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ON MODULAR SEQUENCE SPACES OF NON-ABSOLUTE TYPE

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ABSTRACT. We define the modular sequence space $X\{f_n\}$ determined by the modular $\varrho(x) = \sum\limits_{n=1}^{\infty} f_n\left(\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j\right|\right)$, where $x = \{\xi_j\}_{j=1}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence of Orlicz functions. This generalizes the Cesàro sequence spaces of non-absolute type and modular spaces considered by J. Y. T. Woo in [On modular sequence spaces, Studia Math. 48 (1973), 271–289]. $X\{f_n\}$ is a Banach space; furthermore it is a dense F_σ of the first category subset of the Frèchet space s.

The Cesàro sequence spaces ces_p, $1 \le p < \infty$, consisting of all real sequences $x = \{\xi_j\}_{j=1}^{\infty}$ such that $\sum\limits_{n=1}^{\infty} \frac{1}{n} \sum\limits_{j=1}^{n} |\xi_j|^p < \infty$, were generalized to non-absolute type spaces X_p, where $x \in X_p$ iff $\sum\limits_{n=1}^{\infty} \left|\frac{1}{n}\sum\limits_{j=1}^{n}\xi_j\right|^p < \infty$ [2]. In this note we replace the function $f(t) = t^p$ by a sequence $\{f_n\}_{n=1}^{\infty}$ of Orlicz functions; so we define modular sequence spaces $X\{f_n\}$ determined by the convex modular $\varrho(x) = \sum\limits_{n=1}^{\infty} f_n\left(\left|\frac{1}{n}\sum\limits_{j=1}^{n}\xi_j\right|\right)$. Thus $X\{f_n\}$ generalizes both X_p and $l\{f_n\}$ considered in [3].

A function $f\colon [0,\infty) \to [0,\infty)$ is said to be an Orlicz function if it is continuous, non-decreasing, convex and $\lim\limits_{t\to\infty} f(t) = \infty$. If $f(t) = 0$ for some $t > 0$, then f is said to be a degenerate Orlicz function [1, p. 137].

Let f be a non-degenerate Orlicz function whose right-derivative P satisfies $P(0) = 0$ and $\lim\limits_{t\to\infty} P(t) = \infty$. The right-inverse Q of P given by $Q(u) = \sup\{t\colon P(t) \le u\}$ for $u \ge 0$, is a right-continuous non-decreasing function such that $Q(0) = 0$ and $Q(u) > 0$ for $u > 0$. Put $f^*(t) = \int\limits_0^t Q(u)\,du$ for $t > 0$. Then f^* is also a non-degenerate Orlicz function and it is called the function

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complementary to f. Then $(f^*)^* = f$ and for any $u \geq 0$, $v \geq 0$ the Young's inequality $uv \leq f(u) + f^*(v)$ holds, [1, p. 147].

A sequence $\{f_n\}_{n=1}^{\infty}$ of Orlicz functions is said to satisfy the uniform Δ_2-condition if there exists $K > 0$ and n_0 such that we have $f_n(2t) \leq K f_n(t)$ for each $t \in \left[0, \frac{1}{2}\right]$ and $n \geq n_0$ [1, p. 167]. We will use the following:

Lemma 1. [3, Prop. 3. 2] *If $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition, then there exists $p \geq 1$ and n_0 such that $f_n(t) \geq t^p$ for each $t \in [0,1]$ and $n \geq n_0$.*

In the sequel s denotes the linear space of all real sequences with the usual metric d given by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n - \eta_n|}{1 + |\xi_n - \eta_n|} \qquad \text{for} \quad x = \{\xi_j\}_{j=1}^{\infty} \in s\,, \ \ y = \{\eta_j\}_{j=1}^{\infty} \in s\,.$$

Following [2] we will denote

$$X_p = \left\{ x = \{\xi_j\}_{j=1}^{\infty} \in s \colon \ \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right|^p < \infty \right\}.$$

Then

$$\|x\|_p = \left(\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right|^p \right)^{\frac{1}{p}}$$

is a norm on X_p and $(X_p, \|\cdot\|_p)$ is a Banach space [2].

Now, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions. We put for each $x = \{\xi_j\}_{j=1}^{\infty} \in s$:

$$\varrho(x) = \sum_{n=1}^{\infty} f_n \left(\left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| \right),$$

and

$$\begin{aligned} X\{f_n\} &= \bigl\{x \in s \colon \ \varrho(tx) < \infty \quad \text{for some} \ \ t > 0\bigr\}, \\ X^0\{f_n\} &= \bigl\{x \in s \colon \ \varrho(tx) < \infty \quad \text{for each} \ \ t > 0\bigr\}, \\ X_\alpha\{f_n\} &= \bigl\{x \in s \colon \ \varrho(\alpha x) < \infty\bigr\} \ \text{for} \ \ \alpha > 0\,. \end{aligned}$$

Then ϱ is a convex modular on s and $X\{f_n\}$ is a modular space determined by ϱ; it is normed as usually by the modular norm defined as $\|x\| = \inf\bigl\{t > 0 \colon \varrho\bigl(\frac{x}{t}\bigr) \leq 1\bigr\}$. Assume that $f_n(a_n) = 1$ for $n = 1, 2, \ldots$ Putting $g_n(t) = f_n(a_n t)$ we obtain the sequence $\{g_n\}_{n=1}^{\infty}$ of non-degenerate Orlicz functions with $g_n(1) = 1$ for $n = 1, 2, \ldots$ Now, let $A \colon X\{g_n\} \to X\{f_n\}$ be the map given by $A(x) = \{a_j \xi_j\}_{j=1}^{\infty}$ for $x = \{\xi_j\}_{j=1}^{\infty} \in X\{g_n\}$. It is easy to see that A is a one-to-one linear transformation of $X\{g_n\}$ onto $X\{f_n\}$ and $\varrho'(x) = \varrho\bigl(A(x)\bigr)$ for $x \in X\{g_n\}$, where ϱ' is the modular defined by the sequence $\{g_n\}_{n=1}^{\infty}$.

This means that $X\{f_n\}$ and $X\{g_n\}$ are isometric. So throughout this paper we shall assume that $f_n(1) = 1$ for $n = 1, 2, \ldots$

THEOREM 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions. Then:*

(a) *$X\{f_n\}$ is a Banach space.*

(b) *$\overline{X_{2\alpha}\{f_n\}} \subset \bigcap_{\beta \le \alpha} X_{\beta}\{f_n\}$ for each $\alpha > 0$.*

(c) *$X^0\{f_n\}$ is a closed subspace of $X\{f_n\}$.*

(d) *If $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition, then $X\{f_n\} = X^0\{f_n\}$ and $\lim_{n\to\infty} \frac{1}{n}\xi_n = 0$ for each $x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$.*

Proof. (a) Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $X\{f_n\}$, $x_n = \{\xi_{jn}\}_{j=1}^{\infty}$ and $\varepsilon > 0$.

We choose n_0 such that $\varrho\left(\frac{x_k - x_m}{\varepsilon}\right) \le 1$ for $m, k \ge n_0$. Hence for each natural number p we have

$$\sum_{n=1}^{p} f_n\left(\left|\frac{1}{n\varepsilon}\sum_{j=1}^{n}(\xi_{jk} - \xi_{jm})\right|\right) \le 1 \qquad \text{for} \quad m, k \ge n_0\,, \tag{1}$$

so

$$f_n\left(\left|\frac{1}{n\varepsilon}\sum_{j=1}^{n}(\xi_{jk} - \xi_{jm})\right|\right) \le 1 \qquad \text{for} \quad m, k \ge n_0\,,\ \ n = 1, 2, \ldots$$

Thus, it follows from assumptions on f_n's that

$$\left|\frac{1}{n}\sum_{j=1}^{n}\xi_{jk} - \frac{1}{n}\sum_{j=1}^{n}\xi_{jm}\right| \le \varepsilon \qquad \text{for} \quad m, k \ge n_0\,,\ \ n = 1, 2, \ldots$$

So we have shown that $\left\{\frac{1}{n}\sum\limits_{j=1}^{n}\xi_{jk}\right\}_{k=1}^{\infty}$ is a Cauchy sequence for each $n = 1, 2, \ldots$ Since these sequences are convergent we obtain that also each $\{\xi_{jk}\}_{k=1}^{\infty}$ converges; let us denote

$$\xi_j = \lim_{k\to\infty} \xi_{jk} \qquad \text{for} \quad j = 1, 2, \ldots \quad \text{and} \quad x = \{\xi_j\}_{j=1}^{\infty}\,.$$

Letting $k \to \infty$ in (1) we obtain

$$\sum_{n=1}^{p} f_n\left(\left|\frac{1}{n\varepsilon}\sum_{j=1}^{n}(\xi_j - \xi_{jm})\right|\right) \le 1 \qquad \text{for} \quad m \ge n_0\,,\ \ p = 1, 2, \ldots$$

which implies $\varrho\left(\frac{x - x_m}{\varepsilon}\right) \le 1$ for $m \ge n_0$. In the consequence $x \in X\{f_n\}$ and $\|x - x_m\| \le \varepsilon$ for $m \ge n_0$ which completes the proof of (a).

(b) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of elements of $X_{2\alpha}\{f_n\}$ and $\lim\limits_{n\to\infty}\|x-x_n\|=0$, where $x_n=\{\xi_{jn}\}_{j=1}^{\infty}$, $x=\{\xi_j\}_{j=1}^{\infty}\in X\{f_n\}$. We choose m_0 such that $\|x-x_m\|<\frac{1}{2\alpha}$ for $m\ge m_0$; hence $\varrho\big(2\alpha(x-x_m)\big)\le 1$ for $m\ge m_0$. We fix $m\ge m_0$, then $r=\varrho(2\alpha x_m)<\infty$. Thus for each $k=1,2,\dots$ and $\beta\le\alpha$ we have

$$\sum_{n=1}^{k} f_n\left(\left|\frac{\beta}{n}\sum_{j=1}^{n}\xi_j\right|\right)\le\sum_{n=1}^{k} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}(\xi_j-\xi_{jm})\right|+\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_{jm}\right|\right)$$
$$\le\frac{1}{2}\varrho\big(2\alpha(x-x_m)\big)+\frac{1}{2}\varrho(2\alpha x_m)\le\frac{1}{2}(1+r)\,.$$

Letting $k\to\infty$ we obtain $\varrho(\beta x)\le\frac{1}{2}(1+r)$ which gives $x\in\bigcap\limits_{\beta\le\alpha}X_\beta\{f_n\}$.

(c) Applying (b) we have:

$$\overline{X^0\{f_n\}}=\overline{\bigcap_{\alpha>0}X_{2\alpha}\{f_n\}}\subset\bigcap_{\alpha>0}\bigcap_{\beta\in(0,\alpha)}X_\beta\{f_n\}=\bigcap_{\alpha>0}X_\alpha\{f_n\}=X^0\{f_n\}\,.$$

(d) Let $x=\{\xi_j\}_{j=1}^{\infty}\in X\{f_n\}$; then $\varrho(\alpha x)<\infty$ for some $\alpha>0$. Thus $\varrho(tx)<\infty$ for each $t\in[0,\alpha]$. The Δ_2-condition implies, according to Lemma 1, that $\lim\limits_{n\to\infty}\left|\frac{1}{n}\sum\limits_{j=1}^{n}\xi_j\right|=0$. Hence we can choose n_0 and $K>0$ such that

$$\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j\right|\le\frac{1}{2}\qquad\text{for}\quad n\ge n_0\,,$$

and

$$f_n(2t)\le Kf_n(t)\qquad\text{for}\quad n\ge n_0,\ \ t\in\left[0,\tfrac{1}{2}\right].$$

We fix $\lambda>\alpha$ and we denote $\beta=\min\{\alpha,\frac{1}{2}\}$. Now we take a natural number m with $\lambda\le 2^m\beta$. Then we have for each $n\ge n_0$:

$$f_n\left(\left|\frac{\lambda}{n}\sum_{j=1}^{n}\xi_j\right|\right)\le f_n\left(\left|2^m\frac{\beta}{n}\sum_{j=1}^{n}\xi_j\right|\right)\le K^m f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j\right|\right),$$

so $\varrho(\lambda x)<\infty$ and $x\in X^0\{f_n\}$.

Finally, for a fixed $x=\{\xi_j\}_{j=1}^{\infty}\in X\{f_n\}$ we have $\lim\limits_{n\to\infty}\left(\frac{1}{n}\sum\limits_{j=1}^{n}\xi_j\right)=0$. Since

$$\left|\frac{1}{n}\xi_n\right|=\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j-\frac{1}{n}\sum_{j=1}^{n-1}\xi_j\right|\le\left|\frac{1}{n}\sum_{j=1}^{n}\xi_j\right|+\left|\frac{1}{n-1}\sum_{j=1}^{n-1}\xi_j\right|$$

we obtain $\lim\limits_{n\to\infty}\frac{1}{n}\xi_n=0$ and the proof is completed. ◻

In what follows we shall use the following notations:

$$u_1 = \{1, -1, 0, 0, \dots\}, \ \dots, \quad u_n = \{\underbrace{0, \dots, 0}_{n-1}, n, -n, 0, 0, \dots\}, \quad \text{for } n = 1, 2, \dots$$

It is easy to see that $\varrho(u_n) = 1$ and $\|u_n\| = 1$ for $n = 1, 2, \dots$

Theorem 2. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions which satisfies the uniform Δ_2-condition. Then:*

(a) *the set $M = \{t_1 u_1 + t_2 u_2 + \dots + t_m u_m : m = 1, 2, \dots ; t_1, t_2 \dots, t_m \in \mathbb{Q}\}$ is dense in $X\{f_n\}$, so $X\{f_n\}$ is a separable space;*

(b) *for each $x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$ we have $x = \sum\limits_{n=1}^{\infty} \Big(\frac{1}{n} \sum\limits_{j=1}^{n} \xi_j\Big) u_n$.*

Proof. (a) Let $x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$, $\varepsilon > 0$. Since $\varrho\big(\frac{x}{\varepsilon}\big) < \infty$ we can choose m such that

$$\sum_{n=m+1}^{\infty} f_n\left(\left| \frac{1}{n\varepsilon} \sum_{j=1}^{n} \xi_j \right|\right) < \frac{1}{2}.$$

By the continuity of $f_1, f_2, \dots, f_m$ there exists $\delta > 0$ such that for all $\xi'_1, \xi'_2, \dots, \xi'_m$ satisfying

$$\left| \frac{1}{n\varepsilon} \sum_{j=1}^{n} (\xi_j - \xi'_j) \right| < \delta \qquad \text{for} \quad n = 1, 2, \dots, m$$

we have

$$\sum_{n=1}^{m} f_n\left(\left| \frac{1}{n\varepsilon} \sum_{j=1}^{n} (\xi_j - \xi'_j) \right|\right) < \frac{1}{2}.$$

We take $t_1, t_2, \dots, t_m \in \mathbb{Q}$ such that $|\xi_j - t_j| < \varepsilon\delta$ for $n = 1, 2, \dots, m$ and we put $y = t_1 u_1 + \frac{1}{2}(t_1 + t_2) u_2 + \dots + \Big(\frac{1}{m} \sum\limits_{j=1}^{m} t_j\Big) u_m$. Then $y \in M$,

$$x - y = \left\{ \xi_1 - t_1, \dots, \xi_m - t_m, \xi_{m+1} + \sum_{j=1}^{m} t_j, \xi_{m+2}, \xi_{m+3}, \dots \right\} \qquad \text{and}$$

$$\varrho\Big(\frac{x-y}{\varepsilon}\Big) = \sum_{n=1}^{m} f_n\left(\left| \frac{1}{n\varepsilon} \sum_{j=1}^{n} (\xi_j - t_j) \right|\right) + \sum_{n=m+1}^{\infty} f_n\left(\left| \frac{1}{n\varepsilon} \sum_{j=1}^{n} \xi_j \right|\right) < 1,$$

so $\|x - y\| \le \varepsilon$ and (a) is proved.

(b) For the fixed $x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$ and $\varepsilon > 0$ we choose $m > 1$ with

$$\sum_{n=m+1}^{\infty} f_n\left(\left| \frac{1}{n\varepsilon} \sum_{j=1}^{n} \xi_j \right|\right) < 1.$$

Now putting $y = \xi_1 u_1 + \frac{1}{2}(\xi_1 + \xi_2)u_2 + \dots + \Big(\frac{1}{m}\sum\limits_{j=1}^{m} \xi_j\Big)u_m$ we have

$$x - y = \Big\{\underbrace{0, \dots, 0}_{m}, \sum_{j=1}^{m+1} \xi_j, \xi_{m+2}, \xi_{m+3}, \dots \Big\}$$

and

$$\varrho\Big(\frac{x-y}{\varepsilon}\Big) = \sum_{n=m+1}^{\infty} f_n\left(\left|\frac{1}{n\varepsilon}\sum_{j=1}^{n} \xi_j\right|\right) < 1 .$$

Thus $\|x - y\| \le \varepsilon$ and the proof is completed. □

It is an immediate consequence of definitions that $X_1 \subset X\{f_n\}$. Moreover we have:

Proposition 1. *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-degenerate Orlicz functions.*

(a) $1_\infty \subset X\{f_n\}$ *if and only if there exists $t > 0$ such that* $\sum\limits_{n=1}^{\infty} f_n(t) < \infty$.

(b) *If $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition, then there exists $p \ge 1$ such that $X\{f_n\} \subset X_p$.*

Proof. (a) If $1_\infty \subset X\{f_n\}$ then $x = \{1, 1, 1, \dots\} \in X\{f_n\}$. Thus for some $t > 0$ we have $\infty > \varrho(tx) = \sum\limits_{n=1}^{\infty} f_n(t)$. Conversely, let $t > 0$ be such that $\sum\limits_{n=1}^{\infty} f_n(t) < \infty$ and let $x = \{\xi_j\}_{j=1}^{\infty} \in 1_\infty$. We can choose $\alpha > 0$ such that $\alpha \sup\big\{|\xi_j| \colon j = 1, 2, \dots\big\} < t$, so $\varrho(\alpha x) < \infty$ and $x \in X\{f_n\}$.

(b) It is a consequence of Lemma 1. □

Corollary 1. *If $\{f_n\}_{n=1}^{\infty}$ satisfies the uniform Δ_2-condition, then $1_\infty \not\subset X\{f_n\}$.*

Theorem 3. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of non-degenerate Orlicz functions, then for each $\alpha > 0$, $X_\alpha\{f_n\}$ is a dense F_σ-set of the first category in s.*

Proof. Let us put

$$C_{mk} = \left\{x = \{\xi_j\}_{j=1}^{\infty} \in s \colon \ \sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n} \xi_j\right|\right) > k\right\} \qquad \text{for } m, k = 1, 2, \dots$$

and

$$C = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} C_{mk} .$$

For $w = \{\frac{1}{\alpha}, \frac{1}{\alpha}, \dots\} \in s$ we have $\varrho(\alpha w) = \sum_{n=1}^{\infty} f_n(1) = \infty$, so $w \in C$. Hence

$$C \neq \emptyset\,,\ C \cap X_\alpha\{f_n\} = \emptyset \quad \text{and} \quad C \cup X_\alpha\{f_n\} = s\,. \tag{1}$$

Let $x_0 = \{\xi_j^0\}_{j=1}^{\infty} \in C_{mk}$; we take $\varepsilon > 0$ such that

$$\sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j^0\right|\right) - \varepsilon > k\,.$$

Since $f_1, f_2, \dots, f_m$ are continuous functions we can choose $\delta > 0$ such that for each $\xi_1, \xi_2, \dots, \xi_m$ satisfying $|\xi_j - \xi_j^0| < \delta$ for $j = 1, 2, \dots, m$ it holds

$$\sum_{n=1}^{m}\left| f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j\right|\right) - f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j^0\right|\right)\right| < \varepsilon\,.$$

Let us take $x = \{\xi_j\}_{j=1}^{\infty}$ with $d(x, x_0) < \frac{\delta}{1+\delta}\cdot 2^{-m}$. Then for each $j = 1, 2, \dots, m$ we obtain

$$\frac{|\xi_j - \xi_j^0|}{1 + |\xi_j - \xi_j^0|} < \frac{\delta}{1+\delta}2^{-m+j} \le \frac{\delta}{1+\delta}$$

and hence $|\xi_j - \xi_j^0| < \delta$ for $j = 1, 2, \dots, m$. So

$$\sum_{n=1}^{m}\left| f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j\right|\right) - f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j^0\right|\right)\right| < \varepsilon$$

and then

$$\sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j\right|\right) \ge \sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j^0\right|\right)$$
$$- \sum_{n=1}^{m}\left| f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j^0\right|\right) - f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j\right|\right)\right| \ge \sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j^0\right|\right) - \varepsilon > k\,.$$

Thus we have shown that $\left\{x \in s \colon d(x, x_0) < \frac{\delta}{1+\delta}2^{-m}\right\} \subset C_{mk}$; so C_{mk} is an open set. This fact and (1) imply

$$X_\alpha\{f_n\} \text{ is an } F_\sigma\text{-set in } s\,. \tag{2}$$

Now, let $x = \{\xi_j\}_{j=1}^{\infty} \in X_\alpha\{f_n\}$, $y = \{\eta_j\}_{j=1}^{\infty} \in C$ and $\varepsilon > 0$. We fix n_0 such that $\sum_{n=n_0+1}^{\infty} 2^{-n} < \varepsilon$ and we assume

$$\tau_j = \xi_j \qquad \text{for} \quad j = 1, 2, \dots, n_0\,,$$
$$\tau_{n_0+1} = \sum_{j=1}^{n_0+1}\eta_j - \sum_{j=1}^{n_0}\xi_j\,,$$
$$\tau_{n_0+i} = \eta_{n_0+i} \qquad \text{for} \quad i = 2, 3, \dots\,,$$

and $z = \{\tau_j\}_{j=1}^{\infty}$. Since

$$\lim_{m\to\infty} \sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\eta_j\right|\right) = \infty$$

and

$$\sum_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\tau_j\right|\right) = \sum_{n=1}^{n_0} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\xi_j\right|\right) + \sum_{n=n_0+1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum_{j=1}^{n}\eta_j\right|\right) \quad \text{for } m > n_0\,,$$

we obtain $\lim\limits_{m\to\infty} \sum\limits_{n=1}^{m} f_n\left(\left|\frac{\alpha}{n}\sum\limits_{j=1}^{n}\tau_j\right|\right) = \infty$, which means $z \in C$. Furthermore $d(x,z) \le \sum\limits_{n=n_0+1}^{\infty} 2^{-n} < \varepsilon$; so we have shown that

$$\text{the set } C \text{ is dense in } s\,. \tag{3}$$

Because C is a G_δ-set in s, it follows from (3) and (1) that

$$X_\alpha\{f_n\} \text{ is of the first category in } s\,. \tag{4}$$

Finally we shall show that $X_\alpha\{f_n\}$ is dense in s. Let $x = \{\xi_j\}_{j=1}^{\infty} \in s$ and $\varepsilon > 0$. We fix $y = \{\eta_j\}_{j=1}^{\infty} \in X_\alpha\{f_n\}$, n_0 such that $\sum\limits_{n=n_0+1}^{\infty} 2^{-n} < \varepsilon$. Now we assume

$$\begin{aligned}
\tau_j &= \xi_j \qquad \text{for} \quad j = 1,2,\dots,n_0\,,\\
\tau_{n_0+1} &= \sum_{j=1}^{n_0+1}\eta_j - \sum_{j=1}^{n_0}\xi_j\,,\\
\tau_{n_0+i} &= \eta_{n_0+i} \qquad \text{for} \quad i = 2,3,\dots\,,
\end{aligned}$$

and $z = \{\tau_j\}_{j=1}^{\infty}$. Then $z \in X_\alpha\{f_n\}$ and $d(x,z) < \varepsilon$; thus $X_\alpha\{f_n\}$ is dense in s and the proof is completed. □

Corollary 2. *$X\{f_n\}$ is a dense F_σ-set of the first category in s.*

P r o o f. Since $X\{f_n\} = \bigcup\limits_{m=1}^{\infty} X_{\frac{1}{m}}\{f_n\}$, the conclusion follows from Theorem 3. □

If $f_n(t) = t^p$ for each $n = 1,2,\dots$ and some $p \ge 1$, then Theorem 2 and Corollary 2 imply

Corollary 3. (a) *X_p is separable;*
(b) *X_p is a dense F_σ-set of the first category in s.*

Let $\{f_n\}_{n=1}^\infty$ be a sequence of non-degenerate Orlicz functions satisfying the uniform Δ_2-condition and $f_n'(o) = 0$ for $n = 1, 2, \dots$, and let $\{f_n^*\}_{n=1}^\infty$ be the sequence of complementary functions. For $y = \{\eta_j\}_{j=1}^\infty \in s$ we denote

$$|||y|||^* = \sup \sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_n|,$$

where the supremum is taken over all $x = \{\xi_j\}_{j=1}^\infty \in X\{f_n\}$ with $\varrho(x) \le 1$, and

$$\widetilde{X}\{f_n^*\} = \left\{ y = \{\eta_j\}_{j=1}^\infty \in s : \ |||y|||^* < \infty \right\}.$$

Theorem 4. *The $|||\cdot|||^*$ is a norm on $\widetilde{X}\{f_n^*\}$ and $\left(\widetilde{X}\{f_n^*\},\ |||\cdot|||^*\right)$ is a Banach space.*

P r o o f. It is easy to verity that $\widetilde{X}\{f_n^*\}$ is a linear space and $|||\cdot|||^*$ is a norm on $\widetilde{X}\{f_n^*\}$. Furthermore, $X\{f_n^*\} \subset \tilde{X}\{f_n^*\}$, so $\widetilde{X}\{f_n^*\} \neq \emptyset$. Let $\{y_n\}_{n=1}^\infty$ be a Cauchy sequence in $\tilde{X}\{f_n^*\}$, $y_n = \{\eta_{jn}\}_{j=1}^\infty$. For $\varepsilon > 0$ we take k_0 such that $|||y_m - y_k|||^* < \varepsilon$ for $k, m \ge k_0$. Then

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_{nk} - \eta_{nm}| < \varepsilon \tag{1}$$

for each $k, m \ge k_0$ and $x = \{\xi_j\}_{j=1}^\infty \in X\{f_n\}$ with $\varrho(x) \le 1$. Putting in (1) $x = u_n$ for $n = 1, 2, \dots$ we obtain

$$|\eta_{jk} - \eta_{jm}| < \varepsilon \qquad \text{for} \quad k, m \ge k_0, \quad j = 1, 2, \dots$$

Hence $\{\eta_{jn}\}_{n=1}^\infty$ is a Cauchy sequence for $j = 1, 2, \dots$. We denote $\eta_j = \lim\limits_{n\to\infty} \eta_{jn}$ and $y = \{\eta_j\}_{j=1}^\infty$. Since

$$\sum_{n=1}^{p} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_{nk} - \eta_{nm}| < \varepsilon \qquad \text{for} \quad k, m \ge k_0, \quad p = 1, 2, \dots$$

and $x = \{\xi_j\}_{j=1}^\infty \in X\{f_n\}$ with $\varrho(x) \le 1$; by letting $m \to \infty$ and $p \to \infty$ we obtain

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_{nk} - \eta_n| \le \varepsilon \qquad \text{for} \quad x \in X\{f_n\}, \quad \varrho(x) \le 1.$$

This implies $|||y_k - y|||^* \le \varepsilon$ for $k \ge k_0$ and $y \in \widetilde{X}\{f_n^*\}$, which finishes the proof. ∎

PROPOSITION 2. *For each* $x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$ *and* $y\{\eta_j\}_{j=1}^{\infty} \in \widetilde{X}\{f_n^*\}$ *it holds*

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_n| \le \|x\| \, |||y|||^*.$$

THEOREM 5. *For each* $y = \{\eta_j\}_{j=1}^{\infty} \in \widetilde{X}\{f_n^*\}$ *the formula*

$$\varphi_y(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) \eta_n \qquad \text{for} \quad x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\} \tag{1}$$

defines a continuous functional on $X\{f_n\}$.

Conversely, for each continuous functional φ *on* $X\{f_n\}$ *there exists exactly one element* $y = \{\eta_j\}_{j=1}^{\infty} \in \tilde{X}\{f_n^*\}$ *such that*

$$\varphi(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) \eta_n \qquad \text{for} \quad x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$$

and

$$\|\varphi\| = |||y|||^* .$$

P r o o f. Let $y = \{\eta_j\}_{j=1}^{\infty} \in \widetilde{X}\{f_n^*\}$; then according to Proposition 2, for each $x = \{\xi_j\}_{j=1}^{\infty} \in X\{f_n\}$ we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) \eta_n < \infty .$$

Thus φ_y given by (1) is a well defined linear functional on $X\{f_n\}$. Moreover, by Proposition 2 it holds $|\varphi_y(x)| \le |||y|||^* \|x\|$ for each $x \in X\{f_n\}$. Hence φ_y is continuous and

$$\|\varphi_y\| \le |||y|||^* \tag{2}$$

Conversely, let φ be a continuous functional on $X\{f_n\}$ and let $\eta_j = \varphi(u_j)$ for $j = 1, 2, \dots$, $y = \{\eta_j\}_{j=1}^{\infty}$. We put $\alpha_n = \operatorname{sign}\left(\sum_{j=1}^{n} \xi_j \right) \eta_n$; then for each $m = 1, 2, \dots$ we have

$$\begin{aligned} \sum_{n=1}^{m} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_n| &= \sum_{n=1}^{m} \alpha_n \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) \eta_n = \varphi\left(\sum_{n=1}^{m} \alpha_n \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) u_n \right) \\ &\le \|\varphi\| \left\| \sum_{n=1}^{m} \alpha_n \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) u_n \right\| = \|\varphi\| \left\| \sum_{n=1}^{m} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j \right) u_n \right\| . \end{aligned}$$

Hence letting $m \to \infty$ and applying Theorem 2(b) we obtain

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} \xi_j \right| |\eta_n| \le \|\varphi\| \, \|x\| .$$

This implies $|||y|||^* \le \|\varphi\|$ and $y \in \widetilde{X}\{f_n^*\}$. Since $x = \lim_{m\to\infty} \sum_{n=1}^{m} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j\right) u_n$ and φ is continuous, we have

$$\varphi(x) = \lim_{m\to\infty} \sum_{n=1}^{m} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j\right) \varphi(u_n) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{j=1}^{n} \xi_j\right) \eta_n$$

which finishes the proof. □

REFERENCES

[1] LINDENSTRAUSS, J.—TZAFRIRI, L.: *Classical Banach Spaces. Sequence spaces*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[2] NG PENG-NUNG—LEE PENG-YEE: *Cesàro sequence spaces of non-absolute type*, Ann. Soc. Math. Polon. Ser. I: Comment Math. Prace Mat. **20** (1978), 429–433.

[3] WOO, J. Y. T.: *On modular sequence spaces*, Studia Math. **48** (1973), 271–289.

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