$ and $(Z, \mathscr{U}(Z))$ be as above. Let $F\colon T \to Z$ be a multivalued function, and let $t_0 \in T$.

Definition 1. F is called *approximately upper* (resp. *lower*) *semicontinuous at* $t_0 \in T$ *with respect to* $\mathscr{F}(T)$ if there exists a set $A \in \mathscr{M}(T)$ such that $D(t_0, A) = 1$ and the restriction $F|_A$ is upper (resp. lower) semicontinuous at t_0.

F is called *approximately upper* (resp. *lower*) *semicontinuous with respect to* $\mathscr{F}(T)$ if it is approximately upper (resp. lower) semicontinuous with respect to $\mathscr{F}(T)$ at each point $t \in T$.

If F is simultaneously approximately upper semicontinuous and approximately lower semicontinuous with respect to $\mathscr{F}(T)$, then it is called *approximately continuous with respect to* $\mathscr{F}(T)$.

3. Continuity relative to the density topology

Throughout this section let $(T, d, \mathscr{M}(T))$ and $(Z, \mathscr{U}(Z))$ be as it was assumed in Section 2. Let $F\colon T \to Z$ be a multivalued function. For fixed $t_0 \in T$, $\varrho \in \mathscr{P}$ and $\varepsilon > 0$ let us put

$$A_u(t_0, \varrho, \varepsilon) = \{t \in T\colon F(t) \subset B(F(t_0), \varrho, \varepsilon)\}$$

and

$$A_l(t_0, \varrho, \varepsilon) = \{t \in T\colon F(t_0) \subset B(F(t), \varrho, \varepsilon)\}.$$

Let us consider the following conditions:

(a_1) F is approximately upper semicontinuous at t_0 with respect to $\mathscr{F}(T)$,

(a_2) F is approximately lower semicontinuous at t_0 with respect to $\mathscr{F}(T)$,

(b_1) $\forall \varrho \in \mathscr{P}\ \forall \varepsilon > 0\ \ D\big(t_0, A_u(t_0, \varrho, \varepsilon)\big) = 1$,

(b_2) $\forall \varrho \in \mathscr{P}\ \forall \varepsilon > 0\ \ D\big(t_0, A_l(t_0, \varrho, \varepsilon)\big) = 1$,

(c_1) $\forall G \in \mathscr{T}(Z)\ \ \big(F(t_0) \subset G \implies D\big(t_0, F^+(G)\big) = 1\big)$,

(c_2) $\forall G \in \mathscr{T}(Z)\ \ \big(F(t_0) \cap G \neq \emptyset \implies D\big(t_0, F^-(G)\big) = 1\big)$.

Theorem 1. *The following implications are valid:*

(i) (a_1) $\Rightarrow$ (b_1), (b_2) $\Rightarrow$ (a_2), (c_1) $\Rightarrow$ (b_1) *and* (b_2) $\Rightarrow$ (c_2).

(ii) *If F has compact values, then* (b_1) $\Rightarrow$ (a_1), (a_2) $\Rightarrow$ (b_2), (b_1) $\Rightarrow$ (c_1) *and* (c_2) $\Rightarrow$ (b_2).

Proof. Fix $\varepsilon > 0$ and $\varrho \in \mathscr{P}$. Suppose F is approximately upper semicontinuous with respect to $\mathscr{F}(T)$ at the point t_0. Then there exists a set $A \in \mathscr{M}(T)$ such that $D(t_0, A) = 1$ and $F|_A$ is upper semicontinuous at t_0. From (4) it follows that there exists an open set $U(t_0)$ containing t_0 such that $M = A \cap U(t_0) \subset A_u(t_0, \varrho, \varepsilon)$. Moreover $D(t_0, M) = 1$, which establishes the implication (a_1) $\Rightarrow$ (b_1).

In the same manner we can see that (a_2) $\Rightarrow$ (b_2).

Let us assume (b_2). Let $\varrho \in \mathscr{P}$ and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers decreasing to zero.

We conclude from (b_2) that for each $n \in \mathbb{N}$ there exists a set $B_n \in \mathscr{M}(T)$ such that $B_n \subset A_l(t_0, \varrho, \varepsilon_n)$ and $D(t_0, B_n) = 1$, hence that

(7) $\forall n \in \mathbb{N}\ \exists\, k_n \in \mathbb{N}\ \forall\, k \in \mathbb{N}\ \ \Big($if $t_0 \in I_k$ and $I_k \subset I_{k_n}$, then

$\frac{\mu(B_n \cap I_k)}{\mu(I_k)} > 1 - \varepsilon_n \Big)$.

Let us choose the sequence $(I_{k_n})_{n\in\mathbb{N}}$. We may suppose that $(I_{k_n})_{n\in\mathbb{N}}$ is decreasing and $I_{k_n} \to t_0$. Let us put

(8) $A = \bigcup_{n\in\mathbb{N}} C_n$, where $C_n = B_n \cap (I_{k_n} \backslash I_{k_{n+1}}) \cup \{t_0\}$ for $n \in \mathbb{N}$.

We will show that $D(t_0, A) = 1$. Suppose $I_k \to t_0$. For each $k \in N$ there exists $n \in \mathbb{N}$ such that $I_{k_{n+1}} \subset I_k \subset I_{k_n}$ and $\mu(I_{k_{n+1}}) < \varepsilon_n \mu(I_k)$. By (7)

(9) $\mu(B_n \cap I_k) > (1 - \varepsilon_n)\mu(I_k)$.

On the other hand

(10) $\mu(B_n \cap I_{k_{n+1}}) \le \mu(I_{k_{n+1}}) < \varepsilon_n \mu(I_k)$.

By (9) and (10) it follows immediately that

$$\begin{aligned}\mu(A \cap I_k) \ge \mu(C_n \cap I_k) &= \mu\big(B_n \cap (I_{k_n} \backslash I_{k_{n+1}}) \cap I_k\big)\\ &= \mu(B_n \cap I_k) - \mu\big(B_n \cap I_{k_{n+1}}\big)\\ &> (1 - \varepsilon_n)\mu(I_k) - \varepsilon_n \mu(I_k)\\ &= (1 - 2\varepsilon_n)\mu(I_k),\end{aligned}$$

hence that $\dfrac{\mu(A \cap I_k)}{\mu(I_k)} > (1 - 2\varepsilon_n)$.

If $I_k \to t_0$, then $\varepsilon_n \to 0$. This establishes $D(t_0, A) = 1$.

Now we show that $F|_A$ is lower semicontinuous at t_0. Let $\varrho \in \mathscr{P}$ and $\varepsilon > 0$. Let $n \in \mathbb{N}$ be chosen such that $\varepsilon_n < \varepsilon$. Finally let $r = \delta\big(I_{k_{n+1}}\big)$ and $t \in A \cap B(t_0, \varrho, r)$. Then $t \in \bigcup_{i\in\mathbb{N}} C_{n+i} \cup \{t_0\}$. Moreover for each $i \in \mathbb{N}$ we have $C_{n+i} \subset A_l(t_0, \varrho, \varepsilon_{n+i}) \subset A_l(t_0, \varrho, \varepsilon)$. Therefore $F(t_0) \subset B(F(t), \varrho, \varepsilon)$ for each $t \in A \cap B(t_0, r)$. By (4) the multifunction $F|_A$ is lower semicontinuous at t_0, which establishes the implication $(\mathrm{b}_2) \Rightarrow (\mathrm{a}_2)$.

Applying this arguments again, with (b_2) replaced by (b_1) and with assumed compactness of values of F, we obtain $(\mathrm{b}_1) \Rightarrow (\mathrm{a}_1)$.

To prove $(\mathrm{c}_1) \Rightarrow (\mathrm{b}_1)$, it is sufficient to observe that $A_u(t_0, \varrho, \varepsilon) = F^+\big(B(F(t_0), \varrho, \varepsilon)\big)$.

Now suppose that F has compact values and (b_1) is valid. Let G be an open set such that $F(t_0) \subset G$. Let $\varrho \in \mathscr{P}$ and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers convergent to zero such that for each $n \in \mathbb{N}$, $B\big(F(t_0), \varrho, \varepsilon_n\big) \subset G$. By (b_1) similarly as before (see (8)) we can construct a $\mathscr{M}(T)$-measurable set A such that $D(t_0, A) = 1$. We have $F(t) \subset B\big(F(t_0), \varrho, \varepsilon_n\big) \subset G$ for each $t \in A$. Thus we have $A \subset F^+(G)$ and $D\big(t_0, F^+(G)\big) = 1$, which establishes the implication $(\mathrm{b}_1) \Rightarrow (\mathrm{c}_1)$.

Suppose that F has compact values and (c_2) is valid. Let $\varrho \in \mathscr{P}$ and $\varepsilon > 0$. There exists a sequence $(z_i)_{i=1,2,\dots,n}$ such that $F(t_0) \subset \bigcup\limits_{i=1,2,\dots,n} B(z_i, \varrho, \frac{\varepsilon}{2})$ and $F(t_0) \cap B(z_i, \varrho, \frac{\varepsilon}{2}) \neq \emptyset$ for each $i = 1, 2, \dots, n$. Let $U_i = F^-\big(B(z_i, \varrho, \frac{\varepsilon}{2})\big)$ for each $i = 1, 2, \dots, n$ and let $U(t_0) = \bigcap\limits_{i=1,2,\dots,n} U_i$. By (c_2) we have $D(t_0, U_i) = 1$ for each $i = 1, 2, \dots, n$. Then $t_0 \in U(t_0)$ and $D\big(t_0, U(t_0)\big) = 1$. It suffices to show that $U(t_0) \subset A_l(t_0, \varrho, \varepsilon)$. Let $t \in U(t_0)$. Then for each $i = 1, 2, \dots, n$ we have $F(t) \cap B(z_i, \varrho, \frac{\varepsilon}{2}) \neq \emptyset$. Thus

(11) $\forall i \in \{1, 2, \dots, n\}\ \exists y_i \in F(t)\ \ z_i \in B(y_i, \varrho, \frac{\varepsilon}{2})$.

Let $z \in F(t_0)$. Then there exists $j \in \{1, 2, \dots, n\}$ such that $\varrho(z, z_j) < \frac{\varepsilon}{2}$. Let y_j be chosen for z_j according to (11). Then $\varrho(y_j, z) < \varepsilon$. Thence $F(t_0) \subset B\big(F(t), \varrho, \varepsilon\big)$. This means $t \in A_l(t_0, \varrho, \varepsilon)$ and the proof of $(c_2) \Rightarrow (b_2)$ is complete.

Now let us suppose that (b_2) is valid and let $G \subset Z$ be an open set such that $F(t_0) \cap G \neq \emptyset$. Let $z \in F(t_0) \cap G$ and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing to zero sequence of positive numbers such that $B(z, \varrho, \varepsilon_n) \subset G$ for each $n \in \mathbb{N}$. Again likewise as before (see (8)) we can construct a $\mathscr{M}(T)$-measurable set A such that $D(t_0, A) = 1$. It remains to prove that $A \subset F^-(G)$. Let $t \in A$. Then there exists $n \in \mathbb{N}$ such that $t \in C_n \subset A_l(t_0, \varrho, \varepsilon_n)$. Thus we conclude that $F(t_0) \subset B\big(F(t), \varrho, \varepsilon_n\big)$. Because $z \in F(t_0)$, then there exists $y \in F(t)$ such that $z \in B(y, \varrho, \varepsilon_n)$. By assumption $B(z, \varrho, \varepsilon_n) \subset G$ it follows that $y \in G$, hence $F(t) \cap G \neq \emptyset$, i.e., $t \in F^-(G)$, which establishes the implication $(b_2) \Rightarrow (c_2)$ and completes the proof of Theorem 1. $\square$

The following examples show that the assumption of compactness of values of F in Theorem 1 (ii) is important.

EXAMPLE 1. Let $F\colon \mathbb{R} \to \mathbb{R}$ be a multivalued function defined as follows:

$$F(x) = \begin{cases} [-x, 1] & \text{if } x > 0\,, \\ (0, 1] & \text{if } x = 0\,, \\ [x, 1] & \text{if } x < 0\,. \end{cases}$$

Then for each $\varepsilon > 0$, $D\big(0, A_u(0, \varepsilon)\big) = 1$, but $F^+\big((0, 2)\big) = 0 \notin \mathscr{T}_d$.

Next let $F\colon \mathbb{R} \to \mathbb{R}$ be given by formula

$$F(x) = \begin{cases} [0, \frac{1}{x}] & \text{if } x > 0\,, \\ (0, \infty] & \text{if } x = 0\,, \\ [0, -\frac{1}{x}] & \text{if } x < 0\,. \end{cases}$$

Then $D\big(0, A_l(0, \varepsilon)\big) = 0$ for each $\varepsilon > 0$ and $F^-(G) = \mathbb{R}$, provided the intersection $F(0) \cap G \neq \emptyset$.

We may summarize the above results as a conclusion.

Conclusion 1. *Let* $(T, d, \mathscr{M}(T), \mu)$ *and* $(Z, \mathscr{U}(Z))$ *be as it was assumed above and let* $F\colon T \to Z$ *be a multivalued function with compact values. Then the following conditions are equivalent:*

(a) F *is approximately upper (resp. lower) semicontinuous with respect to the net structure* $\mathscr{F}(T)$,

(b) $\forall t_0 \in T \;\; \forall \varrho \in \mathscr{P} \;\; \forall \varepsilon > 0 \;\; D\big(t_0, A_u(t_0, \varrho, \varepsilon)\big) = 1$,
(*resp.* $\forall t_0 \in T \;\; \forall \varrho \in \mathscr{P} \;\; \forall \varepsilon > 0 \;\; D\big(t_0, A_l(t_0, \varrho, \varepsilon)\big) = 1$),

(c) $\forall G \in \mathscr{T}(Z) \;\; F^+(G) \in \mathscr{T}_d(T)$ (*resp.* $\forall G \in \mathscr{T}(Z) \;\; F^-(G) \in \mathscr{T}_d(T)$).

4. Connections between approximate continuity and measurability

Let us first consider an example.

Example 2. Let $A, B \subset \mathbb{R}$, $A, B \notin \mathscr{B}(\mathbb{R})$ ($\mathscr{B}(\mathbb{R})$ means the σ-field of Borel subsets of $\mathbb{R}$) be two disjoint, dense zero sets. Let $F\colon \mathbb{R} \to \mathbb{R}$ be given by formula

$$F(x) = \begin{cases} [-1, 3] & \text{if } x \in A\,, \\ [-1, 1] & \text{if } x \in \mathbb{R} \setminus (A \cup B)\,, \\ [-3, 1] & \text{if } x \in B\,, \end{cases}$$

and let $G\colon \mathbb{R} \to \mathbb{R}$ be defined as follows

$$G(x) = \begin{cases} [-3, -2] & \text{if } x \in A\,, \\ [-3, 3] & \text{if } x \in \mathbb{R} \setminus (A \cup B)\,, \\ [2, 3] & \text{if } x \in B\,. \end{cases}$$

Let $\mathscr{E} = \{\emptyset, \mathbb{R}, \mathbb{R} \setminus A, \mathbb{R} \setminus B, \mathbb{R} \setminus (A \cup B)\} \subset \mathscr{T}_d(\mathbb{R})$ and let $U \subset \mathbb{R}$ be open. Because $F^+(U) \in \mathscr{E}$ and $F^-\big((2, \infty)\big) = A$, therefore F is approximately upper (but not lower) semicontinuous, by Conclusion 1. Similarly $G^-(U) \in \mathscr{E}$ and $G^+\big((-4, -1)\big) = A$, so G is approximately lower (but not upper) semicontinuous. In both cases these multivalued functions are neither upper nor lower $\mathscr{B}(\mathbb{R})$-measurable.

Let $(T, d, \mathscr{M}(T), \mu)$ and $(Z, \mathscr{U}(Z))$ be as it was assumed in Section 2. We conclude from (6), that:

Theorem 2. *If a multivalued function* $F\colon T \to Z$ *with compact values is* μ*-almost everywhere approximately upper (resp. lower) semicontinuous with respect to* $\mathscr{F}(T)$, *then it is upper (resp. lower)* $\mathscr{M}(T)$*-measurable.*

P r o o f. Let $G \subset Z$ be open and let $t_0 \in F^+(G)$. If F is approximately upper semicontinuous at t_0 with respect to $\mathscr{F}(T)$, then $D\big(t_0, F^+(G)\big) = 1$, by Conclusion 1. Thus the set $F^+(G)$ has density 1 with respect to $\mathscr{F}(T)$ at μ-almost each of its points, and finally $F^+(G) \in \mathscr{M}(T)$, by (6).

We conclude similarly that $F^-(G) \in \mathscr{M}(T)$. The proof is complete. □

Let $\big(T, d, \mathscr{M}(T), \mu\big)$ be as above and let $\big(Z, \mathscr{U}(Z)\big)$ be second countable.

THEOREM 3. *If $F\colon T \to Z$ is lower $\mathscr{M}(T)$-measurable, then F is μ-almost everywhere approximately lower semicontinuous with respect to $\mathscr{F}(T)$.*

Proof. Let $\mathscr{B} = \{U_1, U_2, \dots\}$ be a countable base of Z and let $\varepsilon > 0$. By lower $\mathscr{M}(T)$-measurability of F, for each $n \in \mathbb{N}$, $A_n = F^-(U_n) \in \mathscr{M}(T)$. Let for $n \in \mathbb{N}$, G_n be an open subset of Z such that $A_n \subset G_n$ and $\mu(G_n) \leq \mu(A_n) + \frac{\varepsilon}{2^n}$. Let $B_n = G_n \setminus A_n$ for $n \in \mathbb{N}$, and let $B = \bigcup\limits_{n\in\mathbb{N}} B_n$. Then $B \in \mathscr{M}(T)$ and $\mu(B) \leq m\varepsilon$. We will show, that $F\big|_{T\setminus B}$ is lower semicontinuous.

Let G be an open subset of Z. Then $G = \bigcup\limits_{n\in\mathbb{N}} U_{k_n}$ and

$$\begin{aligned} F^-(G)\setminus B &= \bigcup_{n\in\mathbb{N}} F^-(U_{k_n})\setminus B = \bigcup_{n\in\mathbb{N}} A_{k_n}\setminus B \\ &= \bigcup_{n\in\mathbb{N}} (G_{k_n}\setminus B_{k_n})\setminus B = \bigcup_{n\in\mathbb{N}} G_{k_n}\setminus B \end{aligned}$$

(because $B_{k_n} \subset B$), hence $F^-(G)$ is open in $T\setminus B$. Therefore $F\big|_{T\setminus B}$ is lower semicontinuous. Finally, by the density property of $\mathscr{F}(T)$, $D(t, T\setminus B) = 1$ for μ-almost every point T, i.e., F is μ-almost everywhere approximately lower semicontinuous with respect to $\mathscr{F}(T)$, which completes the proof. □

We conclude from (1) and Theorem 3 that upper $\mathscr{M}(T)$-measurability of F implies its μ-almost everywhere approximate lower semicontinuity with respect to $\mathscr{F}(T)$.

REFERENCES

[1] BRUCKNER, A. M.: *Differentiation of integrals*, Amer. Math. Monthly **78** (1971), P II, 1–54.

[2] DENJOY, A.: *Sur les fonctions dérivées sommables*, Bull. Soc. Math. France **43** (1916), 161–248.

[3] FRANKLIN, D. TALL: *The density topology*, Pacific J. Math. **62** (1976), 275–284.

[4] GOFFMAN, C.—NEUGEBAUER, C.—NISHURA, T.: *Density topology and approximate continuity*, Duke Math. J. **28** (1961), 497–506.

[5] GOFFMAN, C.—WATERMAN, D.: *Approximately continuous transformations*, Proc. Amer. Math. Soc. **12** (1961), 116–121.

[6] KWIECIŃSKA, G.—ŚLĘZAK, W.: *On complete measurability of multifunctions defined on product spaces*, Acta Math. Univ. Comenian. **LXVI, 2** (1997), 293–305.

[7] LECHICKI, A.: *On continuous and measurable multifunctions*, Ann. Soc. Math. Polon. Ser. I. Comment. Mat. **XXI** (1980), 141–156.

[8] SCHUBERT, H.: *Topology*, London, 1968.

Received October 15, 1998

University of Gdańsk
Institute of Mathematics
ul. Wita Stwosza 57
PL–80–952 Gdańsk
POLAND

E-mail: gkk@ksinet.univ.gda.pl