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A TYPICAL PROPERTY OF SOME SYMMETRIC SPACES

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ABSTRACT. In the paper it is shown that the typical symmetric (symmetrically continuous) function has its set of discontinuity points c-dense. This answers some problems raised by B. S. Thomson.

1. Introduction and preliminaries

A function $f\colon \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}$ — the real line) is said to be *symmetric* (*symmetrically continuous*) providing $\lim\limits_{h\to 0}\bigl(f(x+h)+f(x-h)-2f(x)\bigr) = 0$ ($\lim\limits_{h\to 0}\bigl(f(x+h)-f(x-h)\bigr) = 0$) holds for each $x \in \mathbb{R}$. B. S. Thomson raised in his monograph [Th. p. 422] the following problems:

18. In the space of bounded, symmetric functions with supremum norm, does the typical function have a c-dense set of discontinuities?
19. In the space of bounded, symmetrically continuous functions with supremum norm, does the typical function have a c-dense set of discontinuities?

An affirmative answer to the above questions was given by H. Shi in [Sh]. The aim of the present paper is to give an affirmative answer to these questions in a stronger form. We shall do it in a more general setting. In his paper [T2] T. C. Tran has introduced classes of weighted symmetric functions in the following way:

Definition 1. A set of real numbers

$$W_n = \{w_{-n}, \dots, w_{-1}, w_0, w_1, \dots, w_n\}$$

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such that $\sum_{k=-n}^{n} w_k = 0$ and $|w_n| + |w_{-n}| > 0$ is said to be a *weight system of order* n $(n = 1, 2, \dots)$.

A weight system W_n of order n is said to be *even* if $w_{-k} = w_k$, $k = 0, 1, \dots, n$, $\sum_{k=1}^{n} w_k \neq 0$ and $w_0 = -2 \sum_{k=1}^{n} w_k \neq 0$.

A weight system W_n of order n is said to be *odd* if $w_{-k} = -w_k$, $k = 0, 1, \dots, n$, $\sum_{k=1}^{n} w_k \neq 0$ and $w_0 = 0$.

Definition 2. Let $f \colon \mathbb{R} \to \mathbb{R}$. The expression

$$f(x; W_n, h) = \sum_{k=-n}^{n} w_k f(x + kh/2)$$

is said to be the *symmetric difference* with respect to a weight system W_n of order n for a function f.

The function f will be called *symmetric* with respect to a weight system W_n if $\lim_{h \to 0} f(x; W_n, h) = 0$ holds for each $x \in \mathbb{R}$.

Let $S(W_n)$ stand for the class of all functions $f \colon \mathbb{R} \to \mathbb{R}$ symmetric with respect to a weight system W_n.

Proposition 1. ([T2]) *If W_1 is even (odd), then $S(W_1)$ is the class of all symmetric (symmetrically continuous) functions.*

Proposition 2. ([T2]) *If f is symmetric (symmetrically continuous), then f is symmetric with respect to any even (odd) weight system of any order.*

Theorem T. ([T2]) *Given a weight system W_n of order n, the functions symmetric with respect to W_n form a linear space $S(W_n)$. The uniform limit of a sequence in $S(W_n)$ belongs to $S(W_n)$. Moreover, if $bS(W_n)$ is the subspace of $S(W_n)$ consisting of bounded functions, then $bS(W_n)$ is a Banach space with the norm $\|f\| = \sup_{x \in \mathbb{R}} \{|f(x)|\}$.*

Further we shall deal with functions $f \colon X \to \mathbb{R}$ where X is a metric space. Let $C(f)$ $\big(D(f)\big)$ denote the set of all continuity (discontinuity) points of f. The notion of porosity we shall use in the usual sense (see, e.g., [Z1], [Va], or [Z2]): Let (F, d) be a metric space. For $f \in F$ we denote by $S(f, \varepsilon)$ the open sphere about f of radius ε. Let $f \in F$, $F_0 \subset F$, and fix $\varepsilon > 0$. Denote by $\gamma(f, \varepsilon, F_0)$ the supremum of all real numbers $\delta > 0$ for which there exists a $g \in X$ such that $S(g, \delta) \subset S(f, \varepsilon)$ and $S(g, \delta) \cap F_0 = \emptyset$. The porosity of F_0 at f is then given by

$$p(F_0, f) = \limsup_{\varepsilon \to 0+} 2\gamma(f, \varepsilon, F_0)/\varepsilon .$$

The set F_0 is porous at f if $p(F_0, f) > 0$. A set is called porous if it is porous at each of its points. In a general metric space the class of porous sets is a proper subset of the nowhere dense sets. If the "limsup" in the definition of porous is replaced by "liminf" we have the definition of a set being very porous at a point and a set being very porous. Let $p_v(F_0, f)$ stand for very porosity of F_0 at f. A countable union of porous sets or very porous sets is a σ-porous set or a σ-very porous set, respectively.

2. The main result

Theorem. *Let (X, ϱ) be a complete separable metric space with a base $\{B_k\}_{k=1}^{\infty}$, let F be a Banach space of bounded functions $f\colon X \to \mathbb{R}$ endowed with the supremum norm $\|f\| = \sup\limits_{x\in X}\bigl\{|f(x)|\bigr\}$. Suppose that for each $k = 1, 2, \dots$ there exists a function $g_k \in F$ such that $D(g_k) \cap B_k$ is uncountable. Then the set*

$$H = \{f \in F\colon D(f) \text{ is a } c\text{-dense set}\}$$

is a residual G_σ-set in the metric space (F, d) $(d(f,g) = \|f - g\|)$ and $F - H$ is a σ-very porous set.

Proof. Let F_k $(k = 1, 2, \dots)$ be the set of all $f \in F$ for which $D(f) \cap B_k$ is at most a countable set. If $f_m \in F_k$, $m = 1, 2, \dots$, and $d(f_m, f) \to 0$, then $\bigcap\limits_{m=1}^{\infty} C(f_m) \subset C(f)$. Obviously $D(f) \subset \bigcup\limits_{m=1}^{\infty} D(f_m)$, $D(f) \cap B_k \subset \bigcup\limits_{m=1}^{\infty} D(f_m) \cap B_k$ and $f \in F_k$. Hence F_k is closed in F. We show that F_k is very porous. Without loss of generality we can suppose $\|g_k\| = 1$ (otherwise we shall deal with $g_k/\|g_k\|$). Let $o(g, x)$ be the oscillation of g at x. There is $\eta > 0$ such that $o(g_k, x) \ge \eta$ holds for uncountable many $x \in D(g_k) \cap B_k$. Obviously $\eta \le 2$. Let $\varepsilon > 0$, $f \in F_k$ and let $S(f, \varepsilon)$ be the open sphere in F. Put $g_k^\varepsilon = \frac{\varepsilon}{2} g_k$. Obviously $f + g_k^\varepsilon \in S(f, \varepsilon)$. Since there is an uncountable set $D \subset D(f + g_k) \cap B_k$ such that

$$o(f + g_k^\varepsilon, x) \ge \frac{\eta\varepsilon}{2} \tag{1}$$

holds for each $x \in D$, we have $f + g_k^\varepsilon \notin F_k$.

Choose ω, $0 < \omega < \frac{\eta}{2}$ arbitrarily, and $\delta > 0$ such that $\frac{2\delta}{\varepsilon} = \omega < \frac{\eta}{2}$. Since $\eta \le 2$, we have $\delta < \frac{\varepsilon}{2}$. If $\varphi \in S(f + g_k, \delta)$, then $\|\varphi - f\| \le \bigl\|\varphi - (f + g_k^\varepsilon)\bigr\| + \bigl\|(f + g_k^\varepsilon) - f\bigr\| < \delta + \frac{\varepsilon}{2} < \varepsilon$, i.e., $S(f + g_k^\varepsilon, \delta) \subset S(f, \varepsilon)$.

We prove $S(f + g_k^\varepsilon, \delta) \cap F_k = \emptyset$. Let $\varphi \in S(f + g_k, \delta)$ and $x \in D$. We shall show $o(\varphi, x) > 0$. For the function

$$\Psi = \varphi - (f + g_k^\varepsilon) \tag{2}$$

we have $\|\Psi\| = \|\varphi - (f + g_k^\varepsilon)\| < \delta$ and

$$\bigl|\Psi(x') - \Psi(x'')\bigr| < 2\delta \tag{3}$$

holds for any $x', x'' \in X$. Consequently $o(\Psi, x) < \frac{\eta\varepsilon}{2}$. Choose $\alpha > 0$ such that $2\alpha = \frac{\eta\varepsilon}{2} - 2\delta$. Since $\bigl(\varphi(x') - \varphi(x'')\bigr) - \bigl(\Psi(x') - \Psi(x'')\bigr) = (f + g_k^\varepsilon)(x') - (f + g_k^\varepsilon)(x'')$ holds for any $x', x'' \in X$ (see (2)), we have $\bigl|\varphi(x') - \varphi(x'')\bigr| \geq \bigl|(f + g_k^\varepsilon)(x') - (f + g_k^\varepsilon)(x'')\bigr| - \bigl|\Psi(x') - \Psi(x'')\bigr|$. In any neighbourhood of $x \in D$ there are y' and y'' such that $\bigl|(f + g_k^\varepsilon)(y') - (f + g_k^\varepsilon)(y'')\bigr| \geq \frac{\eta\varepsilon}{2} - \alpha$. Consequently $|\varphi(y') - \varphi(y'')| \geq \frac{\eta\varepsilon}{2} - \alpha - 2\delta = \alpha > 0$ (see (3)) and $o(\varphi, x) > 0$. Hence $x \in D(\varphi) \cap B_k$, i.e., $\varphi \notin F_k$.

Let $\gamma(f, \varepsilon, F_k)$ have the introduced meaning. Then $2\gamma(f, \varepsilon, F_k)/\varepsilon \geq 2\delta/\varepsilon = \omega$ and $p_v(F_k, f) = \liminf\limits_{\varepsilon \to 0+} 2\gamma(f, \varepsilon, F_k)/\varepsilon \geq \omega$ holds for each $\omega < \frac{\eta}{2}$. Hence $p_v(F_k, f) \geq \frac{\eta}{2}$, the set F_k is very porous. Consequently, every set F_k, $k = 1, 2, \dots$, is a closed very porous set and $F - \bigcup\limits_{k=1}^{\infty} F_k$ is a residual G_δ-set in F. If $f \in F - \bigcup\limits_{k=1}^{\infty} F_k$ then for each open set $G \subset X$ the F_σ-set $D(f) \cap G$ is uncountable. According to Theorem of Alexandrov and Hausdorff [Ku, p. 459] every uncountable Borel set in a complete separable metric space has cardinality c, hence we have $H = F - \bigcup\limits_{k=1}^{\infty} F_k$, where $\bigcup\limits_{k=1}^{\infty} F_k$ is a σ-very porous set. □

Corollary. *Let W_n be a weight system of order n and let $bS(W_n)$ be the space of all bounded functions symmetric with respect to W_n endowed with the supremum norm. Then*

$$\{f \in bS(W_n) \colon D(f) \text{ is a } c\text{-dense set}\}$$

is a residual G_δ-set and its complement is a σ-very porous set in $bS(W_n)$.

Proof. Suppose W_n to be an even weight system of order n. The statement of Corollary is a consequence of our main result. According to Theorem T, $bS(W_n)$ is a Banach space. It is sufficient to put $X = \mathbb{R}$, $B_k = (a_k, b_k)$, $a_k < b_k$ are all pairs of rationals, and to show that for each $k = 1, 2, \dots$ there is a $g_k \in bS(W_1)$ such that $D(g_k) \cap (a_k, b_k)$ is uncountable. Then, according to Proposition 1 and Proposition 2, $g_k \in bS(W_n)$. It is known that there exists a bounded measurable symmetric function $g \colon \mathbb{R} \to \mathbb{R}$ such that $D(g)$ is uncountable ([T1], [Th, p. 56]). Choose δ, $0 < \delta < (b_k - a_k)/2$. Obviously there exists an integer z such that $\bigl(z\delta, (z+1)\delta\bigr) \cap D(g)$ is an uncountable set. If p is choosen in such a manner that $a_k - z\delta \leq p \leq b_k - (z+1)\delta$ then the function $g_k(x) = g(x - p)$ has the required properties. □

An analogous argumentation can be used for proving an odd version of Corollary. The existence of a bounded measurable symmetrically continuous function f such that $D(f)$ is uncountable is a known fact ([Pr], [T1], [Th, p. 52]).

Remark. A construction of a bounded symmetric function which has c points of discontinuity in every open interval is given in [T1, p. 507].

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