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REMARK ON MEAGER Σ_2^0-SUPPORTED σ-IDEALS ON THE REAL LINE

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ABSTRACT. We consider a condition stating that, for each member $\mathscr{I}$ of some wide class of Σ_2^0-supported σ-ideals on $\mathbb{R}$, there exists a Polish topology on $\mathbb{R}$ making $\mathscr{I}$ meager. We show its independence of ZFC.

Introduction

In [CJ] the authors studied whether, for a given (proper) σ-ideal $\mathscr{I}$ of subsets of X, there exists a possibly good topology τ on X such that $\mathscr{I}$ equals the σ-ideal of meager (i.e., the first category) sets with respect to τ. Such an $\mathscr{I}$ is called briefly a *τ-meager σ-ideal.* In [BR] the above problem was investigated for the case when $\mathscr{I}$ is a Σ_2^0-supported σ-ideal of subsets of a given uncountable Polish space X. Recall that $\mathscr{I}$ is *Σ_2^0-supported* if each set $A \in \mathscr{I}$ is contained in a set $B \in \mathscr{I}$ of type F_σ. (In another notation the class of F_σ sets is written as Σ_2^0, see, e.g., [Ke].) In particular, it is interesting to know whether there exists a Polish topology τ on X such that $\mathscr{I}$ is τ-meager. These studies are connected with a recent theorem of Kechris and Solecki from [KS]. By that theorem, if a Σ_2^0-supported σ-ideal $\mathscr{I}$ on a Polish space is ccc (i.e., each family of disjoint Borel sets which are not in $\mathscr{I}$ is countable), then there exists a countable family $\mathscr{F}$ of closed subsets of X such that $\mathscr{I}$ consists of the sets $A \subseteq X$ for which $A \cap F$ is meager in F for every $F \in \mathscr{F}$. In that case, it is possible to find a Polish topology τ on X such that $\mathscr{I}$ is τ-meager [BR]. (See also [R].) In the present paper we discuss some further conditions under which there exists (or does not) a Polish topology making a given (not necessarily ccc) Σ_2^0-supported σ-ideal meager.

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We use standard set-theoretical notation as in [Ci]. We consider σ-ideals $\mathscr{I}$ of subsets of the real line $\mathbb{R}$. Let $\mathfrak{c} = |\mathbb{R}|$. We say that $\mathscr{A} \subseteq \mathscr{I}$ is a *base* of $\mathscr{I}$ if

$$(\forall B \in \mathscr{I})(\exists A \in \mathscr{A})(B \subseteq A).$$

We denote:

$$\begin{aligned} \operatorname{add}(\mathscr{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \text{ and } \textstyle\bigcup \mathscr{A} \notin \mathscr{I}\}, \\ \operatorname{cov}(\mathscr{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \subseteq \mathscr{I} \text{ and } \textstyle\bigcup \mathscr{A} = \mathbb{R}\}, \\ \operatorname{non}(\mathscr{I}) &= \min\{|E| : E \subseteq \mathbb{R} \text{ and } E \notin \mathscr{I}\}, \\ \operatorname{cof}(\mathscr{I}) &= \min\{|\mathscr{A}| : \mathscr{A} \text{ is a base of } \mathscr{I}\}. \end{aligned}$$

Recall that $\operatorname{add}(\mathscr{I}) \leq \min\{\operatorname{cov}(\mathscr{I}), \operatorname{non}(\mathscr{I})\} \leq \max\{\operatorname{cov}(\mathscr{I}), \operatorname{non}(\mathscr{I})\} \leq \operatorname{cof}(\mathscr{I})$.

Let $\mathscr{M} \subseteq \mathscr{P}(\mathbb{R})$ stand for the σ-ideal of meager (in the natural topology) subsets of $\mathbb{R}$, and let $\mathscr{E} \subseteq \mathscr{P}(\mathbb{R})$ denote the σ-ideal generated by closed sets of Lebesgue measure zero. Clearly, $\mathscr{M}$ and $\mathscr{E}$ are Σ^0_2-supported σ-ideals.

Results

In the reminder of the paper the following statement will be denoted as $(*)$.

For each σ-ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{R})$ such that

(a) $[\mathbb{R}]^{\leq\omega} \subseteq \mathscr{I}$,

(b) $\mathscr{I}$ is Σ^0_2-supported,

(c) $(\forall A \in \mathscr{I})(\exists B \in [\mathbb{R} \setminus A]^{\mathfrak{c}})(B \in \mathscr{I})$,

we have

($\dagger$) there exists a Polish topology τ on $\mathbb{R}$ such that $\mathscr{I}$ is τ-meager.

Remarks. 1^0 It is evident that each ccc σ-ideal satisfies condition (c).
2^0 The σ-ideal $[\mathbb{R}]^{\leq\omega}$ satisfies (a) and (b) but it does not satisfy (c), and ($\dagger$) is false for it since $\mathbb{R}$ with any Polish topology τ contains a τ-nowhere dense uncountable (τ-perfect) set.

At first let us observe (using methods presented in [BR]) that, in some models of ZFC, the statement $(*)$ is valid.

THEOREM 1. *Assume that* $\operatorname{add}(\mathscr{M}) = \operatorname{cof}(\mathscr{M}) = \kappa$. *If a σ-ideal* $\mathscr{I} \subseteq \mathscr{P}(\mathbb{R})$ *satisfies the conditions* (a), (c), *and* $\operatorname{add}(\mathscr{I}) = \operatorname{cof}(\mathscr{I}) = \kappa$ *then* ($\dagger$) *holds true.*

Proof. Using $\operatorname{add}(\mathscr{M}) = \operatorname{cof}(\mathscr{M}) = \operatorname{add}(\mathscr{I}) = \operatorname{cof}(\mathscr{I}) = \kappa$, condition (c), and Sierpiński–Erdős type argument (see [O, Th. 19.5], [BR, Th. 2.1], or

[BJ, Th. 2.1.8]) we can find bases $\{B_\alpha : \alpha < \kappa\}$, $\{D_\alpha : \alpha < \kappa\}$ of $\mathscr{M}$ and $\mathscr{I}$, respectively, such that

- $B_\alpha \subseteq B_\gamma$ and $D_\alpha \subseteq D_\gamma$ for any $\alpha < \gamma < \kappa$,
- $|B_0| = |D_0| = \left|B_{\alpha+1} \setminus B_\alpha\right| = \left|D_{\alpha+1} \setminus D_\alpha\right| = \mathfrak{c}$ for each $\alpha < \kappa$, and
- $B_\alpha = \bigcup_{\gamma<\alpha} B_\gamma$ and $D_\alpha = \bigcup_{\gamma<\alpha} D_\gamma$ for any limit ordinal $\alpha < \kappa$.

Then a bijection $f \colon \mathbb{R} \to \mathbb{R}$ such that $f[B_0] = D_0$ and $f\left[B_{\alpha+1} \setminus B_\alpha\right] = D_{\alpha+1} \setminus D_\alpha$ for each $\alpha < \kappa$ is an isomorphism between $\mathscr{M}$ and $\mathscr{I}$. This means that for each $E \subseteq \mathbb{R}$ the conditions $E \in \mathscr{M}$ and $f[E] \in \mathscr{I}$ are equivalent. Hence the metric ρ given by $\rho(x,y) = \left|f(x) - f(y)\right|$ for $x, y \in \mathbb{R}$ generates a Polish topology on $\mathbb{R}$ that makes $\mathscr{I}$ meager. $\square$

Corollary 1. *The continuum hypothesis CH implies* $(*)$. *In particular* $(*)$ *is consistent with ZFC.*

Theorem 2. *If* $\mathrm{cov}(\mathscr{M}) = \omega_1 < \mathrm{cof}(\mathscr{M})$ *then* $(*)$ *is false.*

Proof. Since $\mathrm{cov}(\mathscr{M}) = \omega_1$, we can find a family $\{F_\alpha : \alpha < \omega_1\}$ of F_σ sets such that $\bigcup_{\alpha<\omega_1} F_\alpha = \mathbb{R}$. Replacing F_α by $\bigcup_{\gamma<\alpha} F_\gamma$, if necessary, we can additionally require that $F_\alpha \subseteq F_\beta$ if $\alpha < \beta$. By further modification, we can also ensure that $|F_\beta \setminus F_\alpha| = \mathfrak{c}$ for every $\alpha < \beta < \omega_1$.

Let $\mathscr{I}$ consist of the sets $A \subseteq \mathbb{R}$ such that $A \subseteq F_\alpha$ for some $\alpha < \omega_1$. Then the σ-ideal $\mathscr{I}$ satisfies the conditions (a), (b), and (c). Also observe that $\mathrm{cof}(\mathscr{I}) = \omega_1$. Suppose that $\mathscr{I}$ satisfies (†) and let τ be a Polish topology on $\mathbb{R}$ that makes $\mathscr{I}$ meager. It is known that there exists a Borel isomorphism between Polish spaces $\langle \mathbb{R}, \tau \rangle$ and $\mathbb{R}$ (with natural topology) that preserves Baire category [CKP, Th. 3.15]. We thus have $\mathrm{cof}(\mathscr{I}) = \mathrm{cof}(\mathscr{M})$ which contradicts $\mathrm{cof}(\mathscr{I}) = \omega_1 < \mathrm{cof}(\mathscr{M})$. $\square$

Since the condition $\mathrm{cov}(\mathscr{M}) = \omega_1 < \mathrm{cof}(\mathscr{M})$ is consistent with ZFC (see, e.g., [BJ]) we can conclude that

Corollary 2. *The condition* $(*)$ *is independent of ZFC.*

Another way of obtaining the consistency of $\neg(*)$ is the following theorem.

Theorem 3. *The condition* $(*)$ *implies the existence of a Luzin set in* $\mathbb{R}$.

Proof. Let $\mathscr{J} \subseteq \mathbb{R}$ be the σ-ideal generated by $\left(\mathscr{M} \cap \mathscr{P}([0,1])\right) \cup \left[\mathbb{R} \setminus [0,1]\right]^{\leq\omega}$. Then $\mathscr{J}$ satisfies conditions (a), (b), and (c). Hence, by $(*)$, there exists a Polish topology τ on $\mathbb{R}$ making $\mathscr{J}$ meager. It is clear that $X = \mathbb{R} \setminus [0,1]$ is a τ-Luzin set, which means that $|A \cap X| \leq \omega$ for each τ-meager set A. Consider a Borel isomorphism φ between $\langle \mathbb{R}, \tau \rangle$ and $\mathbb{R}$ with the natural topology, such that φ preserves Baire category. Then $\varphi[X]$ is a Luzin set in $\mathbb{R}$. $\square$

By Theorem 3 the condition $(*)$ is false in any model of ZFC in which there is no Luzin set. In particular, since there is no Luzin set under Martin's axiom MA and the negation of continuum hypothesis (see [Ku]) we can conclude the following corollary.

COROLLARY 3. MA $+\neg$CH *implies the negation of* $(*)$.

The next corollary gives a partial answer to a question of Balcerzak and Rogowska [BR].

COROLLARY 4. *The existence of a Polish topology making the ideal $\mathscr{E}$ meager is independent of* ZFC.

Proof. Since $\mathscr{E}$ satisfies conditions (a), (b), and (c), it follows from Corollary 1 that, under CH, there exists a Polish topology which makes $\mathscr{E}$ meager. On the other hand, it was proved in [BS] that there are models of ZFC in which $\mathrm{non}(\mathscr{E}) < \mathrm{non}(\mathscr{M})$ and such in which $\mathrm{cov}(\mathscr{E}) > \mathrm{cov}(\mathscr{M})$. In those models the existence of a Polish topology τ that makes $\mathscr{E}$ meager is impossible since the respective Borel isomorphism preserving Baire category would witness that $\mathrm{non}(\mathscr{E}) = \mathrm{non}(\mathscr{M})$ and $\mathrm{cov}(\mathscr{E}) = \mathrm{cov}(\mathscr{M})$. □

The proof of the above corollary is based on the fact that the cardinal numbers non or cov distinguish between $\mathscr{E}$ and $\mathscr{M}$. But what if the numbers add, cov, non, and cof are the same for $\mathscr{M}$ and some σ-ideal $\mathscr{I}$ on $\mathbb{R}$? Does it imply, in ZFC, that $\mathscr{I}$ can be made meager by a Polish topology? This leads to the following open problem.

Problem 1. Is it provable in ZFC that the property (†) holds true for every σ-ideal $\mathscr{I} \subset \mathscr{P}(\mathbb{R})$ fulfilling conditions (a), (b), (c) and the equalities $\mathrm{add}(\mathscr{I}) = \mathrm{add}(\mathscr{M})$, $\mathrm{cov}(\mathscr{I}) = \mathrm{cov}(\mathscr{M})$, $\mathrm{non}(\mathscr{I}) = \mathrm{non}(\mathscr{M})$, $\mathrm{cof}(\mathscr{I}) = \mathrm{cof}(\mathscr{M})$?

It is shown in [CJ, Th.3.11] that, under CH, for any σ-ideal $\mathscr{I}$ on a set X of cardinality $\mathfrak{c}$ there exists a Hausdorff topology on X making $\mathscr{I}$ meager. For the ideal $\mathscr{E}$ we have even better situation — it follows from Corollary 1 that, under CH, $\mathscr{E}$ can be made meager by a Polish topology. However the following general question of Balcerzak and Rogowska [BR] remains open.

Problem 2. Is it provable in ZFC that there is a Hausdorff topology on $\mathbb{R}$ making $\mathscr{E}$ meager?

REFERENCES

[BR] BALCERZAK, M.—ROGOWSKA, D.: *Making some ideals meager on sets of size of the continuum*, Topology Proc. **21** (1996), 1–13.

[BJ] BARTOSZYŃSKI, T.—JUDAH, H.: *Set Theory: on the structure of the real line* (A. K. Peters, ed.), Wellesley, Mass., 1995.

[BS] BARTOSZYŃSKI, T.—SHELAH, S.: *Closed measure zero sets*, Ann. Pure Appl. Logic **58** (1992), 93–110.

[CKP] CICHOŃ, J.—KHARAZISHVILI, A.—WĘGLORZ, B.: *Subsets of the Real Line*, Łódź University Press, Łódź, 1995.

[Ci] CIESIELSKI, K.: *Set Theory for the Working Mathematician*, London Math. Soc. Student Texts, Vol 39, Cambridge Univ. Press, 1997.

[CJ] CIESIELSKI, K.—JASINSKI, J.: *Topology making a given ideal nowhere dense or meager*, Topology Appl. **63** (1995), 277–298.

[Ke] KECHRIS, A. S.: *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1994.

[KS] KECHRIS, A. S.—SOLECKI, S.: *Approximation of analytic by Borel sets and definable countable chain condition*, Israel J. Math. **89** (1995), 343–356.

[Ku] KUNEN, K.: *Luzin spaces*, Topology Proc. **1** (1976), 191–199.

[O] OXTOBY, J. C.: *Measure and Category*, Springer-Verlag, New York, 1971.

[R] ROGOWSKA, D.: *On a metric making a σ-ideal of subsets of an uncountable Polish space meager*, Tatra Mt. Math. Publ. **8** (1996), 169–173.

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