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ON SOME PROPERTIES CONNECTED WITH CONTINUITY AND MEASURABILITY

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ABSTRACT. In this article we introduce and investigate some set-theory generalizations of the continuity and the cliquishness. Our investigations concern also the relations between these generalized continuities and the measurability of such maps with respect to the σ-field containing all sets from considered families (being generalized topologies) and all sets from a σ-ideal.

Let X be a nonempty set and let (Y,ρ) be a metric space with the metric ρ. Denote by $\mathscr{W}$ a family of nonempty subsets of X. Throughout the paper we suppose that

$$\forall_{x\in X}\ \exists_{W\in\mathscr{W}}\ x\in W\,.$$

If $f\colon X\to Y$ is a function then we will write:

(a) f has the property $\mathscr{W}(x)$ at a point $x\in X$ if for each positive real ε there is a set $U(x)\in\mathscr{W}$ with $x\in U(x)$ and $\operatorname{osc} f<\varepsilon$ on $U(x)$;

(b) f has the property $\mathscr{U}(x)$ at a point $x\in X$ if for each positive real ε and for each set $W\in\mathscr{W}$ containing x there is a set $U\in\mathscr{W}$ such that $\emptyset\neq U\subset W$ and $\operatorname{osc} f<\varepsilon$ on U.

Let

$$\mathscr{W}(f)=\{x\in X\colon f \text{ has the property } \mathscr{W}(x) \text{ at a point } x\}$$

and

$$\mathscr{U}(f)=\{x\in X\colon f \text{ has the property } \mathscr{U}(x) \text{ at a point } x\}.$$

Observe that if the family $\mathscr{W}$ satisfies the condition

(1) if $U,V\in\mathscr{W}$ and $x\in U\cap V$ then there is a set $W\in\mathscr{W}$ with $W\subset U\cap V$,

then for every function $f\colon X\to Y$ the inclusion

$$\mathscr{W}(f)\subset\mathscr{U}(f)$$

is true.

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Remark 1. For any function $f\colon X \to Y$ the set

$$\mathscr{W}(f) = \bigcap_{n=1}^{\infty} A_n ,$$

where all sets A_n, $n = 1, 2, \dots$, are the unions of some subfamilies of subsets of the family $\mathscr{W}$.

P r o o f. Denote by $K(y, r)$ the ball with the center $y \in Y$ and with the radius $r > 0$. For $n = 1, 2, \dots$ and $y \in Y$ let

$$A(y, n) = f^{-1}\Big(K\big(y, \tfrac{1}{n}\big)\Big) ,$$

let $W\big(A(y, n)\big)$ be the union of all sets from the family $\mathscr{W}$ which are contained in $A(y, n)$, and let

$$A_n = \bigcup_{y \in Y} W\big(A(y, n)\big) .$$

Observe that

$$\mathscr{W}(f) = \bigcap_{n=1}^{\infty} A_n .$$

This completes the proof. □

Remark 2. If the space Y is separable and if all sets $W\big(A(y, n)\big)$, $y \in Y$, $n = 1, 2, \dots$, belong to $\mathscr{W}$ then a simple modification of the proof of the last Remark shows that the sets A_n, $n = 1, 2, \dots$, can be countable unions of sets from $\mathscr{W}$.

Now we assume that $\mathscr{I}$ is an σ-ideal of subsets of X such that for every family $\{W_s\}_{s \in S}$ of subsets of $\mathscr{W}$ the following implication

$$X \setminus \bigcup_{s \in S} W_s \in 2^X \setminus \mathscr{I} \implies \exists_{W \in \mathscr{W}}\ W \subset X \setminus \bigcup_{s \in S} W_s$$

is true.

EXAMPLES. As some examples of pairs $\big(\mathscr{I}, \{W_s\}_{s \in S}\big)$ in the space $X = \mathbb{R}^n$, where $\mathbb{R}$ denotes the space of all reals, we can take:

(1) $\mathscr{I}$ — the σ-ideal of all sets of Lebesgue measure zero;
$\{W_s\}_{s \in S}$ — the family of all nonempty sets of the density topologies $\mathscr{T}_d$ (with respect to the ordinary differentiation basis or the strong differentiation basis).

(2) $\mathscr{I}$ as above, and the family $\{W'_s\}_{s \in S}$ of all nonempty measurable (in the Lebesgue sense) sets in $\mathbb{R}^n$ which have positive upper density (ordinary or strong) at every point. Such family $\{W'_s\}_{s \in S}$ is not any topology. For the proof that arbitrary union $\bigcup_{s \in S' \subset S} W'_s$ is measurable, see [2] for the case $X = \mathbb{R}$. In the case $X = \mathbb{R}^n$, where $n > 1$, the proof is analogous.

THEOREM 1. *For any function $f\colon X \to Y$ with $X = \mathscr{U}(f)$ the set $X \setminus \mathscr{W}(f)$ belongs to $\mathscr{I}$.*

P r o o f. Fix a positive integer n and denote by A_n the union of all sets $W \in \mathscr{W}$ such that $\operatorname{osc} f < \frac{1}{n}$ on W. Since $X = \mathscr{U}(f)$, we obtain by the hypothesis on $\mathscr{I}$ that
$$X \setminus A_n \in \mathscr{I}.$$
Observe that
$$A = \bigcap_{n=1}^{\infty} A_n \subset \mathscr{W}(f)$$
and
$$X \setminus A = X \setminus \bigcap_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (X \setminus A_n) \in \mathscr{I}.$$
Since
$$X \setminus \mathscr{W}(f) \subset X \setminus A \in \mathscr{I},$$
the proof is completed. □

Now we suppose that $\mathscr{M}$ is the smallest σ-field of subsets of X containing all sets from $\mathscr{W} \cup \mathscr{I}$.

Remind that a function $f\colon X \to Y$ is called $\mathscr{M}$-measurable if $f^{-1}(V) \in \mathscr{M}$ for each open set $V \subset Y$.

THEOREM 2. *Suppose that the space Y is separable, $\mathscr{W} \cap \mathscr{I} = \emptyset$ and for each set $A \in \mathscr{M} \setminus \mathscr{I}$ there is a set $B \in \mathscr{W}$ with $B \subset A$. Then for every $\mathscr{M}$-measurable function $f\colon X \to Y$ the equality $\mathscr{U}(f) = X$ is true.*

P r o o f. Since $\mathscr{W} \cap \mathscr{I} = \emptyset$, the space X is not in $\mathscr{I}$. Fix a point $x \in X$, a set $W \in \mathscr{W}$ containing x and a positive real ε. Since the space Y is separable, there is a countable family of open balls K_n, $n = 1, 2, \dots$, with the diameters smaller than ε such that $Y = \bigcup_{n \in \mathbb{N}} K_n$. From the $\mathscr{M}$-measurability of the function f follows that all sets
$$E_n = f^{-1}(K_n) \in \mathscr{M}$$
and $\operatorname{osc} f < \varepsilon$ on E_n for $n = 1, 2, \dots$. Since W is not in $\mathscr{I}$ and
$$W = \bigcup_{n=1}^{\infty} (W \cap E_n),$$
there is a positive integer k such that $W \cap E_k$ is not in $\mathscr{I}$. But $W \cap E_k \in \mathscr{M}$, so there is a set $G \subset W \cap E_k$ belonging to $\mathscr{W}$. Obviously, $\operatorname{osc} f < \varepsilon$ on the set G and the proof is completed. □

Theorem 3. *Suppose that at least one of the following conditions is satisfied:*

(a) *for every subfamily* $\{W_s\}_{s\in S} \subset \mathscr{W}$ *the union*

$$\bigcup_{s\in S} W_s \in \mathscr{W},$$

(b) *if the sets of a family*

$$\{W_s\}_{s\in S}$$

are such that

$$W_{s_1} \cap W_{s_2} = \emptyset \qquad \textit{for} \quad s_1 \neq s_2\,, \quad s_1, s_2 \in S\,,$$

then the family $\{W_s\}_{s\in S}$ *is countable.*

Then every function $f\colon X \to Y$ *with* $X = \mathscr{U}(f)$ *is* $\mathscr{M}$*-measurable.*

Proof. We will prove that for each real $\varepsilon > 0$ there is a $\mathscr{M}$-measurable function $g\colon X \to Y$ such that $\rho\big(f(x), g(x)\big) \le \varepsilon$ for each point $x \in X$. Fix a positive real ε. By transfinite induction we find a transfinite sequence of sets $W_\alpha \in \mathscr{W}$, $\alpha < \alpha_o$, such that for $\alpha < \beta < \alpha_o$ we have:

- $W_\alpha \cap W_\beta = \emptyset$;
- $\operatorname{osc} f < \varepsilon$ on the set W_α;
- the complementary

$$X \setminus \bigcup_{\alpha<\alpha_o} W_\alpha \in \mathscr{I}\,.$$

For $\alpha < \alpha_o$ fix a point $x_\alpha \in W_\alpha$ and put

$$g(x) = \begin{cases} f(x_\alpha)\,, & \text{for } x \in W_\alpha\,,\ \alpha < \alpha_o\,, \\ f(x)\,, & \text{for } x \in X \setminus \bigcup\limits_{\alpha<\alpha_o} W_\alpha\,. \end{cases}$$

Then the function g is $\mathscr{M}$-measurable and $\rho\big(f(x), g(x)\big) < \varepsilon$ for each point $x \in X$. So, the proof is completed. □

Observe that the both pairs $\big(\mathscr{I}, \{W_s\}_{s\in S}\big)$ and $\big(\mathscr{I}, \{W_s'\}_{s\in S}\big)$ from the Examples, satisfy the hypotheses (a) and (b) of Theorem 3.

Now we suppose that

- the metric space Y is separable;
- for every subfamily $\{W_s\}_{s\in S} \subset \mathscr{W}$ the union

$$\bigcup_{s\in S} W_s \in \mathscr{M}$$

and we introduce yet a family $\mathscr{A} \subset \mathscr{M} \setminus \mathscr{I}$ of nonempty sets such that:

- $\mathscr{W} \subset \mathscr{A}$;
- if $\{A_s\}_{s\in S} \subset \mathscr{A}$ then
$$\bigcup_{s\in S} A_s \in \mathscr{A};$$
- if $A \in \mathscr{A}$, $W \in \mathscr{W}$ and $A \cap W \neq \emptyset$ then there is a set $E \in \mathscr{A}$ contained in $A \cap W$;
- if a set $E \in \mathscr{M} \setminus \mathscr{I}$ then there is a set $D \in \mathscr{A}$ contained in E.

We will say that a function $f\colon X \to Y$ has the property $\mathscr{G}$ ($f \in \mathscr{G}$) if for every point $x \in X$, every set $A \in \mathscr{A}$ containing x and every positive real ε there is a set $W \in \mathscr{W}$ such that $W \cap A \neq \emptyset$ and $\rho\big(f(t), f(x)\big) < \varepsilon$ for each point $t \in A \cap W$.

Observe that if for arbitrary subsets $W, V \in \mathscr{W}$ with $W \cap V \neq \emptyset$ there exist a set $B \subset W \cap V$ such that $B \in \mathscr{W}$, then for every function $f \in \mathscr{G}$ we have $\mathscr{U}(f) = X$.

Remark 3. By a similar proof as that of Theorem 2 we obtain that every function $f \in \mathscr{G}$ is $\mathscr{M}$-measurable.

THEOREM 4. *If $f \in \mathscr{G}$ then $X \setminus \mathscr{W}(f) \in \mathscr{I}$.*

P r o o f . Let $\{K_n\}_n$ be a basis of open balls in Y. If a point $x \in X \setminus \mathscr{W}(f)$ then there is a positive integer $n(x)$ such that $f(x) \in K_{n(x)}$ and for each set $W \in \mathscr{W}$ containing x there is a point $t \in W$ with $f(t) \in Y \setminus K_{n(x)}$. For $n = 1, 2, \dots$ let
$$A_n = \{x \in X \setminus \mathscr{W}(f) \colon n(x) = n\}$$
and let
$$B_n = f^{-1}(K_n)\,.$$
Observe that
$$A_n = B_n \setminus \bigcup\{W \in \mathscr{W} \colon W \subset B_n\}$$
for $n = 1, 2, \dots$ Since the function f is $\mathscr{M}$-measurable, each set A_n, $n = 1, 2, \dots$, belongs to $\mathscr{M}$. If
$$\mathscr{W}(f) = \bigcup_{n=1}^{\infty} A_n \in 2^X \setminus \mathscr{I}$$
then there is a positive integer k such that
$$A_k \in 2^X \setminus \mathscr{I}\,.$$

Let $A \in \mathscr{A}$ be a set contained in A_k and let $x \in A$ be a fixed point. Then $f(x) \in K_k$ and for every set $W \in \mathscr{W}$ for which $A_k \cap W \neq \emptyset$ there is a point t_W with $f(t_W) \in Y \setminus K_k$. Let $Q \subset K_k$ be a closed ball such that $f(x) \in \operatorname{int}(Q)$ (int denotes the interior operation in Y). Since $f \in \mathscr{G}$, there is a set $W_o \in \mathscr{W}$ such that

- $W_o \cap A \neq \emptyset$;
- $f(W_o \cap A) \subset \operatorname{int}(Q)$.

Let $B \subset A \cap W_o$ be a set of the family $\mathscr{A}$ and let $u \in B$ be a fixed point. Let $\mathscr{V}$ be the family of all sets $W \in \mathscr{W}$ for which $W \cap A \neq \emptyset$. If $W \in \mathscr{V}$ then there is a point $t_W \in W$ with $f(t_W) \in Y \setminus K_k$. Consequently, for each pair (W, t_W), $W \in \mathscr{V}$, there is a set $U_W \in \mathscr{W}$ such that

$$W \cap U_W \neq \emptyset \qquad \text{and} \qquad f(W \cap U_W) \subset Y \setminus Q .$$

For $W \in \mathscr{V}$ let $D_W \in \mathscr{A}$ be a set contained in $W \cap U_W$ and let

$$E = B \cup \bigcup_{W \in \mathscr{V}} D_W .$$

Then for each $W \in \mathscr{V}$ we have $f(D_W) \subset Y \setminus Q$, $E \in \mathscr{A}$, $f(u) \in \operatorname{int}(Q)$ and $u \in E$. From the relation $f \in \mathscr{G}$ follows the existence of a set $W_1 \in \mathscr{W}$ such that

- $W_1 \cap E \neq \emptyset$;
- $f(E \cap W_1) \subset \operatorname{int}(Q) \subset K_k$.

From the construction of E we obtain $W_1 \cap E \subset B \cap W_1$.

On the other hand,

$$D_{W_1} \subset E \cap W_1 \qquad \text{and} \qquad D_{W_1} \subset W_1 \setminus B .$$

This contradiction completes the proof. □

In the case $Y = \mathbb{R}$ (with the Euclidean metric) we can prove a generalization of Theorem 3 in which the hypothesis $f \in \mathscr{G}$ is replaced by $f \in \mathscr{H}$.

We will say that a function $f \colon X \to \mathbb{R}$ has the property $\mathscr{H}$ ($f \in \mathscr{H}$) if for every point $x \in X$, every set $A \in \mathscr{A}$ containing x and every positive real ε there are sets $W, V \in \mathscr{W}$ such that $A \cap W \neq \emptyset$, $A \cap V \neq \emptyset$, $f(t) < f(x) + \varepsilon$ for all points $t \in A \cap W$ and $f(u) > f(x) - \varepsilon$ for all points $u \in A \cap V$.

Theorem 5. *If a function $f \colon X \to \mathbb{R}$ has the property $\mathscr{H}$ then f is $\mathscr{M}$-measurable.*

P r o o f. Fix a real a. Let

$$A = \bigcup \{W \in \mathscr{A} \colon f(W) \subset (-\infty, a)\}$$

and

$$B = \bigcup \{W \in \mathscr{A} \colon f(W) \subset [a, \infty)\}\,.$$

If the set $f^{-1}\big((-\infty, a)\big) \notin \mathscr{M}$ then the set

$$E = X \setminus A \setminus B \in \mathscr{M} \setminus \mathscr{I}$$

and consequently there is a set $H \in \mathscr{A}$ contained in E. Evidently,

$$H \cap f^{-1}\big((-\infty, a)\big) \neq \emptyset$$

and

$$H \cap f^{-1}\big([a, \infty)\big) \neq \emptyset\,.$$

So, there is a set $W \in \mathscr{W}$ such that $W \cap H \neq \emptyset$ and

$$f(W \cap H) \subset (-\infty, a)\,.$$

There is a set $K \in \mathscr{A}$ contained in $W \cap H$. Then $K \subset A$. But

$$K \subset H \subset E = X \setminus A \setminus B\,,$$

a contradiction. So the set

$$\{x \in X \colon f(x) < a\} \in \mathscr{M}$$

and the proof is completed. $\square$

Theorem 6. *Suppose that for all sets $A \in \mathscr{A}$ and $U, W \in \mathscr{W}$ with $U \cap W \cap A \neq \emptyset$ there is a set $V \in \mathscr{W}$ such that $V \subset U \cap W$ and $V \cap A \neq \emptyset$.*

If a function $f \colon X \to \mathbb{R}$ has the property $\mathscr{H}$ then

$$X \setminus \mathscr{W}(f) \in \mathscr{I}\,.$$

P r o o f. If

$$X \setminus \mathscr{W}(f) \in \mathscr{M} \setminus \mathscr{I}$$

then similarly as in the proof of Theorem 4 we can show the existence of a closed interval $Q = [a, b]$ $(a < b)$, an open interval $(c, d) \supset [a, b]$ and a set $A \in \mathscr{M} \setminus \mathscr{I}$ such that $f(A) \subset (a, b)$ and for every set $W \in \mathscr{W}$ with $A \cap W \neq \emptyset$ there is a point $t_W \in W$ for which $f(t_W) \in \mathbb{R} \setminus (c, d)$.

The following cases are to consider.

1. If there is a set $W \in \mathscr{W}$ such that

$$A \cap W \neq \emptyset \qquad \text{and} \qquad f(W) \subset (-\infty, b)$$

then we find a set $D_W \in \mathscr{A}$ such that

$$D_W \subset W \qquad \text{and} \qquad f(D_W) \subset (-\infty, a)\,.$$

Let $B \in \mathscr{A}$ be a set such that $B \subset A \cap W$.

If there is a set $U \in \mathscr{W}$ such that

$$U \cap B \neq \emptyset \qquad \text{and} \qquad f(U) \subset (a, \infty)$$

then, by our assumptions, there is a set $Z \in \mathscr{W}$ such that

$$Z \subset W \cap U \qquad \text{and} \qquad Z \cap B \neq \emptyset .$$

Consequently,

$$f(Z) \subset (a, b) \qquad \text{and} \qquad Z \cap A \neq \emptyset ,$$

a contradiction.

If for every set $U \in \mathscr{W}$ with $U \cap B \neq \emptyset$ there is a point $t_U \in U$ such that $f(t_U) \leq a$ then we find a set $G_U \in \mathscr{A}$ such that

$$G_U \subset U \qquad \text{and} \qquad f(G_U) \subset (-\infty, c) .$$

Thus the set

$$F = B \cup \bigcup \{G_U \colon U \in \mathscr{W} \text{ and } U \cap B \neq \emptyset\} \in \mathscr{A} .$$

Since $f \in \mathscr{H}$ and $f(B) \subset (a, b)$, there is a set $U \in \mathscr{W}$ such that

$$F \cap U \neq \emptyset \qquad \text{and} \qquad f(U \cap F) \subset (c, \infty) ,$$

a contradiction with $G_U \subset U$.

2. If there is a set $W \in \mathscr{W}$ such that

$$A \cap W \neq \emptyset \qquad \text{and} \qquad f(W) \subset (c, \infty)$$

then we prove similarly as in the case **1.**

3. If for every set $W \in \mathscr{W}$ there are points $t_W, s_W \in W$ with $f(t_W) < a$ and $f(s_W) > b$ then we find sets $S_W, V_W \in \mathscr{A}$ such that

$$S_W, V_W \subset W \qquad \text{and} \qquad f(S_W) \subset (-\infty, a) \qquad \text{and} \qquad f(V_W) \subset (b, \infty) .$$

Then the set

$$F = B \cup \bigcup \big(\{S_W \cup V_W \colon W \in \mathscr{W} \text{ and } W \cap B \neq \emptyset\} \in \mathscr{A}$$

and there is a set $U \in \mathscr{W}$ such that

$$U \cap F \neq \emptyset \qquad \text{and} \qquad f(U \cap F) \subset (a, \infty) ,$$

a contradiction with $S_U \subset U \cap F$. This completes the proof. $\square$

As some particular examples of families $(\mathscr{W}, \mathscr{A}, \mathscr{I})$ considered in the last part of the paper we can take the Euclidean topology in $\mathbb{R}^n$ (as $\mathscr{W}$), the density topology (as $\mathscr{A}$) and the σ-ideal of sets of Lebesgue measure zero (as $\mathscr{I}$). In these cases some analogous results were obtained in articles [3, 4, 5].

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