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A NOTE ON SAC PROPERTY

FRANCISZEK PRUS-WIŚNIOWSKI — GRZEGORZ SZKIBIEL

ABSTRACT. Some properties of approximately continuous functions are studied. Considered results may lead to the solution of Z. Grande's SAC problem.

The property SAC has been defined by Z. Grande (see [2]) in connection with his investigation of equiderivatives and approximate equicontinuity. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ has the *SAC property*, if for every $\eta > 0$ there is an approximately continuous function $r\colon \mathbb{R} \to \mathbb{R}$ such that for each x and h

$$0 < |h| < r(x) \qquad \Longrightarrow \qquad \left| \frac{1}{h} \int_x^{x+h} f(t)\,dt - f(x) \right| < \eta\,.$$

In his paper, Z. Grande gave an interesting example of a function g having SAC property whose set of discontinuities is of positive measure.

It is not difficult to show that every continuous function f has the SAC property. The idea to construct r is based on a well-known fact that a continuous function on a compact interval is uniformly continuous. Thus one can define $I_n = [n, n+1]$, $n \in \mathbb{Z}$, and because of the uniform continuity, given $\eta > 0$ one obtains

$$\exists_{r_n} \forall_{x,y \in I_n} \left(|x - y| < r_n \;\Rightarrow\; \bigl|f(x) - f(y)\bigr| < \eta \right).$$

Then r is made of r_n's glued with skew lines if necessary.

The above idea does not work if f is approximately continuous but discontinuous. Actually it was Z. Grande who posed a question whether every approximately continuous function has the SAC property. We have not answered the question, but we gained some insight that might lead to a positive answer. Namely we think the following assertion is true:

PROBLEM. *If $p\colon \mathbb{R} \to (0, +\infty)$ is measurable, then there is an approximately continuous function r, such that $0 < r(x) \le p(x)$ if and only if $\operatorname*{ap\,lim\,inf}_{t \to x} p(t) > 0$ at every $x \in \mathbb{R}$.*

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Obviously, a way to utilize the above assertion in the SAC problem would be to define a function p given an approximately continuous function f as follows

$$\begin{aligned} p(x) &= \sup\left\{\gamma\colon\ \forall_{|h|\le\gamma}\ \left|\frac{1}{h}\int\limits_x^{x+h} f(t)\,dt - f(x)\right| \le \eta\right\} \\ &= \inf\left\{|h|\colon\ \left|\frac{1}{h}\int\limits_x^{x+h} f(t)\,dt - f(x)\right| > \eta\right\}. \end{aligned}$$

We will show that so defined function p is measurable. To achieve this, we define a function $H\colon \mathbb{R}\times\mathbb{R}\setminus\{0\}\to\mathbb{R}$ by formula

$$H(x,h) = \frac{1}{h}\int\limits_x^{x+h} f(t)\,dt - f(x).$$

The function H has two obvious properties that imply the measurability of p provided the function f is approximately continuous, finite everywhere and locally summable. It follows directly from continuity of the indefinite integral that if f is finite everywhere and locally summable, then for any x, the function $H(x,\cdot)$ is continuous. On the other hand, if f is approximately continuous and locally summable, then for any h, the function $H(\cdot,h)$ is approximately continuous. Indeed, we put $g(x) = H(x,h)$ and show that g is approximately continuous. Let A be the set with x as its density point and so that $f|_A$ is continuous at x. Let $\varepsilon > 0$ be given. Assume $|x-y| < |h|$ for $y\in A$. One has

$$\bigl|g(x)-g(y)\bigr| \le \frac{1}{|h|}\left(\left|\int\limits_x^{y}\bigl|f(t)\bigr|\,dt\right| + \left|\int\limits_{x+h}^{y+h}\bigl|f(t)\bigr|\,dt\right|\right) + \bigl|f(x)-f(y)\bigr|.$$

By local summability of f we obtain the existence of such δ_1 that

$$|x-y| < \delta_1 \Rightarrow \left|\int\limits_x^{y}\bigl|f(t)\bigr|\,dt\right| < \frac{|h|\varepsilon}{3} \qquad\text{and}\qquad \left|\int\limits_{x+h}^{y+h}\bigl|f(t)\bigr|dt\right| < \frac{|h|\varepsilon}{3}$$

and the approximate continuity of f implies the existence of δ_2 such that

$$|x-y| < \delta_2 \implies \bigl|f|_A(x) - f|_A(y)\bigr| < \frac{\varepsilon}{3}.$$

Thus taking $\delta = \min(\delta_1,\delta_2)$ one obtains an implication

$$|x-y| < \delta \implies \bigl|g|_A(x) - g|_A(y)\bigr| < \varepsilon.$$

The approximate continuity of g follows.

Too see that the function p is measurable, we remark that

$$p(x) = \inf_{r\in\mathbb{Q}\setminus\{0\}} |r|\chi_r(x),$$

where χ_r is the characteristic function of the set $\{y\colon |H(y,r)| > \eta\}$. Thus we have

Observation. *The function p is measurable.*

We conclude this note with an incomplete answer to the problem we formulated above.

Proposition 1. *If p is a positive, approximately lower semicontinuous and of Baire class one, then there exists an increasing sequence (p_n) of approximately continuous (and upper semicontinuous) functions that converge to p pointwise.*

Proof. Let (q_k) be a sequence that consists of all positive rational numbers. The sets $E_k = \{x\colon f(x) > q_k\}$ are d-open and F_σ (by approximate lower semicontinuity and Baire class one). By Zahorski Lemma (see [1], Theorem 6.5 on page 22), there is an upper semicontinuous and approximately continuous function f_k such that

$$0 < f_k \le 1 \quad \text{on } E_k \qquad \text{and} \qquad f_k = 0 \quad \text{on } E_k',$$

where E_k' denotes the complement of the set E_k. For $n \in \mathbb{N}$ define functions $f_n^{q_k}(x) = \min\bigl(q_k, nf_k(x)\bigr)$. They are approximately continuous and upper semicontinuous and for fixed k, the sequence $(f_n^{q_k})$ converges increasingly to $q_k\chi_{E_k}$. It follows that $f_n^{q_k} < p$.

Now define $p_n = \max\limits_{1\le k\le n} f_n^{q_k}$. Then the p_n's are approximately continuous and upper semicontinuous. Moreover, they satisfy the following four properties:

(i) $p_n \le p_{n+1} < p$;

(ii) $\lim p_n \le p$;

(iii) if $p(x) > q$ for some positive rational q then $\lim p_n(x) \ge q$;

(iv) $\lim p_n \ge p$.

Properties (i) and (ii) are obvious. to see (iii), let m be the index of number q in the sequence (q_k). Then $n \ge m$ implies $p_n(x) \ge f_n^{q_m}(x)$. Hence

$$\lim_{n\to\infty} p_n(x) \ge \lim_{n\to\infty} f_n^{q_m}(x) = q_m\chi_{E_{q_m}}(x) = q_m = q\,.$$

The second last equality follows from the fact that $f(x) > q$ implies $x \in E_{q_m}$. Property (iv) is a straightforward consequence of (iii).

The properties (ii) and (iv) imply $\lim\limits_{n\to\infty} p_n = p$. □

Proposition 2. *If p is approximately lower semicontinuous of Baire class one and $p > 0$, then there is an approximately continuous function r with $0 < r \le p$.*

Proof. By Proposition 1, there is an increasing sequence (p_n) of approximately continuous functions convergent to p. Define two sequences of approximately continuous functions:

$$\phi_1 = 1\,, \qquad \psi_1 = \min(\phi_1, p_1)\,,$$

$$\phi_n = \max\left(\psi_{n-1}, \frac{1}{n}\right), \qquad \psi_n = \min(\phi_n, p_n)\,. \tag{$*$}$$

Hence $\phi_n \geq \frac{1}{n} \geq \frac{1}{n+1}$ and $\psi_n \leq \phi_n$. Therefore

$$\phi_{n+1} = \max\left(\psi_n, \frac{1}{n+1}\right) \leq \phi_n\,,$$

that means the sequence (ϕ_n) is non-increasing. Similarly, $\psi_n \leq p_n \leq p_{n+1}$ and $\phi_{n+1} \geq \psi_n$. Thus

$$\psi_{n+1} = \min(\phi_{n+1}, p_{n+1}) \geq \psi_n\,.$$

That implies the sequence (ψ_n) is non-decreasing. Let ϕ and ψ be the pointwise limits of ϕ_n and ψ_n respectively. Passing to the limits in $(*)$ we get

$$\phi = \max(\psi, 0) \qquad \text{and} \qquad \psi = \min(\phi, p)\,.$$

If $\phi(x) \neq \psi(x)$ for some x then $\phi(x) = 0$ and $\psi(x) = p(x)$. Hence $0 = \max(p(x), 0) = p(x)$, a contradiction. Thus $\phi = \psi$.

Denote by r the common limit of (ϕ_n) and (ψ_n). It is an approximately continuous function since r is lower approximately continuous as a limit of increasing sequence (ψ_n) of approximately continuous functions and r is upper approximately continuous as a limit of decreasing sequence (ϕ_n) of approximately continuous functions. It is clear that $0 < r \leq p$. □

During the 15th Summer School on Real Functions Theory in Liptovský Ján in September 1998, Z. Grande and V. A. Skvortsov informed us that they had found an example of an approximately continuous function not having the SAC property. However their example is an unbounded function.

REFERENCES

[1] BRUCKNER, A.: *Differentiation of Real Functions*, CRM monograph Series, American Mathematical Society, Providence, 1994.

[2] GRANDE, Z.: *On equi-derivatives*, Real Anal. Exchange **21** (1995/6), 637–647.

[3] LOJASIEWICZ, S.: *An Introduction to the Theory of Real Functions*, John Wiley & Sons, Chichester, 1988.

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Instytut Matematyki
Uniwersytet Szczeciński
ul. Wielkopolska 15
PL–70–451 Szczecin
POLAND

E-mail: szkibiel@euler.mat.univ.szczecin.pl