Tatra Mt. Math. Publ. **19** (2000), 75–89

Tatra Mountains Mathematical Publications

ON THE CHOQUET INTEGRAL FOR RIESZ SPACE VALUED MEASURE

Miloslav Duchoň — Ján Haluška — Beloslav Riečan

ABSTRACT. The Choquet integral is defined for a real function with respect to a fuzzy measure taking values in a complete Riesz space. As applications there are presented: constructions of belief and plausibility measures, the formulation of an extension principle, and the Möbius transform for vector values measures.

1. Introduction

It is very well known that the *Choquet integral* [5] of a non-negative measurable function can be taken with respect to a very general set function. Indeed, recall that a *fuzzy measure* is a mapping $\mu\colon \mathscr{A} \to \mathbb{R}$ defined on a ring $\mathscr{A}$ of subsets of X and such that

(i) $\mu(\emptyset) = 0$,

(ii) $A, B \in \mathscr{A}$, $A \subset B$ implies $\mu(A) \le \mu(B)$.

We say that $f\colon X \to [0,\infty]$ is measurable with respect to $\mathscr{A}$ if

$$f^{-1}\big([t,\infty)\big) = \big\{x \in X\colon f(x) \ge t\big\} \in \mathscr{A}$$

for any $t \in [0,\infty)$. If we define

$$g(t) = \mu\big(\{x \in X\colon f(x) \ge t\}\big), \qquad t \in [0,\infty),$$

then g is clearly decreasing and nonnegative function. Hence there exists the *Riemann improper integral*

$$\int\limits_0^\infty g(t)\,dt\,,$$

2000 Mathematics Subject Classification: 46G10.
Keywords: Choquet integral, Riesz space, vector measure, belief measure.
This paper has been supported by grants VEGA 5124/98 and 6087/98.

and then the Choquet integral $(C)\int\limits_X f\,d\mu$ can be defined by the formula

$$(C)\int\limits_X f\,d\mu = \int\limits_0^\infty g(t)\,dt\,.$$

If $\mathscr{A}$ is a σ-algebra, it is not difficult to see that

$$(C)\int\limits_X f\,d\mu = \int\limits_0^\infty h(t)\,dt\,,$$

where

$$h(t) = \mu\big(\{x \in X\colon f(x) > t\}\big)\,, \qquad t \in [0,\infty)\,.$$

Denote by C the set of all points $t \in [0,\infty)$ such that g is continuous at t. If $t \in C$, then $g(t) = h(t)$. Evidently, $h(t) \le g(t)$. On the other hand, for every $\varepsilon > 0$, there exists $s > t$ such that

$$g(t) - \varepsilon < g(s) = \mu\big(\{x \in X\colon f(x) \ge s\}\big) \le \mu\big(\{x \in X\colon f(x) > t\}\big) = h(t)\,.$$

Since $g(t) - \varepsilon < h(t)$ for any $\varepsilon > 0$, we obtain $g(t) \le h(t)$. Now,

$$\begin{aligned}
\int\limits_0^\infty g(t)\,dt &= \int\limits_C g(t)\,dt + \int\limits_{[0,\infty)\setminus C} g(t)\,dt \\
&= \int\limits_C h(t)\,dt + \sum_{x\notin C}\int\limits_{\{x\}} g(t)\,dt \\
&= \int\limits_C h(t)\,dt \\
&= \int\limits_0^\infty h(t)\,dt\,.
\end{aligned}$$

The main aim of this paper is the following: to define the Choquet integral in the case that $\mu\colon \mathscr{A} \to Y$ has values in a Riesz space.

In the following a *Riesz space* is a real vector space Y together with a *partial ordering* $\le$ satisfying the following conditions:

(i) $(Y,\le)$ is a lattice;

(ii) if $x, y, z, \in Y$ and $x \le y$, then $x + z \le y + z$;

(iii) if $x, y \in Y$, $\lambda \in \mathbb{R}^+$ and $x \le y$, then $\lambda x \le \lambda y$.

A Riesz space Y is called to be *σ-complete* if every bounded sequence in Y has supremum. To define $(C)\int_X f(t)\,d\mu$ for $f\colon X \to [0,\infty)$, $\mu\colon \mathscr{A} \to Y$, we need first to reinvite the *Riemann–Stieltjes integral* $\int_a^b g\,dh$ for vector valued function g. It will be realized in Section 2. In Section 3, we define the Choquet integral $\int f\,d\mu$ and Sections 4–6 contain some applications.

2. Riemann–Stieltjes integral

For a Riesz space Y, assume that a real function $h\colon [a,b] \to \mathbb{R}$ and a vector function $g\colon [a,b] \to Y$ are given. If $\mathcal{D}\colon a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition and $t_i \in [x_{i-1}, x_i]$ $(i = 1, 2, \ldots, n)$, then we define the integral sums

$$S_g(h,\mathcal{D}) = \sum_{i=1}^{n} h(t_i)\big(g(x_i) - g(x_{i-1})\big)$$

and

$$S_h(g,\mathcal{D}) = \sum_{i=1}^{n} g(t_i)\big(h(x_i) - h(x_{i-1})\big)\,.$$

As usual, $\|\mathcal{D}\| = \max_i (x_i - x_{i-1})$.

Definition. A scalar function h is *strongly integrable with respect to a vector function* g if there exist $c, u \in Y$, $u > 0$, such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|c - S_g(h,\mathcal{D})| < \varepsilon u\,,$$

whenever $\|\mathcal{D}\| < \delta$.

Lemma 1. *If Y is σ-complete, then the element c is defined uniquely.*

Proof. If c_1, c_2 satisfy the assumptions of the preceding Definition, then

$$|c_1 - c_2| \le |c_1 - S_g(h,\mathcal{D})| + |S_g(h,\mathcal{D})| < 2\varepsilon u\,.$$

Put $\varepsilon = \frac{1}{2n}$. Then

$$|c_1 - c_2| < \frac{1}{n} u\,,$$

hence $|c_1 - c_2| = 0$. □

As usual, we denote the uniquely determined element $c \in Y$ as follows:

$$\int_a^b h\,dg = \int_a^b h(t)\,dg(t)\,.$$

Similarly, the integral

$$\int_a^b g\,dh$$

can be defined (for other definitions of Riemann–Stieltjes integral, see [3], [8], [15]).

THEOREM 1. ([6], [8], [15]) *Let $h\colon [a,b] \to \mathbb{R}$, $g\colon [a,b] \to Y$, where Y is a σ-complete Riesz space. Then $\int_a^b h\,dg$ exists if and only if $\int_a^b g\,dh$ exists. Moreover,*

$$\int_a^b h\,dg = h(b)g(b) - h(a)g(a) - \int_a^b g\,dh\,.$$

P r o o f. If we put $t_0 = a$, $t_{n+1} = b$, then

$$\sum_{i=1}^{n} h(t_i)\big(g(x_i) - g(x_{i-1})\big) = h(b)g(b) - h(a)g(a) - \sum_{j=1}^{n} g(x_j)\big(h(t_j) - h(t_{j-1})\big)\,.$$

By this equality, the formula of integration by parts follows. □

THEOREM 2. *Let Y be a complete Riesz space, $g\colon [a,b] \to Y$ be an increasing mapping. Then the integral $\int_a^b h\,dg$ exists if and only if the following Cauchy–Bolzano condition holds: There exists $u \in Y$, $u > 0$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\left|S_g(h,\mathcal{D}_1) - S_g(h,\mathcal{D}_2)\right| < \varepsilon u\,,$$

whenever $\|\mathcal{D}_1\| < \delta$ and $\|\mathcal{D}_2\| < \delta$.

P r o o f. The necessity is clear, we prove the sufficiency.

Let the Cauchy–Bolzano condition hold. For the $\delta > 0$, put

$$a_\delta = \bigwedge_{\|\mathcal{D}\|<\delta} S_g(h,\mathcal{D})\,, \qquad b_\delta = \bigvee_{\|\mathcal{D}\|<\delta} S_g(h,\mathcal{D})\,.$$

Evidently, $0 \le a_\delta \le b_\delta$. Moreover, if $\delta(\varepsilon)$ corresponds to the given ε, then

$$b_{\delta(\varepsilon)} \le a_{\delta(\varepsilon)} + \varepsilon u\,. \tag{1}$$

If $\delta_1 < \delta_2$, then $a_{\delta_2} \le a_{\delta_1} \le b_{\delta_1} \le b_{\delta_2}$. It follows that

$$\bigvee_{\delta>0} a_\delta \le b_{\delta_2}\,.$$

Since the last inequality holds for every $\delta_2 > 0$, we obtain

$$\bigvee_{\delta>0} a_\delta \le \bigwedge_{\delta>0} b_\delta .$$

Let c be any element of Y satisfying the relation

$$\bigvee_{\delta>0} a_\delta \le c \le \bigwedge_{\delta>0} b_\delta . \tag{2}$$

Let $\delta = \delta(\varepsilon) > 0$ correspond to the given $\varepsilon > 0$. Let $\|\mathcal{D}\| < \delta$. Then by (1) and (2),

$$S_g(h,\mathcal{D}) - c \le b_\delta - c \le b_\delta - a_\delta \le \varepsilon u ,$$
$$c - S_g(h,\mathcal{D}) \le c - a_\delta \le b_\delta - a_\delta \le \varepsilon u ,$$

hence $|c - S_g(h,\mathcal{D})| \le \varepsilon u$. □

THEOREM 3. *Let Y be a complete Riesz space, $h\colon [a,b] \to \mathbb{R}$ be a continuous function, $g\colon [a,b] \to Y$ be an increasing (or decreasing) function. Then $\int\limits_a^b h\,dg$ exists.*

P r o o f. We use Theorem 1. Assume that g is increasing. Then we can put $u = g(b) - g(a) > 0$ (the case $g(a) = g(b)$ is trivial). Let ε be an arbitrary positive number. Since h is uniformly continuous, there exists $\delta > 0$ such that $|h(t) - h(s)| < \varepsilon$ whenever $|t - s| < \delta$. Let $\mathcal{D}, \mathcal{D}'$ be partitions of $[a,b]$ with $\|\mathcal{D}\| < \delta$, $\|\mathscr{D}'\| < \delta$. Assume first that $\mathcal{D}'$ is a refinement of $\mathcal{D}$ and denote $x_{i-1} = x_i^0 < x_i^1 < \dots < x_i^{k_i} = x_i$ new dividing points, hence

$$g(x_i) - g(x_{i-1}) = \sum_{j=1}^{k_i} \bigl(g(x_i^j) - g(x_i^{j-1})\bigr) . \tag{3}$$

Choose $t_i \in [x_{i-1}, x_i]$, $t_i^j \in \bigl[x_{i-1}^{j-1}, x_i^j\bigr]$ such that

$$S_g(h,\mathcal{D}) = \sum_{i=1}^{n} h(t_i)\bigl(g(x_i) - g(x_{i-1})\bigr) , \tag{4}$$

$$S_g(h,\mathcal{D}') = \sum_{i=1}^{n} \sum_{j=1}^{k_i} h(t_i^j)\bigl(g(x_i^j) - g(x_{i-1}^j)\bigr) . \tag{5}$$

Then by (3), (4), and (5), we obtain

$$\bigl|S_g(h,\mathcal{D}) - S_g(h,\mathcal{D}')\bigr| \le \sum_{i=1}^{n} \sum_{j=1}^{k_i} \bigl|h(t_i) - h(t_i^j)\bigr| \bigl(g(x_i^j) - g(x_i^j)\bigr) .$$

Since $t_i^j \in \left[x_{i-1}^{j-1}, x_i^j\right] \subset [x_{i-1}, x_i]$, we have $\left|t_i - t_i^j\right| < \delta$, hence $\left|h(t_i) - h(t_i^j)\right| < \varepsilon$. Therefore

$$\begin{aligned}\left|S_g(h,\mathcal{D}) - S_g(h,\mathcal{D}')\right| &\le \varepsilon \sum_{i=1}^{n} \sum_{j=1}^{k_i} \left(g(x_i^j) - g(x_i^{j-1})\right) \\ &= \varepsilon \sum_{i=1}^{n} \left(g(x_i) - g(x_{i-1})\right) \\ &= \varepsilon\big(g(b) - g(a)\big) = \varepsilon u \,.\end{aligned}$$

If $\mathcal{D}_1, \mathcal{D}_2$ are arbitrary partitions with $\|\mathcal{D}_1\| < \delta$, $\|\mathcal{D}_2\| < \delta$, then we can construct the joint refinement $\mathcal{D}'$ of $\mathcal{D}_1$ and $\mathcal{D}_2$ and by preceding

$$\begin{aligned}&\left|S_g(h,\mathcal{D}_1) - S_g(h,\mathcal{D}_2)\right| \\ &\qquad \le \left|S_g(h,\mathcal{D}_1) - S_g(h,\mathcal{D}')\right| + \left|S_g(h,\mathcal{D}') - S_g(h,\mathcal{D}_2)\right| \\ &\qquad \le \varepsilon u + \varepsilon u = 2\varepsilon u \,.\end{aligned}$$

The theorem is proved. □

Corollary 1. *If Y is a complete Riesz space, $h\colon [a,b] \to \mathbb{R}$ a continuous function and $g\colon [a,b] \to Y$ a monotone function, then $\int_a^b g\,dh$ exists.*

P r o o f. It follows by Theorem 1 and Theorem 3. □

3. Choquet integral

Assume again that Y is a complete Riesz space, $\mu\colon \mathscr{A} \to Y$ an Y-valued fuzzy measure defined on an algebra $\mathscr{A}$ of subsets of X and $f\colon X \to \mathbb{R}$ a non-negative real function measurable with respect to $\mathscr{A}$ in the sense that $\{x \in X\colon f(x) > t\} \in \mathscr{A}$ for every $t \in \mathbb{R}$.

Let $A \in \mathscr{A}$. Define $g_A\colon [0,\infty) \to Y$ by the formula

$$g_A(t) = \mu\big(\{x \in A\colon f(x) > t\}\big) \,.$$

The mapping g_A is non-increasing. If we define $h\colon [0,\infty) \to \mathbb{R}$ by the formula $h(t) = t$, then h is continuous on any interval $[0,c]$, hence by Corollary 1 there exists

$$\int_0^c g_A\,dh = \int_0^c g_A(t)\,dt = \int_0^c \mu\big(A \cap \{x\colon f(x) > t\}\big)\,dt \,.$$

Put $\varphi(c) = \int_0^c g_A(t)\,dt \in Y$. Clearly $\varphi \geq 0$ and φ is increasing. Now there are two possibilities. If the function φ is bounded, then there exists

$$\int_0^\infty g_A(t)\,dt = \bigvee_{c>0} \varphi(c) = \bigvee_{c>0} \int_0^c g_A(t)\,dt\,.$$

In the opposite case we define

$$\int_0^\infty g_A(t)\,dt = \infty\,.$$

Definition. If Y is a complete Riesz space, f is a non-negative measurable function, $f\colon (X, \mathscr{A}) \to \mathbb{R}$, $\mathscr{A}$ is an algebra, $\mu\colon \mathscr{A} \to Y$ is a fuzzy measure and $A \in \mathscr{A}$, then we define the *Choquet integral*

$$(C)\int_A f\,d\mu$$

by the formula

$$(C)\int_A f\,d\mu = \int_0^\infty g_A(t)\,dt = \bigvee_{c>0} \int_0^c \mu\big(A \cap \{x\colon f(x) > t\}\big)\,dt\,.$$

Theorem 4. *If $\mu\colon \mathscr{A} \to Y$ is a σ-additive measure defined on a σ-algebra, then*

$$(C)\int_X f\,d\mu = \int_X f\,d\mu\,,$$

where $\int_X f\,d\mu$ is the Lebesgue integral ([1], [10], [15], [18], [19]).

Proof. First, let f be simple, $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$, and the sets A_i be disjoint. Then

$$g(t) = \begin{cases} \mu(A_1 \cup \cdots \cup A_n)\,, & t < \alpha_1\,, \\ \mu(A_{i+1} \cup \cdots \cup A_n)\,, & \alpha_i \leq t < \alpha_{i+1}\,,\ i = 1, \dots, n-1\,, \\ 0\,, & t \geq \alpha_n\,. \end{cases}$$

Therefore,

$$\begin{aligned} \int_0^\infty g(t)\,dt &= \alpha_1 \mu(A_1 \cup \cdots \cup A_n) + \sum_{i=1}^{n-1} (\alpha_{i+1} - \alpha_i)\,\mu(A_{i+1} \cup \cdots \cup A_n) \\ &= \sum_{j=1}^n \alpha_j \mu(A_j) = \int_X f\,d\mu\,. \end{aligned}$$

If f is an arbitrary non-negative measurable function, then there exists a sequence $(f_n)_n$ of simple measurable functions such that $f_n \nearrow f$. Then ([15])

$$\begin{aligned}
\int_X f\,d\mu &= \bigvee_{n=1}^{\infty} \int_X f_n\,d\mu \\
&= \bigvee_{n=1}^{\infty} (C)\int_X f_n\,d\mu \\
&= \bigvee_{n=1}^{\infty} \int_0^{\infty} \mu\big(\{x \in X \colon f_n(x) > t\}\big)\,dt \\
&= \int_0^{\infty} \bigvee_{n=1}^{\infty} \mu\big(\{x \in X \colon f_n(x) > t\}\big)\,dt\,.
\end{aligned}$$

But

$$\begin{aligned}
\bigvee_{n=1}^{\infty} \mu\big(\{x \in X \colon f_n(x) > t\}\big) &= \mu\left(\bigcup_{n=1}^{\infty} \{x \in X \colon f_n(x) > t\}\right) \\
&= \mu\big(\{x \in X \colon f(x) > t\}\big)\,.
\end{aligned}$$

Therefore,

$$\int_X f\,d\mu = \int_0^{\infty} \mu\big(\{x \in X \colon f(x) > t\}\big)\,dt = (C)\int_X f\,d\mu\,.$$

The theorem is proved. □

4. Belief and plausibility measures

A fuzzy measure $\mu \colon \mathscr{A} \to Y$ is called *lover continuous* if

$$A_n \nearrow A\,,\ (A_n) \subset A\,,\ A \in \mathscr{A} \implies \mu(A_n) \nearrow \mu(A)\,.$$

A fuzzy measure $\mu \colon A \to Y$ is called a belief measure if for any $n \in \mathbb{N}$ and any $A_1, \dots, A_n \in \mathscr{A}$ there holds

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_I (-1)^{|I|+1} \mu\left(\bigcap_{i \in I} A_i\right)\,,$$

where the summation is taken over all non-empty subsets I of $\{1, 2, \dots, n\}$ and $|I|$ denotes the cardinal number of I.

A fuzzy measure $\mu\colon \mathscr{A} \to Y$ is called a plausibility measure if for every $n \in \mathbb{N}$ and every $A_1, \dots, A_n \in \mathscr{A}$ there holds

$$\mu\left(\bigcap_{i=1}^{n} A_i\right) \le \sum_{I} (-1)^{|I|+1} \mu\left(\bigcup_{i\in I} A_i\right).$$

THEOREM 5. ([11]) *Let Y be a complete Riesz space, f be a non-negative real function measurable with respect to a measurable space $(X, \mathscr{A})$, $X \in \mathscr{A}$, $\mu\colon \mathscr{A} \to Y$ be a fuzzy measure. Define $\nu\colon \mathscr{A} \to Y$ by the formula*

$$\nu(A) = (C)\int_A f \, d\mu .$$

Then ν is a fuzzy measure, ν is lower continuous, whenever μ is lower continuous. If μ is lower continuous and belief (plausibility) measure, then ν is belief (plausibility) measure, too.

Proof. Evidently, $\nu(\emptyset) = \int_0^\infty \mu(\emptyset)\, dt = 0$. If $A \subset B$, then

$$\begin{aligned}
\nu(A) &= \int_0^\infty \mu\big(A \cap \{x \in X\colon f(x) > t\}\big)\, dt \\
&\le \int_0^\infty \mu\big(B \cap \{x \in X\colon f(x) > t\}\big)\, dt = \nu(B).
\end{aligned}$$

Let now μ be lower continuous and let $A_n \nearrow A$. Then

$$\begin{aligned}
\nu(A) &= \int_0^\infty \mu\big(A \cap \{x \in X\colon f(x) > t\}\big)\, dt \\
&= \int_0^\infty \bigvee_{n=1}^{\infty} \mu\big(A_n \cap \{x \in X\colon f(x) > t\}\big)\, dt \\
&= \bigvee_{n=1}^{\infty} \int_0^\infty \mu\big(A_n \cap \{x \in X\colon f(x) > t\}\big)\, dt \\
&= \bigvee_{n=1}^{\infty} \nu(A_n).
\end{aligned}$$

Now, let μ be lower continuous and plausibility measure. Take simple functions f_n such that $f_n \nearrow f$. Fix n and put $f_n = \sum_{i=1}^{m} \alpha_i \chi_{A_i}$, $0 = \alpha_0 < \alpha_1 < \dots < \alpha_m$,

A_i disjoint. Denote

$$\begin{aligned}
\nu_n(A) &= (C)\int\limits_A f_n\,d\mu \\
&= \int\limits_0^\infty \mu\big(A_n \cap \{x\in X\colon f(x)>t\}\big)\,dt \\
&= \sum_{i=1}^m (\alpha_i - \alpha_{i-1})\,\mu\big(A\cap (A_i \cup A_{i+1}\cdots\cup A_m)\big) \\
&= \sum_{i=1}^m (\alpha_i - \alpha_{i-1})\,\mu(A\cap B_i)\,,
\end{aligned}$$

where $B_i = A_i \cup A_{i+1}\cdots\cup A_m$. Then

$$\begin{aligned}
\nu_n\left(\bigcap_{j=1}^k C_j\right) &= \sum_{i=1}^m (\alpha_i-\alpha_{i-1})\,\mu\Big(\bigcap_{j=1}^k C_j \cap B_i\Big) \\
&\le \sum_{i=1}^m (\alpha_i-\alpha_{i-1}) \sum_I (-1)^{|I|+1}\mu\left(\Big(\bigcup_{j\in I} C_j\Big)\cap B_i\right) \\
&= \sum_I (-1)^{|I|+1} \sum_{i=1}^m (\alpha_i-\alpha_{i-1})\,\mu\left(\Big(\bigcup_{j\in I} C_j\Big)\cap B_i\right) \\
&= \sum_I (-1)^{|I|+1}\nu_n\Big(\bigcup_{j\in I} C_j\Big)\,,
\end{aligned}$$

hence

$$\nu_n\left(\bigcap_{j=1}^k C_j\right) \le \sum_I (-1)^{|I|+1}\nu_n\left(\bigcup_{j\in I} C_j\right) \tag{6}$$

Now

$$\begin{aligned}
\nu(A) &= \int\limits_0^\infty \mu\big(A\cap\{x\in X\colon f(x)>t\}\big)\,dt \\
&= \int\limits_0^\infty \mu\Big(A\cap \bigcup_{n=1}^\infty \{x\in X\colon f_n(x)>t\}\Big)\,dt \\
&= \int\limits_0^\infty \bigvee_{n=1}^\infty \mu\big(A\cap\{x\in X\colon f_n(x)>t\}\big)\,dt
\end{aligned}$$

$$= \bigvee_{n=1}^{\infty} \int_0^{\infty} \mu\big(A \cap \{x \in X : f_n(x) > t\}\big)\, dt$$

$$= \bigvee_{n=1}^{\infty} (C) \int_A f_n \, d\mu$$

$$= \bigvee_{n=1}^{\infty} \nu_n(A) .$$

We have proved that $\nu_n(A) \nearrow \nu(A)$ for every $A \in \mathscr{A}$, hence $\nu_n(A)$ o-converges to $\nu(A)$. By this fact and by (6) we obtain

$$\nu\left(\bigcap_{j=1}^{k} C_j\right) \le \sum_I (-1)^{|I|+1} \nu\left(\bigcup_{j \in I} C_j\right) .$$

The assertion concerning belief measures can be proved similarly. □

5. An extension principle

In [17] M. J. Wierman considered a fuzzy measure μ defined on a family of all subsets of a set and by the help of the Choquet integral he extended μ to the family of all fuzzy subsets of X. In this section we shall show that the Wierman principle can be applied also for Riesz space valued fuzzy measures.

Let $\mathscr{A}$ be an algebra of subsets of X. Let $f \colon X \to \mathbb{R}$ be a non-negative function. As before, we will say that f is measurable with respect to $\mathscr{A}$ if $\{x \in X : f(x) > t\} \in \mathscr{A}$ for every $t \in \mathbb{R}$.

Let $\mu \colon \mathscr{A} \to Y$ be a fuzzy measure, Y being a complete Riesz space. Let $M(\mathscr{A})$ be the set of all $\mathscr{A}$-measurable fuzzy subsets of X. Then we define $\overline{\mu} \colon M(\mathscr{A}) \to Y$ by the formula

$$\overline{\mu}(f) = (C) \int_X f \, d\mu .$$

THEOREM 6. *The mapping $\overline{\mu}$ is an extension of μ. If μ is lower continuous, then $\overline{\mu}$ is lower continuous, too.*

Proof. Evidently,

$$(C) \int_X f \, d\mu = \int_0^{\infty} \mu\big(\{x \in X : f(x) > t\}\big)\, dt$$

$$= \int_0^1 f\, d\mu = \int_0^1 \mu(\{x \in X : f(x) > t\})\, dt\,.$$

If $f = \chi_A$, then

$$\{x \in X : f(x) > t\} = \begin{cases} A\,, & t < 1\,, \\ \emptyset\,, & t \geq 1\,. \end{cases}$$

Therefore

$$\overline{\mu}(\chi_A) = (C)\int_X \chi_A\, d\mu = \int_0^1 \mu(A)\, dt = \mu(A)\,.$$

Let μ be lower continuous and $f_n \in M(\mathscr{A})$, $f_n \nearrow f$. Then

$$\begin{aligned}
\overline{\mu}(f) &= \int_0^1 \mu(\{x \in X : f(x) > t\})\, dt \\
&= \int_0^1 \mu\left(\bigcup_{n=1}^{\infty} \{x \in X : f_n(x) > t\}\right) dt \\
&= \int_0^1 \bigvee_{n=1}^{\infty} \mu(\{x \in X : f_n(x) > t\})\, dt \\
&= \bigvee_{n=1}^{\infty} \int_0^1 \mu(\{x \in X : f_n(x) > t\})\, dt \\
&= \bigvee_{n=1}^{\infty} \overline{\mu}(f_n)\,.
\end{aligned}$$

The theorem is proved. □

The Wierman extension principle can be extended to the case of relations ρ in the Cartesian product $2^U \times 2^V$, i.e., for $\rho \subset 2^U \times 2^V$. Namely, if ρ is a relation between sets, then $\overline{\rho}$ will be a relation between fuzzy subsets of U or V, respectively. If $f : U \to [0,1]$ is a fuzzy subset of U, then denote as usual

$$f^t = \{x \in U : f(x) > t\}\,.$$

Let χ_ρ be the characteristic function of ρ, i.e.,

$$\chi_\rho(A, B) = \begin{cases} 1\,, & A\, \rho\, B\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then the membership function $\chi_{\overline{\rho}}$ of the fuzzy relation $\overline{\rho}$ is defined by the formula

$$\chi_{\overline{\rho}}(f,g) = \int_0^1 \chi_\rho(f^t, g^t)\, dt\,.$$

6. Möbius transform

Assume now that X is a finite set and $\mu\colon 2^X \to Y$ is a fuzzy measure. Then the *Möbius transform* $M_\mu\colon 2^X \to Y$ is defined by the formula

$$M_\mu(A) = \sum_{B\subset A} (-1)^{|A\setminus B|}\mu(B)\,.$$

By the same method as in the real-valued case ([4], [9]) it can be proved that

$$\mu(A) = \sum_{B\subset A} M_\mu(B) \tag{7}$$

for every $A \subset X$.

Now we generalize a theorem by Mesiar ([12]).

Theorem 7. *Let X be a finite set, Y be a complete Riesz space, $f\colon X \to \mathbb{R}$ be a non-negative function, $f_A = \bigwedge_{x\in A} f(x)$ for every $A \subset X$. Then*

$$(C)\int_X f\, d\mu = \sum_{A\subset X} f_A M_\mu(A)\,.$$

Proof. Let $\{a_1, a_2, \dots, a_n\}$ be the range of f, $a_1 \le a_2 \le \dots \le a_n$. Therefore $a_i = f(x_{n_i})$ for some $n_i \in \mathbb{N}$. Then

$$(C)\int_X f\, d\mu = a_1\mu(X) + \sum_{i=2}^n (a_i - a_{i-1})\,\mu\big(X \setminus \{x_{n_1}, \dots x_{n_{i-1}}\}\big)\,.$$

Therefore, by (7),

$$\begin{aligned}(C)\int_X f\, d\mu &= a_1 \sum\{M_\mu(A)\colon A\subset X\}\\ &\quad + \sum_{i=2}^n (a_i - a_{i-1}) \sum\{M_\mu(A)\colon A \subset X\setminus\{x_{n_1},\dots x_{n_{i-1}}\}\}\end{aligned}$$

$$
\begin{aligned}
&= a_1 \sum \{M_\mu(A) \colon x_{n_i} \in A \subset X\} \\
&\quad + \sum_{i=2}^{n} a_i \sum \{M_\mu(A) \colon x_{n_i} \in A \subset X \setminus \{x_{n_1}, \dots x_{n_{i-1}}\}\} \\
&= \sum_{A \subset X} f_A M_\mu(A) .
\end{aligned}
$$

The theorem is proved. □

REFERENCES

[1] BOCCUTO, A.: *Abstract integration in Riesz spaces*, Tatra Mt. Math. Publ. **5** (1995), 107–124.

[2] BOCCUTO, A.—SAMBUCINI, A. R.: *On the De Giorgi–Letta integral with respect to means with values in Riesz spaces*, Real Anal. Exchange **21** (1996), 793–810.

[3] CANDELORO, D.: *Riemann–Stieltjes integration in Riesz spaces*, Rend. Mat. **16** (1996), 563–585.

[4] CHATAUNEUF, A.—JAFFRAY, J. Y.: *Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion*, Math. Social Sci. **17** (1989), 263–283.

[5] CHOQUET, G.: *Theory of capacities*, Ann. Inst. Fourier (Grenoble) **5** (1953), 131–295.

[6] DEBIEVE, C.—DUCHOŇ, M.: *Integration by parts in vector lattices*, Tatra Mt. Math. Publ. **6** (1995), 13–18.

[7] DENNBERG, D.: *Non-additive Measure and Integral*, Kluwer Academic, Dordrecht, 1994.

[8] GRABISCH, M.: *k-order additive fuzzy measures*, in: Proc. IPMU'96, Granada, 1996, pp. 1345–1350.

[9] DUCHOŇ, M.—RIEČAN, B.: *On the Kurzweil–Stieltjes integral in ordered spaces*, Tatra Mt. Math. Publ. **8** (1996), 133–141.

[10] HALUŠKA, J.: *On integration in complete vector lattices*, Tatra Mt. Math. Publ. **3** (1993), 201–212.

[11] HARDING, J.—MARINACCI, M.—NGUYEN, N. T.—WANG, T.: *Local Radon–Nykodým derivatives of set functions*, Internat. J. Uncertain., Fuzziness and Knowledge-Based Systems **5** (1997), 379–394.

[12] MESIAR, R.: *A note to the Choquet integral*, Tatra Mt. Math. Publ. **12** (1997), 241–245.

[13] MARUKAWA, Y.—MUROFUSHI, T.—SUGENO, M.: *Representation of comonotonically additive functional by Choquet integral*, Proc. IPMU'98, Paris, 1998, Vol 2., pp. 1569–1576.

[14] PAP, E.: *Null-additive Set Functions*, Kluwer Academic and Ister Science, Dordrecht and Bratislava, 1995.

[15] RIEČAN, B.—NEUBRUNN, T.: *Integral, Measure, and Ordering*, Kluwer Academic and Ister Science, Dordrecht and Bratislava, 1997.

[16] WANG, Z.—KLIR, G. J.: *Fuzzy Measure Theory*, Plenum Press, New York, 1992.

[17] WIERMAN, M. J.: *Extending set functions and relations*, Int. J. General Systems **26** (1997), 91–96.

[18] VRÁBELOVÁ, M.: *The fundamental theorem of calculus in ordered spaces*, Acta Math. Nitra **3** (1998), 31–37.

[19] VRÁBELOVÁ, M.: *Iterated limits theorem in lattice ordered groups*, Acta Math. Nitra **3** (1998), 15–19.

Received September 30, 1998

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA
E-mail: duchon@mat.savba.sk

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA
E-mail: jhaluska@saske.sk

Department of Mathematics
Faculty of Natural Sciences
Matej Bel University
Tajovského 40
SK–974 01 Banská Bystrica
SLOVAKIA
E-mail: riecan@mat.savba.sk