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MAPPINGS PRESERVING CAUCHY NETS

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ABSTRACT. If $(X,\mathscr{U})$, $(Y,\mathscr{V})$ are uniform spaces then a mapping $f\colon X\to Y$ is said to be Cauchy nets preserving if the image under f of each Cauchy net in X is a Cauchy net in Y. Some properties of mappings preserving Cauchy sequences are investigated.

Let $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ be uniform spaces. We recall that a net $S\colon (D,\succeq)\to X$ in X is a mapping defined on a directed set $(D,\succeq)$ with values in X. A net S in X is Cauchy if for each $U\in\mathscr{U}$ there is $k\in D$ such that $\big(S(m),S(n)\big)\in\mathscr{U}$ whenever $m,n\succeq k$ ([2]). Denote by F_X the family of all Cauchy nets in X. A mapping $f\colon X\to Y$ is said to be Cauchy nets preserving if $f\circ S$ is a Cauchy net in Y whenever S is a Cauchy net in X. Denote by $F(X,Y)$ the family of all Cauchy nets preserving mappings $f\colon X\to Y$, i.e.,

$$F(X,Y)=\{f\colon X\to Y:\ S\in F_X\Rightarrow f\circ S\in F_Y\}.$$

R. F. Snipes in [3] has investigated some properties of mappings preserving Cauchy sequences (if X and Y are metric spaces). In this paper we shall generalize his results and we shall give some further properties of such mappings.

If $(D,\succeq_D)$ and $(E,\succeq_E)$ are directed sets, define $N(E,D)=\{u\colon E\to D:\ \forall m\in D\ \ \exists n\in E\ \ \forall p\in E:\ p\succeq_E n\Rightarrow u(p)\succeq_D m\}$. A net $T\colon (E,\succeq_E)\to X$ is a subnet of a net $S\colon (D,\succeq_D)\to X$ if there is a mapping $u\in N(E,D)$ for which $T=S\circ u$.

Definition 1. Let $(X,\mathscr{U})$ be a uniform space. Let $S\colon (D,\succeq_D)\to X$ and $T\colon (E,\succeq_E)\to X$ be nets in X. We shall say that

– S and T are *parallel* ($S\parallel T$) if $(D,\succeq_D)=(E,\succeq_E)$ and
$$\forall U\in\mathscr{U}\ \ \exists k\in D\ \ \forall m\succeq_D k:\ \big(S(m),T(m)\big)\in U;$$

– S and T are *equivalent* ($S\approx T$) if $(D,\succeq_D)=(E,\succeq_E)$ and
$$\forall U\in\mathscr{U}\ \ \exists k\in D\ \ \forall m,n\succeq_D k:\ \big(S(m),T(n)\big)\in U;$$

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– S and T are *weakly equivalent* ($S \sim T$) if $\forall U \in \mathscr{U}\ \ \exists p \in D\ \ \exists q \in E$
$\forall m \in D\ \ \forall n \in E$: $(m \succeq_D p\ \ \&\ \ n \succeq_E q) \Rightarrow \big(S(m), T(n)\big) \in U$.

Lemma 1. *Let $(X, \mathscr{U})$ be a uniform space. Let S, T, P be nets in X and $x \in X$. Then we have*

(1) $S \| S$ *for each net S in X* and $S \approx S$, $S \sim S$ *for each* $S \in F_X$;

(2) $S \| T \Leftrightarrow T \| S$, $S \approx T \Leftrightarrow T \approx S$, $S \sim T \Leftrightarrow T \sim S$;

(3) $S \approx T \Rightarrow S \| T$;

(4) $S \approx T \Rightarrow S \sim T$;

(5) $S \sim T \Rightarrow S \in F_X$;

(6) $\big(S \| T\ \ \&\ \ S \in F_X\big) \Rightarrow S \approx T$;

(7) $\big(S \| T\ \ \&\ \ T \| P\big) \Rightarrow S \| P$, $(S \approx T\ \ \&\ \ T \approx P) \Rightarrow S \approx P$, $(S \sim T\ \ \&\ \ T \sim P) \Rightarrow S \sim P$;

(8) *if* $S \in F_X$ and T *is a subnet of S then* $S \sim T$;

(9) *S converges to x if and only if $S \sim x$ (here $x\colon \big(\{1\}, \geq\big) \to X$, $x(1) = x$).*

We omit easy proofs.

Proposition 1. *Let $(X, \mathscr{U})$, $(Y, \mathscr{V})$ and $(Z, \mathscr{W})$ be uniform spaces. Then we have*

(10) $\mathrm{id}_X \in F(X, X)$;

(11) *if $f \in F(X, Y)$ and $g \in F(Y, Z)$ then $g \circ f \in F(X, Z)$;*

(12) *if $f \in F(X, Y)$ and $A \subset X$ then $f \restriction A \in F(A, Y)$;*

(13) *if $f, g \in F(X, \mathbb{R})$ and $c \in \mathbb{R}$ then $cf, f{+}g, fg \in F(X, \mathbb{R})$.*

We omit standard proofs.

Denote by $U(X, Y)$ and $C(X, Y)$ the family of all uniformly continuous and continuous mappings $f\colon X \to Y$, respectively. We can investigate the family $\{f\colon X \to Y : S\,\mathrm{R}_1\,T \Rightarrow f \circ S\,\mathrm{R}_2\,f \circ T\}$ where R_1 and R_2 mean the parallelness or the equivalency or the weak equivalency.

Proposition 2. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Then*

$$U(X, Y) = \big\{f\colon X \to Y:\ \ S \| T \Rightarrow f \circ S \| f \circ T\big\}.$$

Proof. Necessity. Let $f \in U(X, Y)$. Let $S\colon (D, \succeq) \to X$ and $T\colon (D, \succeq) \to X$ be nets such that $S \| T$. Let $V \in \mathscr{V}$. Then there is $U \in \mathscr{U}$ such that $(x, y) \in U$ implies $\big(f(x), f(y)\big) \in V$. There is $k \in D$ such that for $m \succeq k$ we have $\big(S(m), T(m)\big) \in U$ and therefore $\big(f\big(S(m)\big), f\big(T(m)\big)\big) \in V$, i.e., $f \circ S \| f \circ T$.

Sufficiency. Let $f \notin U(X, Y)$. Then there is $V \in \mathscr{V}$ such that for each $U \in \mathscr{U}$ there are $x_U, y_U \in X$ with $(x_U, y_U) \in U$ and $\big(f(x_U), f(y_U)\big) \notin V$.

Let $W \succeq U$ if $W \subset U$. Then $(\mathscr{U}, \succeq)$ is a directed set. Put $S(U) = x_U$, $T(U) = y_U$ for each $U \in \mathscr{U}$. If $U \in \mathscr{U}$ then for each $W \in \mathscr{U}$, $W \succeq U$ we have $\big(S(W), T(W)\big) = (x_W, y_W) \in W \subset U$, therefore $S \| T$. However the nets $f \circ S$ and $f \circ T$ are not parallel. $\square$

Proposition 3. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Then*

$$\begin{aligned} F(X,Y) &= \{f \colon X \to Y : \ S \approx T \Rightarrow f \circ S \approx f \circ T\} \\ &= \{f \colon X \to Y : \ S \sim T \Rightarrow f \circ S \sim f \circ T\}. \end{aligned}$$

Proof. Let $f \in F(X,Y)$. Let $S \colon (D, \succeq_D) \to X$ and $T \colon (E, \succeq_E) \to X$ be equivalent nets in X. Put $A = \{1,2\}$. If we define $(i,p) \succeq (j,q)$ if $p \succeq_D q$ for all $(i,p),(j,q) \in A \times D$ then $(A \times D, \succeq)$ is a directed set. Define a net $P \colon (A \times D, \succeq) \to X$ as follows:

$$P\big((1,p)\big) = S(p) \qquad \text{and} \qquad P\big((2,p)\big) = T(p) \quad \text{for each } p \in D.$$

We shall show that $P \in F_X$. Let $U \in \mathscr{U}$. Then there is a symmetric $W \in \mathscr{U}$ with $W \circ W \subset U$. Since $S \approx T$, there is $k \in D$ such that $\big(S(n), T(m)\big) \in W$ for $n, m \succeq_D k$. Therefore for $n, m \succeq_D k$ we obtain $\big(P\big((1,n)\big), P(1,m))\big) = \big(S(n), S(m)\big) \in W \circ W \subset U$, $\big(P\big((2,n)\big), P\big((2,m)\big)\big) = \big(T(n), T(m)\big) \in W \circ W \subset U$, $\big(P\big((1,n)\big), P\big((2,m)\big)\big) = \big(T(n), S(m)\big) \in W \subset W \circ W \subset U$, $\big(P\big((2,n)\big), P\big((1,m)\big)\big) = \big(T(n), S(m)\big) \in U$. For $(i,n),(j,m) \in A \times D$, (i,n), $(j,m) \succeq (1,k)$ we get $\big(P\big((i,n)\big), P\big((j,m)\big)\big) \in U$, i.e., $P \in F_X$.

Since $f \in F(X,Y)$ we obtain $f \circ P \in F_Y$. Now let $V \in \mathscr{V}$. Then there is $(a,r) \in A \times D$ such that $\big((f \circ P)\big((i,n)\big), (f \circ P)\big((j,m)\big)\big) \in V$ for all $(i,n),(j,m) \succeq (a,r)$. Therefore for $n, m \succeq_D r$ we get $\big((f \circ S)(n), (f \circ T)(m)\big) = \Big(f\big(S(n)\big), f\big(T(m)\big)\Big) = \Big(f\big(P\big((1,n)\big)\big), f\big(P\big((2,m)\big)\big)\Big) = \Big((f \circ P)\big((1,n)\big), (f \circ P)\big((2,m)\big)\Big) \in V$, i.e., $f \circ S \approx f \circ T$ and $f \in \{f \colon X \to Y : S \approx T \Rightarrow f \circ S \approx f \circ T\}$.

Let $f \in \{f \colon X \to Y : \ S \approx T \Rightarrow f \circ S \approx f \circ T\}$ and let $S \colon (D, \succeq_D) \to X$ and $T \colon (E \succeq_E) \to X$ be nets in X such that $S \sim T$. Then $(D \times E, \succeq)$, where $(a,b) \succeq (c,d)$ if and only if $a \succeq_D c$ and $b \succeq_E d$, is a directed set. Define nets $P \colon (D \times E, \succeq) \to X$ and $Q \colon (D \times E, \succeq) \to X$ in X as follows

$$P\big((p,q)\big) = S(p), \qquad Q\big((p,q)\big) = T(q).$$

We shall show that $P \approx Q$. Let $U \in \mathscr{U}$. Since $S \sim T$, there are $a \in D$, $b \in E$ such that $\big(S(m), T(n)\big) \in U$ for each $n \in D$, $m \in E$ with $n \succeq_D a$ and $m \succeq_E b$. Now let $(p,q),(r,s) \in D \times E$, $(p,q),(r,s) \succeq (a,b)$. Then $\big(P\big((p,q)\big), Q\big((r,s)\big)\big) = \big(S(p), T(s)\big) \in U$, i.e., $P \approx Q$. Hence $f \circ P \approx f \circ Q$. Let $V \in \mathscr{V}$. Then there is $(c,d) \in D \times E$ such that $\big((f \circ P)\big((p,q)\big), (f \circ Q)\big((r,s)\big)\big) \in V$ for each $(p,q),(r,s) \in D \times E$ with $(p,q),(r,s) \succeq (c,d)$.

If $m \in D$, $n \in E$ are such that $m \succeq_D c$ and $n \succeq_E d$, then $\big((f \circ S)(m), (f \circ T)(n)\big) = \Big(f\big(S(m)\big), f\big(T(n)\big)\Big) = \Big(f\big(P((m,d))\big), f\big(Q((c,n))\big)\Big) = \Big((f \circ P)\big((m,d)\big), (f \circ Q)\big((c,n)\big)\Big) \in V$, i.e., $f \circ P \approx f \circ Q$. This yields $f \circ S \sim f \circ T$ and $f \in \{f\colon X \to Y:\ S \sim T \Rightarrow f \circ S \sim f \circ T\}$.

If $f \in \{f\colon X \to Y:\ S \sim T \Rightarrow f \circ S \sim f \circ T\}$ and $S \in F_X$, then $S \sim S$ and $f \circ S \sim f \circ S$ by assumptions. Now (5) implies $f \circ S \in F_Y$ and $f \in F(X,Y)$. This completes the proof. □

We say that a mapping $f\colon X \to Y$ (where X is a set and $(Y, \mathscr{V})$ is a uniform space) is pseudoconstant if for an arbitrary $V \in \mathscr{V}$ and each $x, y \in X$ we have $\big(f(x), f(y)\big) \in V$. Evidently, every constant mapping is pseudoconstant and if $(Y, \mathscr{V})$ is a Hausdorff uniform space then pseudoconstant mappings are constant.

Proposition 4. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Then $f \in \{f\colon X \to Y:\ S \| T \Rightarrow f \circ S \approx f \circ T\}$ if and only if f is pseudoconstant.*

Proof. Assume that f is not pseudoconstant. Therefore there is $V \in \mathscr{V}$ and points $a, b \in X$ such that $\big(f(x), f(y)\big) \notin V$. Define sequences S and T as: $S(n) = a$ for n even, $S(n) = b$ for n odd and $T = S$. Then S and T are parallel but $f \circ S$ and $f \circ T$ are not equivalent. □

Proposition 5. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Then $F(X,Y) = \{f\colon X \to Y:\ S \approx T \Rightarrow f \circ S \| f \circ T\}$.*

Proof. According to Proposition 3 and (3), $F(X,Y) \subset \{f\colon X \to Y:\ S \approx T \Rightarrow f \circ S \| f \circ T\}$. Now assume that $f \notin F(X,Y)$. Therefore there is a net $P = (P, D) \in F_X$ such that $f \circ P \notin F_Y$. Thus there is $V \in \mathscr{V}$ such that for each $p \in D$ there are $n_p, m_p \in D$ such that $n_p, m_p \succeq_D p$ and $\big((f \circ P)(n_p), (f \circ P)(m_p)\big) \notin V$. Let $u(p) = n_p$ and $v(p) = m_p$ for $p \in D$. If $k \in D$, then for $p \succeq_D n_k$ we have $u(p) = n_p \succeq_D p \succeq_D n_k \succeq_D k$ and this yields $u \in N(D,D)$, similarly $v \in N(D,D)$. The nets $P \circ u$ and $P \circ v$ are subnets of the Cauchy net P and therefore by (8) and (7) we obtain $P \sim P \circ u$, $P \sim P \circ v$, $P \circ u \sim P \circ v$ and $P \circ u \approx P \circ v$. However $\big((f \circ P \circ u)(p), (f \circ P \circ v)(p)\big) = \big((f \circ P)(n_p), (f \circ P)(m_p)\big) \notin V$ for each $p \in D$ and therefore $f \notin \{f\colon X \to Y:\ S \approx T \Rightarrow f \circ S \| f \circ T\}$. □

If $f \in F(X,Y)$ then obviously f preserves Cauchy sequences but not conversely. However, if a uniformity on X has a countable base then also converse implication is true.

Example 1. Let X be the set of all ordinals which are less than the first uncountable ordinal Ω and let $Y = \mathbb{R}$ with the usual metric. If we denote $U_a = \{(x,y)\colon x = y \text{ or } x \geq a \text{ and } y \geq a\}$, then the family of all sets of the form U_a is a base for a uniformity $\mathscr{U}$ on X ([2; p. 204]). Define $f\colon X \to Y$ as follows: $f(a) = 1$ if a is a limit ordinal and $f(a) = 0$ if a is not a limit ordinal.

Then it is easy to see that f preserve Cauchy sequences (only Cauchy sequences in X are constant) however, for the Cauchy net $T\colon (X, \geq) \to X$, $T(a) = a$, the image $f \circ T$ is not a Cauchy net in Y.

THEOREM 1. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. If the uniformity $\mathscr{U}$ has a countable base then $f \in F(X,Y)$ if and only if f preserves Cauchy sequences.*

Proof. Necessity is obvious. Sufficiency. Let $\mathscr{B} = \{U_n \colon n \in \mathbb{N}\}$ be a symmetric base for $\mathscr{U}$ such that $U_{n+1} \subset U_n$ for each $n \in \mathbb{N}$. Assume that $f \notin F(X,Y)$. Then according to Proposition 3 there are nets $S\colon (D, \succeq) \to X$, $T\colon (D, \succeq) \to X$ such that S and T are equivalent but $f \circ S$ and $f \circ T$ are not equivalent. Therefore there is $V \in \mathscr{V}$ such that for each $k \in D$ we can find $i_k, m_k \in D$ such that $i_k, m_k \succeq k$ and $\big((f \circ S)(i_k), (f \circ T)(m_k)\big) \notin V$. Further we can find a sequence (p_n), $p_n \in D$, such that $p_{n+1} \succeq p_n$ and $\big(S(i), T(m)\big) \in U_n$ for all $i, m \in D$, $i, m \succeq p_n$.

Put $P(n) = S(i_{p_n})$ and $Q(n) = T(m_{p_n})$ for each $n \in \mathbb{N}$. If $U \in \mathscr{U}$ then there is $U_k \in \mathscr{B}$ with $U_k \subset U$. For $r, s \succeq k$ we have $i_{p_r} \succeq p_r \succeq p_k$, $m_{p_s} \succeq p_s \succeq p_k$ and hence $\big(P(r), Q(s)\big) = \big(S(i_{p_r}), Q(m_{p_s})\big) \in U_k \subset U$.

Put $R(2n) = P(n)$ and $R(2n-1) = Q(n)$. It is easy to see that the sequence R is Cauchy. Hence $f \circ R$ is a Cauchy sequence in Y. Therefore there is $a \in \mathbb{N}$ such that $(f \circ R)(b)$, $(f \circ R)(c) \in V$ for $b, c \geq a$. Especially, $\Big(f\big(R(2a)\big), f\big(R(2a-1)\big)\Big) = \Big(f\big(P(a)\big), f\big(Q(a)\big)\Big) = \Big(f\big(S(i_{p_a})\big), f\big(T(m_{p_a})\big)\Big) \in V$, a contradiction. □

THEOREM 2. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Then*

$$U(X,Y) \subset F(X,Y) \subset C(X,Y).$$

Proof. The inclusion $U(X,Y) \subset F(X,Y)$ follows from Propositions 2 and 5. Let $f \in F(X,Y)$ and $x \in X$. If S is a net in X converging to x, then according to Lemma 1 we obtain $S \sim x$, $f \circ S \sim f(x)$ and therefore $f \circ S$ converges to $f(x)$, i.e., f is continuous. □

Remark 1. These inclusions are proper in general. In [1], a characterization of spaces X, for which the equalities $U(X,Y) = F(X,Y)$ or $F(X,Y) = C(X,Y)$ hold, is given (X is assumed to be a metric space and Y is assumed to be a nontrivial normed linear space).

THEOREM 3. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Let $f\colon X \to Y$ be a bijection such that $f \in F(X,Y)$ and $f^{-1} \in F(Y,X)$. Then $(X, \mathscr{U})$ is a complete uniform space if and only if $(Y, \mathscr{V})$ is a complete uniform space.*

Proof. Suppose that $(X, \mathscr{U})$ is a complete uniform space and $(Y, \mathscr{V})$ is not complete. Then there is a Cauchy net T in Y which does not converge. The net

$S = f^{-1} \circ T$ is Cauchy in X and hence converges to some point $x \in X$. However f is continuous and thus $f \circ S = T$ converges to $f(x) \in Y$, a contradiction. □

Lemma 2. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Let $f \colon X \to Y$ be a mapping and let $S \colon (D, \succeq_D) \to X$ be a Cauchy net in X. Then the following conditions are equivalent:*

(14) *If T is weakly equivalent to S then $f \circ S$ is weakly equivalent with $f \circ T$;*

(15) *For each $V \in \mathscr{V}$ there is $U \in \mathscr{U}$ and $k \in D$ such that $\big(f\big(S(m)\big), f(y)\big) \in V$ for each $m \succeq k$ and each $y \in X$ with $\big(S(m), y\big) \in U$.*

Proof. (15) $\Rightarrow$ (14): Let $T \colon (E \succeq_E) \to X$ be a net such that $S \sim T$. Let $V \in \mathscr{V}$. Then there are $U \in \mathscr{U}$ and $k \in D$ such that $\big(f\big(S(m)\big), f(y)\big) \in V$ for each $m \succeq_D k$ and each $y \in X$ with $\big(S(m), y\big) \in U$. There are $n \in D$ and $a \in E$ such that $\big(S(m), T(b)\big) \in U$ for $m \succeq_D n$ and $b \succeq_E a$. Let $m \succeq_D k$, $m \succeq_D n$ and $b \succeq_E a$. Then $\big(S(m), T(b)\big) \in U$ and hence $\big(f\big(S(m)\big), f\big(T(b)\big)\big) \in V$, i.e., $f \circ S \sim f \circ T$.

(14) $\Rightarrow$ (15): Assume that (15) does not hold. Therefore there is $V \in \mathscr{V}$ such that for each $U \in \mathscr{U}$ and each $k \in D$ we can find $n_{U,k} \succeq_D k$ and $y_{U,k} \in X$ for which

$$\big(S(n_{U,k}), y_{U,k}\big) \in U \quad \text{and} \quad \big(f\big(S(n_{U,k})\big), f(y_{U,k})\big) \notin V .$$

For $(W, a), (Z, b) \in \mathscr{U} \times D$, let $(Z, b) \succeq (W, a)$ if $Z \subset W$ and $b \succeq_D a$. Then $\mathscr{U} \times D$ is a directed set. Put

$$T(U, k) = y_{U,k} .$$

We shall show that $S \sim T$. Let $U \in \mathscr{U}$. Let $Z \in \mathscr{U}$ be a symmetric set such that $Z \circ Z \subset U$. There is $p \in D$ such that $\big(S(i), S(m)\big) \in Z$ for $i, m \succeq_D p$. If $i \succeq_D p$ and $(W, m) \succeq (Z, p)$, then $n_{W,m} \succeq_D m \succeq_D p$ and hence $\big(S(i), S(n_{W,m})\big) \in Z$. Further $\big(S(n_{W,m}), y_{W,m}\big) \in W \subset Z$ and therefore $\big(S(i), T\big((W, m)\big)\big) = \big(S(i), y_{W,m}\big) \in Z \circ Z \subset U$.

From (14) we get $f \circ S \sim f \circ T$. Therefore there are $r \in D$ and $(Z, s) \in \mathscr{U} \times D$ such that $\big((f \circ S)(i), (f \circ T)(m)\big) \in V$ for $i \succeq_D r$ and $(W, m) \succeq (Z, s)$. If $k \in D$ is such that $k \succeq_D r$ and $k \succeq_D s$, then $n_{Z,k} \succeq_D k \succeq_D r$ and $(Z, k) \succeq (Z, s)$ and therefore $\big(f\big(S(n_{Z,k})\big), f(y_{Z,k})\big) = \big((f \circ S)(n_{Z,k}), (f \circ T)\big((Z, k)\big)\big) \in V$, a contradiction. □

Proposition 6. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces.. Let $f \colon X \to Y$ be a mapping. Then $f \in F(X, Y)$ if and only if for each net $S \colon (D, \succeq_D) \to X$, $S \in F_X$ and for each $V \in \mathscr{V}$ there are $U \in \mathscr{U}$ and $k \in D$ such that $\big((f \circ S)(m), (f \circ S)(n)\big) \in V$ for all $n, m \succeq_D k$ with $\big(S(m), S(n)\big) \in U$.*

Proof. Necessity follows from Proposition 3 and Lemma 2.
Sufficiency. Let $S \colon (D, \succeq_D) \to X$, $S \in F_X$ and let $V \in \mathscr{V}$. Then there are

$U \in \mathscr{U}$ and $k \in D$ such that $\big((f \circ S)(m), (f \circ S)(n)\big) \in U$ for all $n, m \succeq_D k$ with $\big(S(m), S(n)\big) \in U$. There is $p \in D$ such that $\big(S(n), S(m)\big) \in U$ for all $n, m \succeq_D p$. Therefore for $m, n \in D$, $n, m \succeq_D k$, $n, m \succeq_D p$ we have $\Big(f\big(S(n)\big), f\big(S(m)\big)\Big) \in V$, i.e., $f \circ S \in F_Y$ and $f \in F(X, Y)$. $\square$

Proposition 7. *Let $(X, \mathscr{U})$ be a uniform space and let $(Y, \mathscr{V})$ be a complete Hausdorff space. Let $A \subset X$ and $f \in F(A, Y)$. Then there is a unique mapping $\bar{f} \in F(\operatorname{Cl} A, Y)$ for which $\bar{f} \restriction A = f$.*

Proof. Let $x \in \operatorname{Cl} A$. Then there is a net S in A converging to x. Then $f \circ S \in F_Y$ and hence $f \circ S$ converges to some $y \in Y$. Since Y is Hausdorff, y is uniquely determined. Put $\bar{f}(x) = y$. If T is another net in A converging to x, then by Lemma 1 $S \sim x$, $T \sim x$ and $S \sim T$. Hence $f \circ S \sim f \circ T$. Since $f \circ S \sim y$, also $f \circ T \sim y$. Therefore $f \circ T$ converges to y.

Now we shall show that $\bar{f} \in F(\operatorname{Cl} A, Y)$. Let $S\colon (D \succeq_D) \to \operatorname{Cl} A$ be a Cauchy net in $\operatorname{Cl} A$. Then for each $p \in D$ there is a net $T_p\colon (D, \succeq_{D_p}) \to A$ in A which converges to $S(p)$. Therefore for each $U \in \mathscr{U}$ there is $n_{p,U} \in D_p$ such that $\big(T_p(n), S(p)\big) \in U$ for each $n \succeq_{D_p} n_{p,U}$. For $(p, U), (q, V) \in D \times \mathscr{U}$, let $(q, V) \succeq (p, U)$ if $q \succeq_D p$ and $V \subset U$. Then $(D \times \mathscr{U}, \succeq)$ is a directed set. Put $P\big((p, U)\big) = T_p(n_p, U)$, $Q\big((p, U)\big) = S(p)$. It is easy to see that $S \sim Q$. Hence $Q \in F_X$. Let $U \in \mathscr{U}$ and $p \in D$. Then $\Big(P\big((m, W)\big), Q\big((m, W)\big)\Big) = \big(T_m(n_{m,W}), S(m)\big) \in W \subset U$ for $(m, W) \succeq (p, U)$, therefore $P \| Q$. By Lemma 1 we get $P \in F_X$. Since P is a net in A we get $f \circ P \in F_Y$. Since $f \circ T_p$ converges to $\bar{f}\big(S(p)\big)$, we can similarly show that $f \circ P \| \bar{f} \circ S$. Hence $\bar{f} \circ S \in F_Y$ and $\bar{f} \in F(\operatorname{Cl} A, Y)$.

Since dense sets are determining sets for continuous functions (if Y is Hausdorff), by Theorem 2, such function $\bar{f}$ is unique. $\square$

Proposition 8. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Let $f \in F(X, Y)$. Then $f \restriction A \in U(A, Y)$ for each totally bounded set A in X.*

Proof. Assume that there is a totally bounded set $A \subset X$ and $f \in F(X, Y)$ such that $f \restriction A$ is not uniformly continuous. Therefore there is $V \in \mathscr{V}$ such that for each $U \in \mathscr{U}$ we can find points $x_U, y_U \in A$ with $(x_U, y_U) \in U$ and $\big(f(x_U), f(y_U)\big) \notin V$. For $U, W \in \mathscr{U}$, let $W \succeq U$ if $W \subset U$. Then $(\mathscr{U}, \succeq)$ is a directed set. Put $S(U) = x_U$, $T(U) = y_U$ for each $U \in \mathscr{U}$. Then S and T are nets in A and $S \| T$. Since A is totally bounded, the net S has a Cauchy subnet $S \circ u$ (where $u \in N(E, D)$ for some directed set E). Then $S \circ u \| T \circ u$ and $S \circ u \approx T \circ u$. Hence $f \circ S \circ u \approx f \circ T \circ u$, a contradiction. $\square$

Corollary 1. *If $f \in F(X, Y)$ and A is a totally bounded subset of X then $f(A)$ is a totally bounded subset of Y.*

Remark 2. The statement in Proposition 8 cannot be converted. If f and X are as in Example 1, then $f \notin F(X,Y)$. However, each totally bounded set in X is finite and therefore f is uniformly continuous on every totally bounded set in X. Further, if X is a metric space, then also converse statement is true (cf. Theorem 4).

For subsets A and B of a uniform space $(X, \mathscr{U})$ define $A \ \delta \ B$ if for each $U \in \mathscr{U}$ there are $a \in A$, $b \in B$ such that $(a,b) \in U$. (In particular, if (X,d) is a metric space, then $A \ \delta \ B$ if $d(A,B) = \inf\{d(a,b) \colon a \in A,\ b \in B\} = 0$).

PROPOSITION 9. *Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Let $f \in F(X,Y)$. If A and B are totally bounded subsets of X and $A \ \delta \ B$, then $f(A) \ \delta \ f(B)$.*

Proof. Let A, B be totally bounded subsets of X and $A \ \delta \ B$. Then for each $U \in \mathscr{U}$ there are points $a_U \in A$, $b_U \in B$ such that $(a_U, b_U) \in U$. For $U, V \in \mathscr{U}$, let $V \succeq U$ if $V \subset U$. Then $(\mathscr{U}, \succeq)$ is a directed set. If $S(u) = a_U$, $T(u) = b_U$ then S and T are parallel nets in A. Since A is totally bounded, the net S has a Cauchy subnet $S \circ u$ (where $u \in N(E,D)$ for some directed set E). By Lemma 1 we have $S \circ u \approx T \circ u$. By Proposition 5 we obtain $f \circ (S \circ u) \| f \circ (T \circ u)$. Therefore for each $V \in \mathscr{V}$ there is $p \in D$ such that $\Big(f\big(S\big(u(p)\big)\big),\ \big(T\big(u(p)\big)\big)\Big) \in V$, i.e., $f(A) \ \delta \ f(B)$. □

Remark 3. The statement in Proposition 9 cannot be converted. Let f and X be as in Example 1. If A and B are totally bounded in X and $A \ \delta \ B$, then $A \cap B \neq \emptyset$ and hence $f(A) \ \delta \ f(B)$, but $f \notin F(X,Y)$. However if X and Y are metric spaces, then also converse statement is true (cf. Theorem 4).

The well-known Blumberg's Theorem says that for each Baire metric space X and each real function $f \colon X \to \mathbb{R}$ there is a dense set D in X such that $f \restriction D$ is continuous. However we cannot require $f \restriction D$ to be Cauchy sequences preserving (cf. Theorem 4). We recall that a mapping $f \colon X \to Y$ (where X and Y are topological spaces) is quasicontinuous at $x \in X$ if for each neighbourhood V of $f(x)$ and for each neighbourhood U of x there is a nonempty open subset G of U such that $f(G) \subset V$; f is quasicontinuous if it is quasicontinuous at each point.

THEOREM 4. *Let (X, d_X) and (Y, d_Y) be pseudometric spaces and let $f \colon X \to Y$ be a mapping. Then the following conditions are equivalent:*

(16) $f \in F(X,Y)$;

(17) $f \restriction A \in F(A,Y)$ *for each totally bounded set A in X;*

(18) $f \restriction A$ *is uniformly continuous for each totally bounded set A in X;*

(19) *if A and B are totally bounded sets in X and $d_X(A,B) = 0$ then $d_Y\big(f(A), f(B)\big) = 0$;*

(20) f *is quasicontinuous and there is a dense set* H *in* X *such that* $f \restriction H \in F(H,Y)$.

Proof.

(16)$\Rightarrow$(18): From Proposition 8.

(18)$\Rightarrow$(17): From Theorem 2.

(17)$\Rightarrow$(16): Let S be a Cauchy sequence in X Then the set $A = \{S(n)\colon n \in \mathbb{N}\}$ is totally bounded and hence $f \restriction A \in F(A,Y)$. Now $S \in F_A$ implies $f \circ S \in F_Y$. According to Theorem 1, $f \in F(X,Y)$.

(16)$\Rightarrow$(19): From Proposition 9.

(19)$\Rightarrow$(16): Assume that $f \notin F(X,Y)$. Then there is a Cauchy sequence S in X such that $f \circ S \notin F_Y$. Therefore there is $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ we can find $m_n, i_n \geq n$ with $d_Y\big(f\big(S(m_n)\big), f\big(S(i_n)\big)\big) \geq 3\varepsilon$. Denote $C = \{S(n)\colon n \in \mathbb{N}\}$. Then C is a totally bounded set. For each $b \in C$ we put

$$\begin{aligned} V(b) &= \big\{a \in C\colon d_Y\big(f(a), f(b)\big) < \varepsilon\big\}\,, \\ W(b) &= \big\{a \in C\colon d_Y\big(f(a), f(b)\big) \geq 3\varepsilon\big\}\,. \end{aligned}$$

For each $b \in C$, the sets $V(b)$ and $W(b)$ are totally bounded and $V(b)$ is nonempty. If $W(b)$ is nonempty then $d_Y\big(f\big(V(b)\big),\, f\big(W(b)\big)\big) \geq \varepsilon$ and $d_X\big(V(b), W(b)\big) > 0$ by assumptions.

Now we shall construct by induction sequences (δ_k), (y_k) and (z_k) such that (δ_k) is a decreasing sequence of positive reals converging to zero and $y_k, z_k \in C$ are points such that $d_X(y_k, z_k) < \delta_k$ and $d_Y\big(f(y_k), f(z_k)\big) \geq 3\varepsilon$.

Put $\delta_1 = 1/2$. If $j_1 \in \mathbb{N}$, $m_{j_1}, i_{j_1} \geq j_1$ are such that $d_X\big(S(m_{j_1}), S(i_{j_1})\big) < \delta_1$ and $d_Y\big(f\big(S(m_{j_1})\big), f(i_{j_1})\big) \geq 3\varepsilon$ we put $y_1 = S(m_{j_1})$, $z_1 = S(i_{j_1})$.

Assume that we have constructed δ_i, y_i, z_i for $i \leq k$. Since $d_X(y_k, z_k) < \delta_k$ and $d_Y\big(f(y_k), f(z_k)\big) \geq 3\varepsilon$, we have $y_k \in W(z_k)$ and therefore $d_X\big(V(z_k), W(z_k)\big) > 0$. Similarly $d_X\big(V(y_k), W(y_k)\big) > 0$. Put

$$\delta_{k+1} = \min\big\{\delta_k/2,\, d_X\big(V(y_k), W(y_k)\big),\, d_X\big(V(z_k), W(z_k)\big)\big\}\,.$$

Then $0 < \delta_{k+1} < \delta_k \leq 2^{-k}$. Since S is Cauchy, there is $j_{k+1} \in \mathbb{N}$ such that

$d_X(S(i), S(m)) < \delta_{k+1}$ for all $i, m \geq j_{k+1}$. Then $d_Y\Big(f(S(m_{j_{k+1}})),\ f(S(i_{j_{k+1}}))\Big)$ $\geq 3\varepsilon$. Put

$$y_{k+1} = S(m_{j_{k+1}}), \qquad z_{k+1} = S(i_{j_{k+1}}).$$

Then $d_X(y_{k+1}, z_{k+1}) < \delta_{k+1}$, $d_Y(f(y_{k+1}), f(z_{k+1})) \geq 3\varepsilon$ and $y_{k+1}, z_{k+1} \in C$.

Put $A = \{y_k : k \in \mathbb{N}\}$ and $B = \{z_k : k \in \mathbb{N}\}$. Then A and B are totally bounded sets and $d_X(A, B) = 0$.

We shall show that $d_Y(f(A), f(B)) \geq \varepsilon$. Suppose that $d_Y(f(A), f(B)) < \varepsilon$. Then there are $i, j \in \mathbb{N}$ such that $d_Y(f(y_i), f(z_j)) < \varepsilon$. Since $d_Y(f(y_i), f(z_i)) \geq 3\varepsilon$, we have $i \neq j$. For example, let $j < i$. Then $y_i \in V(z_j)$. We obtain

$$d_X(y_i, z_i) < \delta_i \leq \delta_{j+1} \leq d_X(V(z_j), W(z_j)).$$

If $z_i \in W(z_j)$, then $d_X(V(z_j), W(z_j)) \leq d_X(y_i, z_i)$. Therefore $z_i \notin W(z_j)$ and this implies $d_Y(f(z_i), f(z_j)) < 2\varepsilon$ and thus $d_Y(f(y_i), f(z_i)) \leq d_Y(f(y_i), f(z_j)) + d_Y(f(z_j), f(z_i)) < \varepsilon + 2\varepsilon = 3\varepsilon$, a contradiction. Similarly for $j > i$. Therefore $d_Y(f(A), f(B)) \geq \varepsilon > 0$.

$(16) \Rightarrow (20)$: Proposition 1, Theorem 2 and the fact that the continuity implies the quasicontinuity.

$(20) \Rightarrow (16)$: Let S be a Cauchy sequence in X and let $\varepsilon > 0$. Fix $k \in \mathbb{N}$. Since f is quasicontinuous at $S(k)$, there is nonempty open set $G_k \subset \{y \in X : d_X(S(k), y) < 1/k\}$ such that $d_Y(f(S(k)), f(y)) < \varepsilon/3$ for each $y \in G_k$. Since H is dense in X, there is $y_k \in G_k \cap H$. Then $d_X(S(k), y_k) < 1/k$ and $d_Y(f(S(k)),\ f(y_k)) < \varepsilon/3$.

Put $T(k) = y_k$. Then T is a sequence in H and $S \| T$. According to Lemma 1, we obtain $T \in F_H$. Hence $f \circ T \in F_Y$. Therefore there is $m \in \mathbb{N}$ such that $d_Y((f \circ T)(i), (f \circ T)(j)) < \varepsilon/3$ for each $i, j \geq m$. Now for $i, j \geq m$ we obtain

$$\begin{aligned} & d_Y((f \circ S)(i), (f \circ S)(j)) \\ & \quad \leq d_Y(f(S(i)), f(T(i))) + d_Y(f(T(i)), f(T(j))) + d_Y(f(T(j)), f(S(j))) \\ & \qquad\qquad\qquad\qquad < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon, \end{aligned}$$

i.e, $f \in F(X, Y)$. □

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