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REMARKS ON INFINITE SERIES IN LINEAR NORMED SPACES

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ABSTRACT. In this paper we introduce some partial results concerning the theory of infinite series in linear normed spaces. The fundamental facts from this theory can be found in papers [2], [4], [7] (p. 15–19), [8] (p. 18–27) and [13] (p. 85–90).

Introduction

The purpose of this paper is to generalize results of many authors concerning infinite series. These results were mostly proved for series in $\mathbb{R}$ or $\mathbb{R}^n$.

The papers is divided into three sections. In the first one we will deal with functions which preserve convergence of infinite series in linear normed spaces. In the second part we are studying the permutations of the set $\mathbb{N}$ of all positive integers which transform each convergent series into convergent series. These permutations of the set $\mathbb{N}$ are said to be convergence preserving permutations of $\mathbb{N}$. We will find a sufficient and necessary condition for a permutation of $\mathbb{N}$ to be convergence preserving. In the last section we extend a result well-known for subseries of divergent series with real terms to series with terms in a linear normed space.

In the whole paper we denote by E a linear normed space over the field $\mathbb{R}$ (real numbers). In some cases E can also be understood as a linear normed space over the field $\mathbb{C}$ of complex numbers. The norm of the space E will be denoted by $\|\cdot\|$.

1. Functions preserving convergence of series

There are several papers (e.g., [1], [3], [6], [9], [10], [14], [15], [17]) concerning the functions $f\colon \mathbb{R} \to \mathbb{R}$ that preserve convergence of series with real terms

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which means that for any convergent series $\sum_{n=1}^{\infty} x_n$ of real numbers the series $\sum_{n=1}^{\infty} f(x_n)$ is also convergent. Similar methods are discussed in [5] for series in $\mathbb{R}^n$. Functions preserving the absolute convergence of series with real terms and related questions are characterized in [3].

R. Rado proved in [7] that if a function $f\colon \mathbb{R} \to \mathbb{R}$ preserves the convergence of series with real terms then f has the form $f(x) = cx$ (c is a constant) in a certain neighborhood of the point 0. This results has been re-proved in papers [14] and [17]. Using the method of [14] we give here a theorem characterizing the functions $f\colon E \to E$ preserving convergence of series in linear normed spaces. A function f is said to preserve the convergence of series in E if for any convergent series $\sum_{n=1}^{\infty} x_n$ ($x_n \in E$) the series $\sum_{n=1}^{\infty} f(x_n)$ is also convergent.

THEOREM 1.1. *Let E be a linear normed space over the field $\mathbb{R}$. If a function $f\colon E \to E$ preserves convergence of series in E then there exists a neighborhood U of the point $0 \in E$ such that $f(0) = 0$ and f is continuous, additive and homogeneous in U.*

Proof. The space $E \times E$ equipped with the norm

$$\|(x,y)\| = \|x\| + \|y\| \qquad \text{for} \quad (x,y) \in E \times E$$

is a linear normed space by the usual definition of sum and multiplication (by a scalar) of pairs (x,y).

Let us define the function $g\colon E \times E \to E$ in the following way

$$g(x,y) = f(x,y) + f(-x) + f(-y) \qquad \text{for} \quad (x,y) \in E \times E\,. \tag{1}$$

We will show that $g(x,y) = 0$ in a neighborhood of the point $(0,0) \in E \times E$. We will prove this indirectly. Suppose that in every neighborhood of $(0,0)$ there is a point (x,y) such that $g(x,y) \neq 0$. Then there exist sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in E such that $x_n \to 0$, $y_n \to 0$ $(n \to \infty)$ and $g(x_n,y_n) \neq 0$ for all $n \in \mathbb{N}$. Let M_n be the number

$$M_n = \left[\|g(x_n,y_n)\|^{-1}\right] + 1\,, \qquad n = 1,2,3,\dots\,, \tag{2}$$

where $[t]$ denotes the "integer part" of the real number t.

Now let us construct the series

$$\begin{aligned}
&\underbrace{(x_1 + y_1) + (-x_1) + (-y_1) + \cdots + (x_1 + y_1) + (-x_1) + (-y_1)}_{M_1 \text{ summands } (x_1+y_1)+(-x_1)+(-y_1)} \\
+&\underbrace{(x_2 + y_2) + (-x_2) + (-y_2) + \cdots + (x_2 + y_2) + (-x_2) + (-y_2)}_{M_2 \text{ summands } (x_2+y_2)+(-x_2)+(-y_2)} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
+&\underbrace{(x_n + y_n) + (-x_n) + (-y_n) + \cdots + (x_n + y_n) + (-x_n) + (-y_n)}_{M_n \text{ summands } (x_n+y_n)+(-x_n)+(-y_n)} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \quad .
\end{aligned}$$

Denote this series by $\sum_{j=1}^{\infty} a_j$. Let S_k $(k = 1, 2, 3, \dots)$ be the kth partial sum of this series, i.e., $S_k = \sum_{j=1}^{k} a_j$. Then it is clear that

$$S_{3k} = 0\,, \quad S_{3k+1} = x_n + y_n\,, \quad S_{3k+2} = y_n \quad \text{for suitable } n \in \mathbb{N}\,. \tag{3}$$

Therefore this series is convergent and its sum is 0. Since the function f preserves convergence of series in E, the series $\sum_{j=1}^{\infty} f(a_j)$ converges in E which means that the series

$$\begin{aligned}
&\underbrace{f(x_1 + y_1) + f(-x_1) + f(-y_1) + \cdots + f(x_1 + y_1) + f(-x_1) + f(-y_1)}_{M_1 \text{ summands } f(x_1+y_1)+f(-x_1)+f(-y_1)} \\
+&\underbrace{f(x_2 + y_2) + f(-x_2) + f(-y_2) + \cdots + f(x_2 + y_2) + f(-x_2) + f(-y_2)}_{M_2 \text{ summands } f(x_2+y_2)+f(-x_2)+f(-y_2)} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
+&\underbrace{f(x_n + y_n) + f(-x_n) + f(-y_n) + \cdots + f(x_n + y_n) + f(-x_n) + f(-y_n)}_{M_n \text{ summands } f(x_n+y_n)+f(-x_n)+f(-y_n)} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned} \tag{4}$$

is convergent. The convergence of (4) implies the convergence of the series

$$\sum_{n=1}^{\infty} M_n g(x_n, y_n)\,, \qquad \text{see (1)}\,. \tag{4'}$$

But the definition of numbers M_n gives us that $\|M_n g(x_n, y_n)\| > 1$ and therefore (4') cannot be convergent, so we have a contradiction.

Hence there is a neighborhood U of 0 such that for every $x, y \in U$

$$f(x+y) + f(-x) + f(-y) = 0\,. \tag{5}$$

In what follows let U be a ball neighborhood of the point 0. If we put $x = y = 0$ in (5) we obtain

$$f(0) = 0\,. \tag{6}$$

By putting $y = 0$ in (5) we get

$$f(-x) = -f(x) \tag{7}$$

and therefore (5) can be re-written in the following form

$$f(x+y) = f(x) + f(y) \qquad \text{for} \quad x, y \in U\,. \tag{8}$$

Hence the function f is additive in U.

Now we can prove that the function f is continuous at the point 0. Let $x_n \to 0$. Construct the following series

$$x_1 + (-x_1) + x_2 + (-x_2) + \dots + x_n + (-x_n) + \dots$$

This series is convergent and its sum is 0. Then the series

$$f(x_1) + f(-x_1) + f(x_2) + f(-x_2) + \dots + f(x_n) + f(-x_n) + \dots$$

is convergent, as well. Because of (6) the partial sums of this series are

$$f(x_1), 0, f(x_2), 0, f(x_3), 0, \dots$$

which implies that $\lim\limits_{n\to\infty} f(x_n) = 0$, and therefore f is continuous at 0.

Let y be an arbitrary point of U. By (6) and (7) we have (for $x \in U$)

$$f(x-y) = f(x) - f(y)\,. \tag{9}$$

If $x \to y$ we get from (9) $\lim\limits_{x\to y} f(x) = f(y)$. Therefore the function f is continuous on U.

Finally, we prove that f is homogeneous. Let $x \in U$, $r \in \mathbb{Q}$, $|r| \le 1$ ($\mathbb{Q}$ means the set of all rationals). Because U is a ball with the centre in 0, $rx \in U$ and by using standard procedure it can be shown (see [7]) that

$$f(rx) = rf(x)\,. \tag{10}$$

Now, let $x \in U$, $\alpha \in \mathbb{R}$, $|\alpha| \le 1$. Then again $\alpha x \in U$. Choose arbitrary sequence of rationals $(r_n)_{n\in\mathbb{N}}$, $|r_n| \le 1$ and $r_n \to \alpha$. Then $r_n x \to \alpha x$ $(n \to \infty)$ and the continuity of f in U and (10) give

$$f(\alpha x) = \lim_{n\to\infty} f(r_n x) = \lim_{n\to\infty} r_n f(x) = \alpha f(x)\,.$$

So the function f is homogeneous and the proof of the Theorem 1.1. is finished.

□

Now we will show that the converse of the Theorem 1.1. is also true.

THEOREM 1.1'. *Let $f\colon E \to E$ be an additive and continuous function in a neighborhood of the point 0. Then the function f preserves convergence of infinite series in E.*

P r o o f . We can assume that the function f is additive and continuous in the ball $U = \{x \in E\colon \|x\| < \delta\}$ where $\delta > 0$. From additivity of f in U we have $f(0) = 0$. Moreover like in the proof of the previous theorem we get that f is homogeneous in U.

Let $\sum\limits_{k=1}^{\infty} x_k$ be a convergent series and $x_k \in E$ $(k = 1, 2, 3, \dots)$. We will show that the series $\sum\limits_{k=1}^{\infty} f(x_k)$ is also convergent.

According to the Cauchy condition for convergence of series there is a $k \in \mathbb{N}$ such that for every $p \in \mathbb{N}$

$$\|x_{k+1} + x_{k+2} + \dots + x_{k+p}\| < \frac{\delta}{2}. \tag{11}$$

From this we have

$$\Big\| \sum_{j=k+1}^{\infty} x_j \Big\| \le \frac{\delta}{2}. \tag{11'}$$

By (11) and the additivity of f in U we get

$$f\left(\sum_{j=k+1}^{k+p} x_j \right) = \sum_{j=k+1}^{k+p} f(x_j). \tag{12}$$

Consider now that the elements $y_p = \sum\limits_{j=k+1}^{k+p} x_j$ $(p = 1, 2, 3, \dots)$ and $y = \sum\limits_{j=k+1}^{\infty} x_j$ belong to the neighborhood U. The function f is continuous in U, i.e., $f(y_p) \to f(y)$ because $y_p \to y$ for $p \to \infty$. Hence the limit of the left side of the equality (12) exists and is equal to $f\left(\sum\limits_{j=k+1}^{\infty} x_j \right)$. It means that the limit of the right side of (12) exists as well and equals to $f\left(\sum\limits_{j=k+1}^{\infty} x_j \right)$. This guarantees the convergence of the series $\sum\limits_{j=k+1}^{\infty} f(x_j)$ and therefore the series $\sum\limits_{j=1}^{\infty} f(x_j)$ must also be convergent in E. □

Remark 1.1. It follows from the proof of previous theorem that a function $f\colon E \to E$ preserves convergence of series if and only if there exist a continuous linear[1] function $l\colon E \to E$ and a neighborhood U of the point $0 \in E$ such that $f|_U = l|_U$.

[1] Function l is called linear in a real normed space E if it is additive and homogeneous.

Remark 1.2. This gives us that function preserving convergence of series in E need not have the form of $f(x) = cx$ where $c \in \mathbb{R}$ as it is in the real case (see [10], [11], [17]).

EXAMPLE 1.1. *An example of such function can be the function $f\colon \mathcal{M} \to \mathcal{M}$ where $\mathcal{M}$ is the linear normed space of all bounded sequences of real numbers with the* sup *norm and f is defined in the following way: for $x = (x_k)_{k\in\mathbb{N}}$ we put*

$$f(x) = (x_1, 0, 0, \dots, 0, \dots) \in \mathcal{M}\,.$$

2. Permutations of N that preserve convergence of series in linear normed spaces

The notion of rearrangement of series in the linear normed space is defined in the same way as for series with real terms (cf. [7] p. 15, [8] p. 18, [13] p. 88).

DEFINITION 2.1. A permutation $p = (p_n)_{n\in\mathbb{N}}$ of the set $\mathbb{N}$ of positive integers is said to preserve convergence of series in a linear normed space $\big(E, \|\cdot\|\big)$ if for each convergent series $\sum\limits_{n=1}^{\infty} x_n$ in E the series $\sum\limits_{n=1}^{\infty} x_{p_n}$ is also convergent.

Remark 2.1. If we use the notation $\pi\colon \mathbb{N} \to \mathbb{N}$ for a permutation $p = (p_n)_{n\in\mathbb{N}}$ of $\mathbb{N}$ (i.e., we put $\pi(n) = p_n$, $n = 1, 2, \dots$) then

$$\sum_{n=1}^{\infty} x_{p_n} = \sum_{n=1}^{\infty} x_{\pi(n)}\,.$$

The set of all permutations preserving convergence of series in E has been characterized in [1]. This characterization can be extended to permutations of $\mathbb{N}$ preserving convergence of series in an arbitrary linear normed space.

First recall that every set $A \subset \mathbb{N}$ of the form $A = \{i+1, i+2, \dots, i+m\}$ for some $i, m \in \mathbb{N}$ is called a block of positive integers.

THEOREM 2.1. *A permutation $p = (p_n)_{n\in\mathbb{N}}$ of the set $\mathbb{N}$ preserves convergence of series in a linear normed space E if and only if it satisfies the following condition:*

(a) *There exists a positive integer M such that for every $n \in \mathbb{N}$ the set $\{p_1, p_2, \dots, p_n\}$ can be expressed as a union of not more than M blocks of positive integers.*

P r o o f. "$\Longleftarrow$" Let $p = (p_n)_{n\in\mathbb{N}}$ satisfy the condition (a). Let $v = \sum\limits_{k=1}^{\infty} x_k$ be a convergent series in E. We will show that the series $\sum\limits_{k=1}^{\infty} x_{p_k}$ is also convergent and has the same sum v.

Denote by s_j and σ_j the jth partial sum of the series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} x_{p_k}$, respectively.

Let $\varepsilon > 0$ be an arbitrary number. The convergence of $\sum_{k=1}^{\infty} x_k$ implies that there is a positive integer n_0 such that for every $n \geq n_0$

$$\| s_n - v \| < \frac{\varepsilon}{2} \tag{13}$$

holds. Of course n_0 can be chosen so large that for $j \geq n_0$ and $q \in \mathbb{N}$ we also have

$$\| x_{j+1} + x_{j+2} + \cdots + x_{j+q} \| < \frac{\varepsilon}{2M} . \tag{13'}$$

(This is Cauchy condition for convergence of the series $\sum_{k=1}^{\infty} x_k$.)

Choose now a positive integer m_0 sufficiently large such that all numbers $1, 2, \ldots, n_0$ are contained in the sequence $p_1, p_2, \ldots, p_{m_0}$.

Let $m \geq m_0$. The choice of m_0 and the condition (a) give us the expression of σ_m is the following way

$$\sigma_m = \sum_{k=1}^{m} x_{p_k} = s_n + s_{B_1} + s_{B_2} + \cdots + s_{B_r} , \tag{14}$$

where $n \geq n_0$ and s_B (for B a block $B = \{i+1, \ldots, i+l\}$) means the sum $s_B = x_{i+1} + x_{i+2} + \cdots + x_{i+l}$.

On account of (13') we obtain

$$\| s_{B_k} \| < \frac{\varepsilon}{2M} \qquad (k = 1, 2, \ldots, r) . \tag{15}$$

But then from (13), (14) and (15) we get (for $m \geq m_0$)

$$\| \sigma_m - v \| \leq \| \sigma_m - s_n \| + \sum_{k=1}^{r} \| s_{B_k} \| < \frac{\varepsilon}{2} + (M-1) \frac{\varepsilon}{2M} < \varepsilon .$$

Thus $v = \sum_{k=1}^{\infty} x_{p_k}$.

"$\Longrightarrow$" Suppose that a permutation $p = (p_n)_{n \in \mathbb{N}}$ of $\mathbb{N}$ preserves convergence of series in E. Fix a point $x_0 \in E$, $x_0 \neq 0$ and let us construct the linear subspace $E_1 = \{ r x_0 \colon r \in \mathbb{R} \}$ of E. Then p must also preserve convergence of series in E_1. It is now clear that an arbitrary series $\sum_{k=1}^{\infty} a_k x_0$ in E_1 is convergent if and only if the real series $\sum_{k=1}^{\infty} a_k$ is convergent. Therefore p preserves convergence of series with real terms and so (see [1]) it satisfies the condition (a). $\square$

Now we will find a sufficient condition for a permutation $p = (p_n)_{n \in \mathbb{N}}$ preserving convergence of series in a Banach space.

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a Banach space and π a permutation of $\mathbb{N}$ which transforms any convergent series in E into a series with bounded sequence of partial sums (with respect to norm in E). Then π preserves convergence of series in E.*

P r o o f . We are going to proceed indirectly. Suppose that π satisfies the assumption of the theorem but it does not preserve convergence of series in E. It means that there exists a convergent series $\sum_{k=1}^{\infty} x_k$, $x_k \in E$ $(k = 1, 2, \dots)$, such that the series $\sum_{k=1}^{\infty} x_{\pi(k)}$ is not convergent. Since E is a Banach space, the Cauchy condition of convergence is not satisfied for the series $\sum_{k=1}^{\infty} x_{\pi(k)}$. Therefore the following statement holds

$$\exists \varepsilon_0 > 0 \ \forall n_0 \in \mathbb{N} \ \exists m > n_0 \ \exists q \in \mathbb{N} \ \left\| x_{\pi(m+1)} + x_{\pi(m+2)} + \cdots + x_{\pi(m+q)} \right\| \geq \varepsilon_0 . \tag{16}$$

Since $\sum_{k=1}^{\infty} x_k$ is convergent, the Cauchy condition for this series is satisfied, which means

$$\forall k \in \mathbb{N} \ \exists m \in \mathbb{N} \ \forall i, n \in \mathbb{N} (i > n \geq m) \ \left\| \sum_{n \leq j < i} x_j \right\| < \frac{1}{k^3} . \tag{17}$$

From (16) and (17) it follows the existence of a sequence $m_1 < m_2 < \cdots < m_k < \dots$ of positive integers and a sequence of numbers $(q_k)_{k\in\mathbb{N}}$ such that: $m_k + q_k < m_{k+1}$ $(k = 1, 2, \dots)$,

$$\left\| x_{\pi(m_k+1)} + \cdots + x_{\pi(m_k+q_k)} \right\| \geq \varepsilon_0 \tag{18}$$

and

$$\forall k \in \mathbb{N} \ \forall i, n \in \mathbb{N} (i > n \geq m_k) \ \left\| \sum_{n \leq j < i} x_j \right\| < \frac{1}{k^3} . \tag{19}$$

In particular we have

$$\forall k \in \mathbb{N} \ \left\| \sum_{j=m_k}^{m_{k+1}-1} x_j \right\| < \frac{1}{k^3} . \tag{19'}$$

Now let us construct the series $\sum_{j=1}^{\infty} x'_j$ in the following way

$$x'_j = \begin{cases} x_j & \text{for } j = 1, 2, \dots, m_1 , \\ k x_j & \text{for } m_k < j \leq m_{k+1} \ (k = 1, 2, \dots) . \end{cases}$$

The series $\sum_{j=1}^{\infty} x'_j$ is convergent. To prove this it is sufficient to show that $\lim_{m\to\infty} r_m = 0$ where $r_m = \limsup_{k\to\infty} \left\| \sum_{j=m}^{k} x'_j \right\|$.

Let $\eta > 0$. We choose $i_0 \in \mathbb{N}$ such that

$$\sum_{i \geq i_0} \frac{1}{i^2} < \eta \,. \tag{20}$$

Now let $m \geq m_{i_0}$. Then there is an integer $i \geq i_0$ for which $m_i \leq m < m_{i+1}$. We can estimate r_m. By (19) and (19') we get for $k > m$

$$\begin{aligned}\left\| \sum_{j=m}^{k} x'_j \right\| &\leq \left\| \sum_{j=m}^{m_{i+1}-1} x'_j \right\| + \left\| \sum_{j=m_{i+1}}^{m_{i+2}-1} x'_j \right\| + \dots + \left\| \sum_{j=m_{i+l}}^{k} x'_j \right\| \\ &< \frac{1}{i^2} + \frac{1}{(i+1)^2} + \dots + \frac{1}{(i+l)^2} < \sum_{j \geq i} \frac{1}{j^2} \,.\end{aligned}$$

Therefore

$$r_m = \limsup_{k \to \infty} \left\| \sum_{j=m}^{k} x'_j \right\| \leq \sup_{k > m} \left\| \sum_{j=m}^{k} x'_j \right\| \leq \sum_{j \geq i} \frac{1}{j^2} < \eta \,.$$

Thus $\lim\limits_{m \to \infty} r_m = 0$.

To complete the proof we will show that the sequence of partial sums of the series $\sum\limits_{j=1}^{\infty} x'_{\pi(j)}$ is not bounded. Indeed, by the definition of x'_j $(j = 1, 2, \dots)$ we have

$$x'_{m_k+1} + x'_{m_k+2} + \dots + x'_{m_k+q_k} = k(x_{m_k+1} + x_{m_k+2} + \dots + x_{m_k+q_k}) \,,$$

and so by (18)

$$\left\| x'_{\pi(m_k+1)} + x'_{\pi(m_k+2)} + \dots + x'_{\pi(m_k+q_k)} \right\| \geq k\varepsilon_0 \qquad (k = 1, 2, \dots) \,.$$

From this the unboundedness of the sequence $\left(\sum\limits_{j=1}^{n} x'_{\pi(j)} \right)_{n \in \mathbb{N}}$ follows immediately. $\square$

3. Subseries of series in linear normed spaces

Let

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \dots + x_n + \dots \tag{21}$$

be a series with $x_n \in E$ $(n = 1, 2, \dots)$ and E be a linear normed space. If $k_1 < k_2 < \dots < k_n < \dots$ is a sequence of positive integers, then the series

$$\sum_{n=1}^{\infty} x_{k_n} = x_{k_1} + x_{k_2} + \dots + x_{k_n} + \dots \tag{22}$$

is said to be subseries of the series (21). The importance of the concept of subseries is emphasized by the well-known theorem of Orlicz (cf. [13] p. 86, Theorem 10.3) according to which a series $\sum_{n=1}^{\infty} x_n$ unconditionally converges in E if and only if each subseries of $\sum_{n=1}^{\infty} x_n$ converges.

Now we will introduce another form of notation of subseries which enables us to state a one-to-one correspondence between subseries of a given series and points of the interval $(0,1]$. Let

$$t = \sum_{n=1}^{\infty} 2^{-k_n} \in (0,1], \qquad k_1 < k_2 < k_3 < \dots$$

This is a non-terminating dyadic expansion of the number t which can also be written in the form

$$t = \sum_{k=1}^{\infty} c_k(t)\, 2^{-k}, \qquad \text{where } c_k(t) = \begin{cases} 0 & \text{for } k \neq k_i \ (i = 1, 2, \dots), \\ 1 & \text{for } k \in \{k_1, k_2, k_3, \dots\}. \end{cases}$$

Now we can define a correspondence between subseries of the series (21) and real numbers in $(0,1]$ by putting

$$S(t) = \sum_{k=1}^{\infty} c_k(t) x_k . \tag{22'}$$

Obviously there is only a formal difference between the series (22) and (22'). In what follows we will use the form (22') for subseries.

Let $\mathcal{S}$ be any class of subseries (22') of the series (21). The "magnitude" of the class $\mathcal{S}$ can be measured now by the set $\{t \in (0,1] \colon S(t) \in \mathcal{S}\} \subset (0,1]$. We say that almost all subseries (22') of (21) belong to $\mathcal{S}$ if

$$\lambda\bigl(\{t \in (0,1] \colon S(t) \in \mathcal{S}\}\bigr) = 1\,, \qquad \text{where } \lambda \text{ is the Lebesgue measure on } \mathbb{R}\,.$$

The following theorem is an extension of a theorem on subseries of series with real terms[2] (cf. [11], Theorem 4).

THEOREM 3.1. *Suppose that $(E, \|\cdot\|)$ is a Banach space. Then almost all subseries of a divergent series in E are divergent. More precisely, if $\sum_{k=1}^{\infty} x_k$ is a divergent series then*

$$\lambda\bigl(\{t \in (0,1] \colon S(t) \text{ diverges}\}\bigr) = 1\,.$$

To prove this theorem we shall formulate and prove some auxiliary results.

[2] See also R. G. Cooke: *Infinite Matrices and Sequence Spaces* (Russian translation) Gos. Izd. Fiz.-Mat. lit. Moscow (1960) p. 404–405

Lemma 3.1. *Let $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements of a space E. Suppose that $K=\{k_1<k_2<\dots\}$, $L=\{l_1<l_2<\dots\}$ are infinite subsets of $\mathbb{N}$ and $K\cup L=\mathbb{N}$, $K\cap L=\emptyset$. If the series $\sum_{j=1}^{\infty} x_{k_j}$, $\sum_{j=1}^{\infty} x_{l_j}$ are convergent in E then the series $\sum_{j=1}^{\infty} x_j$ is also convergent and its sum is the sum of sums of these series, i.e.,*

$$\sum_{j=1}^{\infty} x_j = \sum_{j=1}^{\infty} x_{k_j} + \sum_{j=1}^{\infty} x_{l_j}\,.$$

P r o o f . The proof of this theorem is obvious and can be left to the reader. □

Theorem 3.2. *Let E be a Banach space and $(x_k)_{k\in\mathbb{N}}$ be a sequence of elements of the space E. Denote by $C(x_1,x_2,\dots)$ and $D(x_1,x_2,\dots)$ the set of all $t\in(0,1]$ for which the series $S(t)$ converges and diverges, respectively. Then $C(x_1,x_2,\dots)$ $(D(x_1,x_2,\dots))$ is an $F_{\sigma\delta}$ $(G_{\delta\sigma})$ set in $(0,1]$. (We assume the classical Euclidean metric in the space $(0,1]$).*

P r o o f . It is sufficient to prove the part of this theorem concerning the set $C(x_1,x_2,\dots)$ since

$$D(x_1,x_2,\dots)=(0,1]\setminus C(x_1,x_2,\dots)\,.$$

In Banach spaces, because the Cauchy criterion holds, a series $S(t)$ converges if and only if

$$\forall k\ge 1\;\exists m\in\mathbb{N}\;\forall p\in\mathbb{N}\;\left\|c_{m+1}(t)x_{m+1}+c_{m+2}(t)x_{m+2}+\dots+c_{m+p}(t)x_{m+p}\right\|\le\frac{1}{k}\,.$$

From this we get

$$C(x_1,x_2,\dots)=\bigcap_{k=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{p=1}^{\infty}A(m,p,k)\,, \tag{23}$$

where:

$$A(m,p,k)=\Big\{t\in(0,1]:\left\|c_{m+1}(t)x_{m+1}+c_{m+2}(t)x_{m+2}+\ldots+c_{m+p}(t)x_{m+p}\right\|\le\tfrac{1}{k}\Big\}.$$

Put $\mathbb{Q}'=(0,1]\setminus\mathbb{Q}$. Then the set $A(m,p,k)\cap\mathbb{Q}'$ is closed in $\mathbb{Q}'$ because it consists of a finite union of disjoint sets of the form $i^{(j)}_{m+p}\cap\mathbb{Q}'$ where: $i^{(j)}_{m+p}=\big(\frac{j}{2^{m+p}}\,,\ \frac{j+1}{2^{m+p}}\big]$, m,p are fixed and j runs over some numbers of the set $\{0,1,2,\dots,2^{m+p}-1\}$.

But then because of (23) we see that $C(x_1,x_2,\dots)\cap\mathbb{Q}'$ is an $F_{\sigma\delta}$ set in $\mathbb{Q}'$ and in $(0,1]$, too (because $\mathbb{Q}'$ itself is an $F_{\sigma\delta}$ set in $(0,1]$). The set $C(x_1,x_2,\dots)\cap\mathbb{Q}$ is countable so the theorem is proved. □

We recall the concept of a homogeneous set. A measurable set $B \subset [0,1]$ is called homogeneous if there is a $d \in [0,1]$ such that for every subinterval I of $[0,1]$

$$\frac{\lambda(I \cap B)}{\lambda(I)} = d$$

holds. It is well-known (cf. [11], [16]) that the measure of each homogeneous set is 0 or 1.

LEMMA 3.2. *Let $B \subset (0,1]$ be a measurable set. Suppose that B satisfies the following condition: For an arbitrary $n \in \mathbb{N}$ and $0 \le k \le k' \le 2^n - 1$*

$$\lambda\big(i_n^{(k)} \cap B\big) = \lambda\big(i_n^{(k')} \cap B\big)$$

holds. Then B is a homogeneous set.

Proof. See Lemma 1 in [11]. □

Proof of Theorem 3.1. The set $C(x_1, x_2, \dots) = C$ is measurable by Theorem 3.2. It is easy to see that it is also homogeneous. To prove this is sufficient to observe that if $t \in (0,1]$ belongs to the set C then all points of the form $t \pm \frac{1}{2^n} \in (0,1]$ belong to C, too. This is an easy consequence of the fact that convergence of a series cannot be changed by changing a finite number of terms of this series.

Hence $\lambda(C) = 0$ or $\lambda(C) = 1$. Suppose that $\lambda(C) = 1$. Define the function $g(t) = 1 - t$ for $t \in \left(0, \frac{1}{2}\right)$. This function maps the interval $\left(0, \frac{1}{2}\right)$ onto the interval $\left(\frac{1}{2}, 1\right)$ and preserves the Lebesgue measure. Therefore

$$\lambda\Big(g\Big(C \cap \left(0, \tfrac{1}{2}\right)\Big)\Big) = \frac{1}{2} = \lambda\Big(C \cap \left(\tfrac{1}{2}, 1\right)\Big) = \lambda\Big(C \cap \left(0, \tfrac{1}{2}\right)\Big).$$

Since the sets $g\big(C \cap (0, \frac{1}{2})\big)$ and $C \cap \left(\frac{1}{2}, 1\right)$ are contained in $\left(\frac{1}{2}, 1\right)$ and each of them has the measure $\frac{1}{2}$, there exists an irrational number $1 - t_0$ contained in both which means that $t_0 \in C \cap \left(0, \frac{1}{2}\right)$ and $1 - t_0 \in C \cap \left(\frac{1}{2}, 1\right)$.

Let $\sum\limits_{j=1}^{\infty} 2^{k_j}$ and $\sum\limits_{j=1}^{\infty} 2^{l_j}$ be the dyadic expansions of the numbers t_0 and $1-t_0$, respectively. The sets $K = \{k_1 < k_2 < \dots\}$, $L = \{l_1 < l_2 < \dots\}$ are infinite disjoint and their union is the set $\mathbb{N}$. Moreover, because both t_0 and $1 - t_0$ are in the set C, the series $\sum\limits_{j=1}^{\infty} x_{k_j}$ and $\sum\limits_{j=1}^{\infty} x_{l_j}$ are convergent. Then according to Lemma 3.1. the series $\sum\limits_{j=1}^{\infty} x_j$ converges, i.e., we obtained a contradiction. Thus we have $\lambda\big(C(x_1, x_2, \dots)\big) = 0$ and the proof is finished. □

This means that the set $C(x_1, x_2, \dots)$ is a "small set" from the metric point of view. Now we will study the same set from the topological point of view.

THEOREM 3.3. *Let $(E, \|\cdot\|)$ be a Banach space. Let $\sum\limits_{j=1}^{\infty} x_j$ be a divergent series. Then $C(x_1, x_2, \dots)$ is a set of the first Baire category in $(0,1]$.*

Proof. If $\liminf\limits_{j\to\infty} \|x_j\| > 0$ then obviously $C(x_1, x_2, \dots) = \emptyset$ and the theorem holds. Hence we can assume that

$$\liminf_{j\to\infty} \|x_j\| = 0\,. \tag{24}$$

Let (as in the proof of Theorem 3.1) $\mathbb{Q}' = (0,1] \setminus \mathbb{Q}$ and define the functions

$$g_m(t) = \sum_{j=1}^{m} c_j(t) x_j \qquad (m = 1, 2, \dots),$$

$$\text{where:} \qquad t = \sum_{j=1}^{\infty} c_j(t)\, 2^{-j} \in (0,1]\,.$$

These functions are continuous on the set $\mathbb{Q}'$ because g_m is constant on every set $i_m^{(j)} \cap \mathbb{Q}'$, $(j = 1, 2, \dots)$.

Further we define on $C \cap \mathbb{Q}'$ $(C = C(x_1, x_2, \dots))$ the function g in the following way

$$g(t) = \sum_{j=1}^{\infty} c_j(t) x_j\,.$$

Then we have $g(t) = \lim\limits_{m\to\infty} g_m(t)$ for $t \in C \cap \mathbb{Q}'$ and therefore g is a Baire 1 function on $C \cap \mathbb{Q}'$.

Now we will prove that g is discontinuous at every point $t \in C \cap \mathbb{Q}'$. From this (according to a well-known theorem about the set of points of discontinuity of a Baire 1 function (cf [13] p. 181, Theorem 8.3.), we can conclude that $C \cap \mathbb{Q}'$ is a set of the first Baire category in $C \cap \mathbb{Q}'$ and therefore also in $(0,1]$.

To finish the proof we shall show that g is discontinuous at each point $t_0 \in C \cap \mathbb{Q}'$. Since the series $\sum\limits_{j=1}^{\infty} x_j$ diverges, by the Cauchy condition for convergence of series there exists an $\varepsilon_0 > 0$ and positive integers m_i, p_i $(i = 1, 2, \dots)$ such that

$$\|x_{m_i+1} + x_{m_i+2} + x_{m_i+p_i}\| \geq \varepsilon_0\,. \tag{25}$$

$$m_1 < m_2 < \dots < m_i < \dots \quad \text{and} \quad m_i + p_i < m_{i+1} \quad (i = 1, 2, \dots).$$

Because $t_0 \in C$ we can find an $s \in \mathbb{N}$ for which

$$\Bigl\| \sum_{j=s+1}^{\infty} c_j(t_0) x_j \Bigr\| \leq \frac{\varepsilon_0}{3}\,. \tag{26}$$

Now let us choose $i_0 \in \mathbb{N}$ such that $m_{i_0} > s$.

Finally we find positive integers k_j such that

$$m_{i_0} + p_{i_0} < k_1 < k_2 < \dots < k_j < \dots, \tag{26'}$$

$$\lim_{n\to\infty} (k_{n+1} - k_n) = +\infty, \tag{26''}$$

$$\sum_{j=1}^{\infty} \| x_{k_j} \| < \frac{\varepsilon_0}{3} \qquad \text{(see (24))}. \tag{26'''}$$

The number t_0 belongs to an interval $i_s^{(v)} = \left(\frac{v}{2^s}, \frac{v+1}{2^s}\right]$ (for suitable v, $0 \le v \le 2^s - 1$). Since $t_0 \in \mathbb{Q}'$, it also belongs to $i_s^{(v)} \cap \mathbb{Q}'$.

The number s can be chosen arbitrarily large, hence the set $i_s^{(v)} \cap \mathbb{Q}'$ can be arbitrarily small neighborhood of the point t_0. The proof will be finished if we show that for each s satisfying (26) there exists a point $t_1 \in i_s^{(v)} \cap \mathbb{Q}'$ such that

$$\big\| g(t_1) - g(t_0) \big\| \ge \frac{\varepsilon_0}{3}.$$

Let s satisfy (26). Define the point t_1 in the following way:

$$t_1 = \sum_{j=1}^{\infty} c_j(t_1) 2^{-j},$$

$$\text{where:} \qquad c_j(t_1) = \begin{cases} c_j(t_0) & \text{for } j = 1, 2, \dots, s, \\ 1 & \text{for } j = m_{i_0}+1, m_{i_0}+2, \dots, m_{i_0}+p_{i_0}, \\ 1 & \text{for } j = k_1, k_2, \dots, k_n, \dots, \\ 0 & \text{for all others } j. \end{cases} \tag{27}$$

Now (27) guarantee that t_1 belongs to $i_s^{(v)} \cap \mathbb{Q}'$. So we can estimate $\|g(t_1) - g(t_0)\|$,

$$g(t_0) = \sum_{j=1}^{s} c_j(t_0) x_j + \sum_{j=s+1}^{\infty} c_j(t_0) x_j, \tag{28}$$

$$g(t_1) = \sum_{j=1}^{s} c_j(t_0) x_j + (x_{m_{i_0}+1} + x_{m_{i_0}+2} + \dots + x_{m_{i_0}+p_{i_0}}) + \sum_{j=1}^{\infty} x_{k_j}. \tag{29}$$

By subtracting (28) from (29) we get

$$g(t_1) - g(t_0) = (x_{m_{i_0}+1} + x_{m_{i_0}+2} + \dots + x_{m_{i_0}+p_{i_0}}) + \sum_{j=1}^{\infty} x_{k_j} - \sum_{j=s+1}^{\infty} c_j(t_0) x_j.$$

From this we obtain

$$\|g(t_1)-g(t_0)\| \geq \|x_{m_{i_0}+1}+x_{m_{i_0}+2}+\dots+x_{m_{i_0}+p_{i_0}}\| - \sum_{j=1}^{\infty} \|x_{k_j}\| - \Big\| \sum_{j=s+1}^{\infty} c_j(t_0)x_j \Big\| .$$

By (25), (26) and (26''') we get from this

$$\|g(t_1) - g(t_0)\| \geq \varepsilon_0 - \frac{\varepsilon_0}{3} - \frac{\varepsilon_0}{3} = \frac{\varepsilon_0}{3} .$$

□

COROLLARY 3.1. (a) *If* $\sum\limits_{j=1}^{\infty} x_j$ *is a divergent series in* E *then* $C(x_1, x_2, \dots)$ *is a null set of the first Baire category in* $(0,1]$.

(b) *If* $\sum\limits_{j=1}^{\infty} x_j$ *is a divergent series in* E *then the set* $D(x_1, x_2, \dots)$ *has the measure* 1 *and is a residual in* $(0,1]$.

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