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INVARIANCE OF ALMOST PARACOMPACTNESS UNDER SOME OPERATIONS

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ABSTRACT. We show that the change of a given topology on the Hashimoto ones preserves almost paracompactness. This is used to give some sufficient conditions for invariance of almost paracompactness under multivalued maps.

In 1969, M. K. Singal and S. P. Arya introduced a weak form of the paracompactness called the almost paracompactness [9]. Namely, a topological space (X,T) is said to be almost paracompact if for every open cover $\mathscr{U}$ of X there is a locally finite family $\mathscr{V}$ of open subsets of X which refines $\mathscr{U}$ and whose union is dense in (X,T). Then the following theorems on invariance of the almost paracompactness were given, [9, Th. 6.3.1 and Th. 6.3.2]:

Theorem A. *If f is a closed open continuous map of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then Y is almost paracompact if X is almost paracompact.*

Theorem B. *If f is a closed continuous and irreducible mapping of a space X onto a space Y such that $f^{-1}(y)$ is compact for each $y \in Y$, then X is almost paracompact if Y is such.*

Recently J. Cao and I. L. Reilly obtained an extension of Theorem A on the case of α-continuous multivalued maps.

Throughout this paper, (X,T) and (Y,τ) are always topological spaces. For a subset A in (X,T) its closure and interior are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively. Then A is called α-open [7] if $A \subset \mathrm{Int}\big(\mathrm{Cl}\big(\mathrm{Int}(A)\big)\big)$. The family T_α of all α-open sets in (X,T) forms a topology on X.

Now, let $F\colon (X,T) \to (Y,\tau)$ be a multivalued map; we always assume $F(x) \neq \emptyset$ for all $x \in X$. For every $A \subset X$, $B \subset Y$ we will denote $F^+(B) = \{x \in X\colon F(x) \subset B\}$, $F^-(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}$ and $F(A) = \cup\{F(x)\colon x \in A\}$; also we will write $F^-(y)$ instead of $F^-\big(\{y\}\big)$ for every

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$y \in Y$. A map F is called surjection if $F(X) = Y$, [2]. A multivalued map $F\colon (X,T) \to (Y,\tau)$ is said to be:

- upper (lower) semicontinuous, if for each open set $W \subset Y$ the set $F^+(W)$, (resp. $F^-(W)$) is open, [2],
- upper (lower) α-continuous, if $F\colon (X,T_\alpha) \to (Y,\tau)$ is upper (lower) semicontinuous [4, 6, 8] (in [6] the α-continuity is called the strong quasicontinuity).

A subset A of a space (X,T) is said to be α-paracompact if for every open cover $\mathscr{U}$ of A in (X,T) there exists a locally finite open cover $\mathscr{V}$ of A which refines $\mathscr{U}$ [1].

According to [3, Th. 4.10] we have the following result:

Theorem C. *Let $F\colon (X,T) \to (Y,\tau)$ be an upper and lower α-continuous surjection from an almost paracompact space (X,T) such that $F(x)$ is α-paracompact for each $x \in X$ and $F^-(y)$ is compact for each $y \in Y$. If $F\colon (X,T) \to (Y,\tau_\alpha)$ is open and closed, then (Y,τ) is almost paracompact.*

As it was shown in [7], the topology T_α can be presented in the form $T_\alpha = \{U \setminus H\colon U \in T,\ H$ is nowhere dense$\}$. But in a similar way, topologies $T(P) = \{U \setminus H\colon U \in T,\ H \in P\}$ can be constructed, where P is some ideal in X. The purpose of this paper is to generalize some results of [9] and [3]; we examine the invariance of the almost paracompactness under multivalued maps using $T(P)$ topologies.

Let P be an ideal of subsets of a topological space (X,T). For a set $A \subset X$ we denote $D_P(A) = \{x \in X\colon U \cap A \notin P$ for each neighbourhood U of $x\}$. Furthermore, if the following is satisfied:

$$A \in P \iff A \cap D_P(A) = \emptyset \iff D_P(A) = \emptyset\,, \qquad (*)$$

then $A \to A \cup D_p(A)$ is the closure operator inducing the topology $T(P) = \{U \setminus H\colon U \in T,\ H \in P\}$, [5]. Now, by $\mathrm{Cl}_P(A)$ and $\mathrm{Int}_P(A)$ we will denote the $T(P)$-closure and $T(P)$-interior of A. Also the following properties will be used:

1. Lemma. [5]. *Let P be an ideal in (X,T) that satisfies $(*)$. Then*

(a) $\mathrm{Cl}(W) = \mathrm{Cl}_P(W)$ *for each set $W \in T(P)$;*

(b) *each set $H \in P$ is $T(P)$-nowhere dense.*

2. Lemma. *Let P be an ideal in (X,T) satisfying $(*)$ and $T \cap P = \{\emptyset\}$. A family $\mathscr{V} = \{U_j \setminus H_j\colon j \in J\}$, where $U_j \in T$, $H_j \in P$ for $j \in J$, is $T(P)$-locally finite if and only if $\mathscr{U} = \{U_j\colon j \in J\}$ is T-locally finite.*

P r o o f. Assume that $\mathscr{V}$ is $T(P)$-locally finite and $x \in X$. Then there are $W \in T$ and $M \in P$ such that the set $J(x) = \{j \in J\colon\ (U_j \setminus H_j) \cap (W \setminus M) \neq \emptyset\}$

is finite. Thus for $j \in J \setminus J(x)$ we have $W \cap U_j \setminus (M \cup H_j) = \emptyset$, so the condition $T \cap P = \{\emptyset\}$ implies $W \cap U_j = \emptyset$. Hence $\{j \in J \colon W \cap U_j \neq \emptyset\} \subset J(x)$ and it means that $\mathscr{U}$ is T-locally finite. The converse follows from $T \subset T(P)$. □

3. Theorem. *Let P be an ideal in (X,T) satisfying $(*)$ and $T \cap P = \{\emptyset\}$. The space (X,T) is almost paracompact if and only if $\big(X, T(P)\big)$ is almost paracompact.*

P r o o f . Assume that (X,T) is almost paracompact. Let $\mathscr{U}$ be a $T(P)$-open cover of X. Then it is of the form $\mathscr{U} = \{U_s \setminus H_s \colon s \in S\}$ with $U_s \in T$, $H_s \in P$ for $s \in S$ and $\mathscr{U}_1 = \{U_s \colon s \in S\}$ is a T-open cover of X. From the assumption there is a T-open T-locally finite refinement $\mathscr{V}_1 = \{V_j \colon j \in J\}$ of $\mathscr{U}_1$ such that $\bigcup\{\mathrm{Cl}(V_j) \colon j \in J\} = X$. For each $j \in J$ we fix $s_j \in S$ such that $V_j \subset U_{s_j}$. Hence $\mathscr{V} = \{V_j \setminus H_{s_j} \colon j \in J\}$ is a $T(P)$-open $T(P)$-locally finite refinement of $\mathscr{U}$. Furthermore, according to Lemma 1 we have

$$\begin{aligned}\bigcup\{\mathrm{Cl}_P(V_j \setminus H_{s_j}) \colon j \in J\} &= \bigcup\{\mathrm{Cl}(V_j \setminus H_{s_j}) \colon j \in J\}\\ &= \bigcup\{\mathrm{Cl}(V_j) \colon j \in J\} = X,\end{aligned}$$

so $\big(X, T(P)\big)$ is almost paracompact.

Conversely, let $\mathscr{U} = \{U_s \colon s \in S\}$ be a T-open cover of X. Since $\big(X, T(P)\big)$ is almost paracompact, there is a $T(P)$-open, $T(P)$-locally finite refinement $\mathscr{V}$ of $\mathscr{U}$, $\mathscr{V} = \{V_j \setminus H_j \colon j \in J\}$, $V_j \in T$, $H_j \in P$, such that $\bigcup\{\mathrm{Cl}_P(V_j \setminus H_j) \colon j \in J\} = X$. Then from Lemma 2, $\mathscr{V}_1 = \{V_j \colon j \in J\}$ is T-locally finite. For each $j \in J$ we fix $s_j \in S$ such that $V_j \setminus H_j \subset U_{s_j}$. Hence $\mathscr{U}_1 = \{V_j \cap U_{s_j} \colon j \in J\}$ is a T-open, T-locally finite refinement of $\mathscr{U}$ and

$$\begin{aligned}\bigcup\{\mathrm{Cl}(V_j \cap U_{s_j}) \colon j \in J\} &\supset \bigcup\{\mathrm{Cl}(V_j \setminus H_j) \colon j \in J\}\\ &= \bigcup\{\mathrm{Cl}_P(V_j \setminus H_j) \colon j \in J\} = X.\end{aligned}$$

Thus (X,T) is almost paracompact and the proof is completed. □

Let us remark that Theorem 3 gives examples of almost paracompact spaces which are not paracompact. For instance, if (R,T) is the space of real numbers with the euclidean topology and P is an ideal satisfying $(*)$ and $T \cap P = \{\emptyset\}$, then $\big(R, T(P)\big)$ is a Hausdorff space which is not regular. Thus $\big(R, T(P)\big)$ is not paracompact, but following Theorem 3 it is almost paracompact.

Evidently, if P is the ideal nowhere dense sets in (X,T), then $T(P) = T_\alpha$. So we obtain

4. Corollary. [3]. *A topological space (X,T) is almost paracompact if and only if (X, T_α) is almost paracompact.*

5. Remark. In [3] the following characterization of the almost paracompactness is given: a topological space (X,T) is almost paracompact if and only if each

T_α-open cover of X has a T-open T-locally finite refinement whose union is dense in (X,T). This is not true in general case.

6. EXAMPLE. Let $(\mathbb{R},T)$ be the space of real numbers with the usual topology, $\mathbb{Q}$ the set of all rational numbers and let P be the ideal of the first category sets in $(\mathbb{R},T)$. Following Theorem 3, $(\mathbb{R},T(P))$ is almost paracompact. For each $n=1,2,\dots$ let us put $A_n=\{n\sqrt{2}+q\colon q\in\mathbb{Q}\}$. Then $\mathscr{U}=\{\mathbb{R}\setminus A_n\colon n=1,2,\dots\}$ is a $T(P)$-open cover of $\mathbb{R}$ but $\operatorname{Int}(\mathbb{R}\setminus A_n)=\emptyset$ for each $n=1,2,\dots$; thus a T-open refinement of $\mathscr{U}$ does not exist.

Recall that a multivalued map $F\colon (X,T)\to(Y\tau)$ is called closed (open) if for each closed (open) set $U\subset X$ the set $F(U)$ is closed (open). Similarly as for functions there is a characterization of closed multivalued maps; we include it as the lemma without a proof.

7. LEMMA. *A multivalued map $F\colon (X,T)\to(Y,\tau)$ is closed if and only if for each point $y\in Y$ and each open set U containing $F^-(y)$ there exists a neighbourhood V of y such that $F^-(V)\subset U$.*

8. THEOREM. *Let P_1,P_2 be ideals in (X,T) and (Y,τ) satisfying $(*)$ and $T\cap P_1=\{\emptyset\}=\tau\cap P_2$, and let $F\colon (X,T)\to(Y,\tau)$ be a surjection with α-paracompact values and $F^-(y)$ compact for $y\in Y$. If (X,T) is almost paracompact, $F\colon \big(X,T(P_1)\big)\to(Y,\tau)$ is upper and lower semicontinuous and $F\colon (X,T)\to\big(Y,\tau(P_2)\big)$ is closed and open, then (Y,τ) is almost paracompact.*

Proof. Let $\mathscr{U}$ be a $\tau(P_2)$-open cover of Y; then $\mathscr{U}=\{U_s\setminus H_s\colon s\in S\}$, where $U_s\in\tau$, $H_s\in P_2$, and $\mathscr{U}_1=\{U_s\colon s\in S\}$ is a τ-open cover of Y. Since F has α-paracompact values, for each $x\in X$ there exists a τ-locally finite τ-open cover $\mathscr{V}_x$ of $F(x)$ which refines $\mathscr{U}_1$. Let V_x be the union of all sets belonging to $\mathscr{V}_x$. The map $F\colon \big(X,T(P_1)\big)\to(Y,\tau)$ is upper semicontinuous, so $\{F^+(V_x)\colon x\in X\}$ is a $T(P_1)$-open cover of X. Then we have $F^+(V_x)=G_x\setminus C_x$ for some $G_x\in T$ and $C_x\in P_1$. Following Theorem 3 the space $\big(X,T(P_1)\big)$ is almost paracompact, so there exists a $T(P_1)$-open, $T(P_1)$-locally finite refinement $\{W_j\setminus M_j\colon j\in J\}$ of $\{F^+(V_x)\colon x\in X\}$, such that $W_j\in T$, $M_j\in P_1$ for $j\in J$ and $\bigcup\big\{\mathrm{Cl}_{P_1}(W_j\setminus M_j)\colon j\in J\big\}=X$. In virtue of Lemma 2 the family $\{W_j\colon j\in J\}$ is T-locally finite. For each $j\in J$ we fix $x_j\in X$ such that $W_j\setminus M_j\subset F^+(V_{x_j})=G_{x_j}\setminus C_{x_j}$. This implies $W_j\setminus M_j=(W_j\cap G_{x_j})\setminus(M_j\cup C_{x_j})$ and then $F(W_j\cap G_{x_j})\cap V_{x_j}\neq\emptyset$. Now we have the family $\big\{F(W_j\cap G_{x_j})\cap V\colon j\in J,\ V\in\mathscr{V}_{x_j}\big\}$ of $\tau(P_2)$-open sets which refines $\mathscr{U}_1$. For each pair (j,V), where $j\in J$, $V\in\mathscr{V}_{x_j}$, we fix $s(j,V)\in S$ such that $F(W_j\cap G_{x_j})\cap V\subset U_{s(j,V)}$. Then

$$\mathscr{G}=\big\{F(W_j\cap G_{x_j})\cap V\setminus H_{s(j,V)}\colon j\in J,\ V\in\mathscr{V}_{x_j}\big\}$$

is a $\tau(P_2)$-open refinement of $\mathscr{U}$. We are going to show that $\mathscr{G}$ is $\tau(P_2)$-locally finite. Let $y \in Y$. Since the family $\{W_j \cap G_{x_j} : j \in J\}$ is T-locally finite and $F^-(y)$ is a T-compact set, we can select $A \in T$ such that $F^-(y) \subset A$ and the set $J_o = \{j \in J \colon A \cap W_j \cap G_{x_j} \neq \emptyset\}$ is at most finite. The map $F\colon (X,T) \to \big(Y, \tau(P_2)\big)$ is closed, so according to Lemma 7, there is a $\tau(P_2)$-neighbourhood E of y with $F^-(E) \subset A$. Thus $\big\{j \in J \colon F(W_j \cap G_{x_j}) \cap E \neq \emptyset\big\} \subset J_o$; consequently E intersects at most a finite number of sets belonging to $\mathscr{G}$. So we have shown that $\mathscr{G}$ is locally finite in $\big(Y, \tau(P_2)\big)$. The map $F\colon \big(X, T(P_1)\big) \to (Y, \tau)$ is lower semicontinuous, therefore $F\big(\mathrm{Cl}_{P_1}(W_j \setminus M_j)\big) \subset \mathrm{Cl}\big(F(W_j \setminus M_j)\big)$ for $j \in J$; moreover $F(W_j \setminus M_j) \subset F(W_j \cap G_{x_j}) \cap V_{x_j}$. Hence we obtain

$$
\begin{aligned}
Y = F(X) &= F\Big(\bigcup\{\mathrm{Cl}_{P\,1}(W_j \setminus M_j) \colon j \in J\}\Big) \subset \bigcup\Big\{\mathrm{Cl}\big(F(W_j \setminus M_j)\big) \colon j \in J\Big\} \\
&\subset \bigcup\Big\{\mathrm{Cl}\big(F(W_j \cap G_{x_j}) \cap V_{x_j}\big) \colon j \in J\Big\} \\
&= \bigcup\Big\{\mathrm{Cl}\big(F(W_j \cap G_{x_j}) \cap V\big) \colon j \in J,\ V \in \mathscr{V}_{x_j}\Big\} \\
&= \bigcup\Big\{\mathrm{Cl}\big(F(W_j \cap G_{x_j}) \cap V \setminus H_{s(j,V)}\big) \colon j \in J,\ V \in \mathscr{V}_{x_j}\Big\}.
\end{aligned}
$$

This means that $\big(Y, \tau(P_2)\big)$ is almost paracompact, so — in virtue of Theorem 3 — the proof is completed. □

9. Corollary. *Let P_1, P_2 be ideals in (X,T) and (Y,τ) respectively, satisfying $(*)$ and $T \cap P_1 = \{\emptyset\} = \tau \cap P_2$, and let $F\colon X \to Y$ be a surjection such that $F(x)$, $F^-(y)$ are compact for all $x \in X$, $y \in Y$. Assume that $F\colon \big(X, T(P_1)\big) \to (Y,\tau)$ is upper and lower semicontinuous, $F\colon (X,T) \to \big(Y, \tau(P_2)\big)$ is open and closed. Then (X,T) is almost paracompact if and only if (Y,τ) is such.*

P r o o f. If (X,T) is almost paracompact, then the conclusion follows from Theorem 8. So assume (Y,τ) to be almost paracompact and let us consider the map $G\colon Y \to X$ given by $G(y) = F^-(y)$. Then G is a surjection and for each $x \in X$, $A \subset X$ and $B \subset Y$ we have $G^-(x) = F(x)$ and $G^-(A) = F(A)$, $G(B) = F^-(B)$. Thus it suffices to apply Theorem 8 once more and the proof is completed. □

As a particular case of the above corollary we obtain a stronger version of Theorem A, namely:

10. Corollary. *Let $f\colon (X,T) \to (Y,\tau)$ be a closed open continuous surjection such that $f^{-1}(y)$ is compact for each $y \in Y$. Then (X,T) is almost paracompact if and only if (Y,τ) is almost paracompact.*

A multivalued surjection $F\colon (X,T) \to (Y,\tau)$ is said to be irreducible if $F(A) \neq Y$ for each proper closed subset $A \subset X$.

11. Theorem. *Let (X,T) be a topological space, (Y,τ) an almost paracompact one and let P be an ideal in X satisfying $(*)$ and $T\cap P=\{\emptyset\}$. If $F\colon (X,T)\to(Y,\tau)$ is an irreducible closed surjection such that:*

(a) *$F(x)$ is τ-compact and $F^-(y)$ is T-compact for each $x\in X$, $y\in Y$;*

(b) *$F\colon \big(X,T(P)\big)\to(Y,\tau)$ is upper and lower semicontinuous;*

then (X,T) is almost paracompact.

Proof. Let $\mathscr{U}=\{U_j\setminus H_j\colon j\in J\}$ with $U_j\in T$, $H_j\in P$, be a $T(P)$-open cover of X. For each $y\in Y$ we select a finite set $J(y)\subset J$ such that $F^-(Y)\subset \bigcup\{U_j\colon j\in J(y)\}$. Since F is closed, for each $y\in Y$ there is a τ-neighbourhood W_y for which it holds $F^-(W_y)\subset\bigcup\{U_j\colon j\in J(y)\}$. Thus $\mathscr{A}=\{W_y\colon y\in Y\}$ is a τ-open cover of Y, so there is a τ-locally finite family of τ-open sets $\mathscr{G}=\{G_s\colon s\in S\}$ which refines $\mathscr{A}$ and $\bigcup\{\mathrm{Cl}(G_s)\colon s\in S\}=Y$. Now, for each $s\in S$ we fix $y_s\in Y$ such that $G_s\subset W_{y_s}$; then $F^-(G_s)\subset\bigcup\{U_j\colon j\in J(y_s)\}$. Let us put

$$\begin{aligned} V_{s,j}&=F^-(G_s)\cap U_j \qquad \text{for } s\in S,\ j\in J(y_s),\\ \mathscr{V}&=\big\{V_{s,j}\setminus H_j\colon s\in S,\ j\in J(y_s)\big\}.\end{aligned}$$

By the lower semicontinuity of F, $\mathscr{V}$ is a family of $T(P)$-open sets which refines $\mathscr{U}$. Let $x_0\in X$; since $F(x_0)$ is compact and $\mathscr{G}$ is locally finite, a τ-open set D can be choosen such that $F(x_0)\subset D$ and the set $S(x_0)=\{s\in S\colon D\cap G_s\neq\emptyset\}$ is finite. Hence $F^+(D)$ is a $T(P)$-neighbourhood of x_0. If $F^+(D)\cap V_{s,j}\neq\emptyset$, then for some $z\in X$ we have $F(z)\subset D$ and $F(z)\cap G_s\neq\emptyset$. This implies $s\in S(x_0)$. Thus $F^+(D)$ intersects at most finitely many of the sets G_s; in the consequence $F^+(D)$ intersects only a finite number of sets belonging to $\mathscr{V}$. So we have shown that $\mathscr{V}$ is $T(P)$-locally finite. Furthermore, we have

$$\begin{aligned} &\bigcup\{\mathrm{Cl}_P(V_{s,j}\setminus H_j)\colon s\in S,\ j\in J(y_s)\} =\bigcup\{\mathrm{Cl}(V_{s,j})\colon s\in S,\ j\in J(y_s)\}\\ &=\bigcup\Big\{\mathrm{Cl}\Big(\bigcup\{V_{s,j}\colon j\in J(y_s)\}\Big)\colon s\in S\Big\}=\bigcup\Big\{\mathrm{Cl}\big(F^-(G_s)\big)\colon s\in S\Big\}.\end{aligned}$$

Using the fact that F is closed we obtain

$$\begin{aligned} &F\Big(\bigcup\{\mathrm{Cl}_P(V_{s,j}\setminus H_j)\colon s\in S,\ j\in J(y_s)\}\Big)=\bigcup\Big\{F\big(\mathrm{Cl}(F^-(G_s))\big)\colon s\in S\Big\}\\ &\quad\supset\bigcup\Big\{\mathrm{Cl}\big(FF^-(G_s)\big)\colon s\in S\Big\}\supset\bigcup\{\mathrm{Cl}(G_s)\colon s\in S\}=Y.\end{aligned}$$

But F is irreducible, so we have

$$\bigcup\{\mathrm{Cl}_P(V_{s,j}\setminus H_j)\colon s\in S,\ j\in J(y_s)\}=X.$$

This means $\big(X,T(P)\big)$ is almost paracompact, so according to Theorem 3 the proof is completed. $\square$

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