Tatra Mt. Math. Publ. **19** (2000), 127–133

TATRA MOUNTAINS
Mathematical Publications

A-INTEGRABLE MARTINGALE SEQUENCES AND APPLICATION TO HAAR SERIES

Valentin Skvortsov

ABSTRACT. It is proved that the uniform A-integrability of partial sums of a Haar series convergent almost everywhere to an A-integrable function f is a sufficient condition for this series to be A-Fourier series of f. At the same time there exists a Haar series which is everywhere convergent to an A-integrable function and which is not A-Fourier series of this function.

A well known theorem in martingale theory states that a martingale sequence $\{X_n, \mathcal{F}_n,\ n = 1, 2, \dots\}$ is closed on the right by the last element $X_\infty(\omega) = \lim\limits_{n\to\infty} X_n(\omega)$ iff $\{X_n, \mathcal{F}_n\}$ is a uniformly integrable sequence (see [1, p. 300], [3, p. 239] or [7, p. 60]). The conditional expectation in this theory is defined in terms of the Lebesgue integral.

Meanwhile there are some other versions of the notion of the mathematical expectation which involve integration more general than the Lebesgue integration. One of such generalization was introduced by Kolmogorov in [5] who defined generalized mathematical expectation as a non-absolutely convergent integral which later became known as the Kolmogorov A-integral (see [2], [6]). In [8] we have extended this definition to the case of the conditional expectation and have applied this extension to the investigation of A-integrable martingales.

We recall some definitions.

Definition 1. A random variable (r. v.) X defined on a probability space $(\Omega, \mathscr{B}, P)$ is said to be *A-integrable* over a set $B \in \mathscr{B}$ if

$$P\bigl(\{\omega \in B \colon |X(\omega)| > C\}\bigr) = \bar{o}(1/C) \quad \text{as} \quad C \to \infty \tag{1}$$

and if there exists a finite limit

$$\lim_{C\to\infty} \int\limits_{\{\omega\in B\colon\ |X(\omega)|\le C\}} X\, dP = I\,.$$

2000 Mathematics Subject Classification: 26A39, 42C10, 60G46.

Keywords: A-integral, uniform A-integrability, martingale sequence, Haar series, dyadic Henstock-Kurzweil integral.

Supported by RFFI (grant 99–01–00355).

Then I is called *the A-integral* of X over B and is denoted by $(A)\int_B X\,dP$.

Note that if a r. v. X is A-integrable over some set $B \in \mathscr{B}$ and $X(\omega) \geq 0$ a. s. on B then X is also L-integrable and $(A)\int_B X\,dP = (L)\int_B X\,dP$. This fact implies that if an $\mathcal{F}$-measurable r. v. X ($\mathcal{F}$ being a sub-σ-field of $\mathscr{B}$) is A-integrable over any $\mathcal{F}$-measurable subset of some $\mathcal{F}$-measurable set B then X is L-integrable over B. Indeed, put $B_+ = \{\omega \in B\colon X(\omega) \geq 0\}$ and $B_- = \{\omega \in B\colon X(\omega) < 0\}$. Then, being the r. v. X A-integrable over B_+ and over B_-, it must be L-integrable over each of these sets and consequently over B. We use this observation in the following definition.

Definition 2. Let a r. v. X be A-integrable over any set $B \in \mathcal{F}$ where $\mathcal{F}$ is a sub-σ-field of $\mathscr{B}$. *The conditional A-expectation of X with respect to $\mathcal{F}$* is defined as an $\mathcal{F}$-measurable r. v. $AE(X\,|\,F)$ such that for every $B \in \mathcal{F}$ we have

$$\int_B AE(X\,|\,\mathcal{F})\,dP = (A)\int_B X\,dP\,. \tag{2}$$

Definition 3. An A-integrable r. v. X_∞ is said to be *the last element* of a martingale sequence $\{X_n, \mathcal{F}_n, n = 1, 2, \ldots\}$ if $X_n = AE(X_\infty\,|\,\mathcal{F}_n)$ for each $n = 1, 2, \ldots$. We also say in this case that X_∞ *closes the martingale sequence from the right.*

We have omitted the "A"-sign in front of the left hand side of (2) because of the above observation, meaning that the Lebesgue integral can be used here.

It follows from the same observation that the use of the A-integral in the definition of a martingale sequence gives an essentially more general notion only for the last element and only in the case where each σ-field $\mathcal{F}_n$ is a proper subset of the σ-field $\mathcal{F}_\infty$ generated by $\cup_n \mathcal{F}_n$.

A sufficient condition for a martingale sequence to be closed on the right by the A-integrable last element is given in [8]. This condition is formulated in terms of uniform A-integrability which is a non-absolute analogue of the uniform Lebesgue integrability.

Definition 4. A family of r. v. $\{X_\gamma\}_{\gamma\in\Gamma}$, defined on $(\Omega, \mathscr{B}, P)$ (Γ is some index set) is said to be *uniformly A-integrable on $B \in \mathscr{B}$* iff the sets $D_\gamma(C) = \{\omega \in B\colon |X_\gamma(\omega)| > C\}$ satisfy the conditions: $P\big(D_\gamma(C)\big) = \bar{o}(1/C)$ uniformly in γ as $C \to \infty$ and

$$\sup_{\gamma\in\Gamma}\left|(A)\int_{D_\gamma(C)} X_\gamma\,dP\right| \longrightarrow 0 \qquad \text{as} \quad C \to \infty\,.$$

Theorem A. *Let $\{X_n, \mathcal{F}_n,\ n = 1, 2, \dots\}$ be a martingale sequence convergent a. s. to a r. v. X_∞ and let the sequence $\{X_n\}$ be uniformly A-integrable on any $B \in \cup_n \mathcal{F}_n$ in the sense of Definition 4. Then X_∞ is A-integrable on each $B \in \cup_n \mathcal{F}_n$ and closes the martingale sequence $\{X_n, \mathcal{F}_n\}$ on the right, i.e., $\{X_n, \mathcal{F}_n, n = 1, 2, \dots, \infty\}$ is an A-integrable martingale with $X_n = AE(X_\infty | \mathcal{F}_n)$, X_∞ being its last element.*

It is known (see [11] for example) that for some orthogonal systems, the partial sums of series with respect to those systems are martingale sequences. In particular this is the case for the Haar system.

Remind that the Haar functions are defined as follows: $h_0(x) \equiv 1$, and for $n = 2^{i-1} + j$

$$h_n(x) := \begin{cases} 2^{(i-1)/2}, & x \in \triangle^i_{2j}, \\ -2^{(i-1)/2}, & x \in \triangle^i_{2j+1}, \\ 0, & \text{otherwise}, \end{cases}$$

where $0 \le j \le 2^{i-1} - 1$, $i = 1, 2, \dots$ and $\triangle^i_j = \left(\frac{j}{2^i}, \frac{j+1}{2^i}\right)$. At points of discontinuity Haar functions are defined so that $h(x) = \big(h(x+0) + h(x-0)\big)/2$ and at 0 and 1 they are defined to be continuous from inside of $[0, 1]$.

The sequence of partial sums S_n of any Haar series forms a martingale sequence $\{S_n, \mathcal{F}_n\}$ where the σ-algebra $\mathcal{F}_n$ is generated by dyadic intervals on which S_n is constant.

A relation between martingale theory and the theory of Haar series is shown by the following theorem (see [11]):

Theorem B. *A Haar series is the Fourier series of a function f iff the partial sums of the series form a martingale sequence $\{S_n, \mathcal{F}_n\}$ which is closed from the right by the last element f.*

This assertion is true not only for the Lebesgue integral but also for many of its generalizations and in particular for A-integral.

Combining this theorem with Theorem A we get:

Theorem 1. *Let the sequence of the partial sums of a Haar series converge almost everywhere to a function f and be uniformly A-integrable. Then f is A-integrable and the series is the A-Fourier series of f.*

Thus the coefficients of a Haar series under the above conditions are uniquely defined by the sum f as the A-Fourier coefficients.

Note that unlike to the case of the uniform Lebesgue integrability, the above condition in terms of the uniform A-integrability is not necessary for a Haar series to be the A-Fourier series of its sum f. This can be shown by constructing an example of a Haar series such that it is the A-Fourier series of an A-integrable function and its partial sums are not uniformly A-integrable. (See [11] for details.)

Another condition for the unique definition of the coefficients of a Haar series by the sum is given by the pointwise convergence everywhere on $[0,1]$. In this case the coefficients can be recovered uniquely from the sum by Fourier formulae where the integral sign is understood in the sense of the dyadic Perron integral (see [9]) or equivalently in the sense of the dyadic Henstock-Kurzweil integral (see [10]).

In this respect it is of interest to investigate the following problem: Is a Haar series everywhere convergent to an A-integrable function the A-Fourier series of this function? In other words: Is the A-integral compatible with the dyadic Henstock-Kurzweil integral on the class of sums of everywhere convergent Haar series?

In what follows we are giving a negative answer to these questions. We are doing this by constructing an appropriate example of a Haar series.

We start by defining on $[0,1]$ for each $k = 1, 2, \ldots$ and each $m = 1, \ldots, 2^{k-1}$ a function

$$Q_{k,m}(x) = \begin{cases} 2^{2^k+2m}, & x \in \triangle^{2^k+2m+k}_{(2^k-m)2^{2^k+2m-k+1}}, \\ 2^{2^k+2m-1}, & x \in \triangle^{2^k+2m+k-1}_{(2^k-m)2^{2^k+2m-k}+1}, \\ -2^{2^{k-1}+m}, & x \in \triangle^{2^{k-1}+m+k-1}_{(2^k-m+1)2^{2^{k-1}+m-k}-1}, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

At points of discontinuity let $Q_{k,m}(x) = \big(Q_{k,m}(x+0) + Q_{k,m}(x-0)\big)/2$ and let $Q_{k,m}$ be continuous from inside at 0 and 1.

It is easy to see that $Q_{k,m}$ can be represented as a polynomial with respect to Haar functions whose supports are included into the interval $\overline{\triangle^{2k-1}_{2^k-m}}$. So the supports of Haar functions taken from two different polynomials $Q_{k,m}$ and $Q_{k',m'}$ are pairwise disjoint.

The series $\sum\limits_{k=1}^{\infty} \sum\limits_{m=1}^{2^{k-1}} Q_{k,m}$ can be rearranged into Haar series

$$\sum_{n=0}^{\infty} a_n h_n(x) \tag{4}$$

which at each point x is in fact a finite sum. So the series (4) is convergent everywhere on $[0,1]$ and its sum is a function f such that

$$f(x) = Q_{k,m}(x) \qquad \text{if} \quad x \in \triangle^{2k-1}_{2^k-m}, \tag{5}$$

$f(0) = 0$ and f is defined at the endpoints of each $\triangle_{2^k-m}^{2k-1}$ in such a way that $f(x) = \big(f(x+0) + f(x-0)\big)/2$.

Note that in series (4)

$$a_0 = 0\,. \tag{6}$$

As it is mentioned above any everywhere convergent Haar series is the Haar-Fourier series if the dyadic Henstock-Kurzweil integral is used in Fourier formulae. Hence in view of (6)

$$(H_d)\int_0^1 f = 0\,. \tag{7}$$

(H_d stands here for Henstock-Kurzweil dyadic.)

Now we prove that the function f defined above is A-integrable.

First notice that according to (3) and (5) all non-trivial values of f are exhausted (up to a countable set of values at the points of discontinuity) by the sequences $\{2^{2^k+p}\}$, $\{-2^{2^k+p-1}\}$, $k = 1, 2, \dots,\ p = 1, 2, \dots, 2^k$ and

$$P\Big(\big\{x\colon f(x) = 2^{2^k+p}\big\}\Big) = P\Big(\big\{x\colon f(x) = -2^{2^k+p}\big\}\Big) = 2^{-(2^k+p+k)}\,, \tag{8}$$

$k = 1, 2, \dots,\ p = 1, 2, \dots, 2^k$,

$$P\Big(\big\{x\colon f(x) = -2^2\big\}\Big) = 2^{-2}\,. \tag{9}$$

(This time P stands for Lebesgue measure on $[0,1]$.)

Take any $C > 4$ and choose k and p such that

$$2^{2^k+p-1} \le C < 2^{2^k+p}\,, \qquad k = 1, 2, \dots, \quad p = 1, 2, \dots, 2^k\,.$$

Then according to (3), (5) and (8)

$$\begin{aligned} P\big(\{x\colon |f(x)| > C\}\big) &\le P\Big(\big\{x\colon |f(x)| > 2^{2^k+p-1}\big\}\Big) \\ &= \sum_{i=2^k+p}^{\infty} P\Big(\big\{x\colon |f(x)| 2^i\big\}\Big) = 2\sum_{i=2^k+p}^{\infty} 2^{-(i+k)} \\ &= 2^{-(2^k+p+k-2)} < C^{-1}2^{2-k} = \bar{o}(1/C)\,. \end{aligned}$$

So (1) is fulfilled for f.

Furthermore in view of (8) and (9) we have for any $C > 4$

$$\int_{\{x\colon |f(x)|<C\}} f = -4P(\{x\colon f(x) = -4\}) = -1\,.$$

Thus $(A)\int_0^1 f = -1$.

Now imply (6) and (7)

$$(A)\int_0^1 f \neq a_0 = (H_d)\int_0^1 f\,.$$

So we have proved

THEOREM 2. *There exists a Haar series which converges everywhere to an A-integrable function f and which is not A-Fourier series of f.*

In particular

THEOREM 3. *A-integral and (H_d)-integral are noncompatible.*

Of course those integrals agree in view of Theorem 1 on the class of sums of everywhere convergent series if the partial sums of this series are uniformly A-integrable.

REFERENCES

[1] ASH, R. B.: *Real Analysis and Probability*, Academic Press, New York, 1972.

[2] BARY, N. K.: *A Treatise on Trigonometric Series*, Macmillan, New York, 1964.

[3] CHOW, Y. S.—TEICHER, H.: *Probability Theory*, 2nd edition, Springer-Verlag, Berlin, 1988.

[4] GUNDY, R. F.: *Martingale theory and positive convergence of certain orthogonal series*, Trans. Amer. Math. Soc. **124** (1966), 228–248.

[5] KOLMOGOROV, A. N.: *Foundations of the Theory of Probability*, Chelsea Publishing Company, New York, 1950.

[6] LEE, L. P.: *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989.

[7] LOEVE, M.: *Probability Theory II*, 4th edition, Springer-Verlag, Berlin, 1978.

[8] SKVORTSOV, V. A.: *Martingale closure theorem for A-integrable martingale sequences*, Real Anal. Exchange **24** (1998–99), 815–820.

[9] SKVORTSOV, V. A.: *Haar series with convergent subsequences of partial sums*, Soviet Math. Dokl. **9** (1968), 1464–1471.

[10] SKVORTSOV, V. A.: *Some properties of dyadic derivatives*, in: Lecture Notes in Math., Springer-Verlag, Berlin, Vol. 1419, 1989, pp. 167–170.

[11] SKVORTSOV, V. A.—KOSTIN, V. V.: *Martingale Sequences in the Theory of Orthogonal Series*, Vestnik of Moscow University (to appear).

Received November 14, 1998

Department of Mathematics
Moscow State University
R–119899 Moscow
RUSSIA

E-mail: vaskvor@nw.math.msu.su

Current address:
Instytut Matematyki, WSP
Plac Weyssenhoffa 11
PL–85–072 Bydgoszcz
POLAND

E-mail: skworcow@wsp.bydgoszcz.pl