Tatra Mt. Math. Publ. **19** (2000), 51–62

TATRA MOUNTAINS
Mathematical Publications

STATISTICAL CONVERGENCE AND POROSITY OF SETS

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ABSTRACT. The porosity of the class of statistically convergent sequences of real numbers is investigated in this paper. This is done in the space of all bounded sequences (with the sup-norm) and also in the space of all sequences of real numbers endowed with Fréchet's metric.

Introduction

The concept of statistical convergence was introduced in papers [4], [12] and further studied in several other papers (e.g., [1], [2], [3], [5], [6], [8], [10], [11], [14]). The structure of some sequence spaces from the point of view of statistical convergence has been investigated in papers [11], [14].

Definitions and notations

We shall use the concept of asymptotic density of sets of positive integers. If $A \subseteq N$ then $A(n)$ denotes the number of elements of $a \in A$ for which $a \leq n$. Then we put

$$d(A) = \lim_{n\to\infty} \frac{A(n)}{n}$$

if this limit exists. The number $d(A)$ is called the *asymptotic density* of A. The numbers

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(n)}{n}\,, \quad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(n)}{n}$$

are called *lower* and *upper asymptotic density* of A, respectively. Obviously each of these numbers belongs to $[0,1]$ (cf [7], xix).

2000 Mathematics Subject Classification: 40A05, 54E35, 54E52.
Keywords: asymptotic density, statistical convergence, h-porosity.

The work on this paper was supported by grants VEGA 1/4314/97 and S2M01G.

Definition A. A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers *converges statistically* to $\xi \in \mathbb{R}$ if for every $\varepsilon > 0$ we have $d(A(\varepsilon)) = 0$, where

$$A(\varepsilon) = \{n : |x_n - \xi| \geq \varepsilon\}.$$

If x converges statistically to ξ, then we write lim-stat $x_n = \xi$ (cf [4], [11]).

Obviously the statistical convergence is a natural generalization of the usual convergence of sequences.

The position of statistically convergent sequences in the spaces m (of all bounded sequences) and s (of all sequences) was described in [11]. The aim of this paper is to complete the results from [11] by investigation of porosity of sets C^0_{stat} (the set of all bounded statistically convergent sequences) in m and C_{stat} (set of all statistically convergent sequences in the space s). The concept of porosity we shall use by analogy with [13], [15].

We introduce the concept of h-porosity which will be useful for investigation of porosity of the set C_{stat} in s.

Let (Y, ρ) be a metric space. Suppose that $h : [0, +\infty) \to \mathbb{R}$ is a non-negative continuous function, increasing on $[0, +\infty)$ with $h(0) = 0$, $h(x) > 0$ for $x > 0$.

If $y \in Y$, $\delta > 0$, we denote $B(y, \delta) = \{x \in Y : \rho(x, y) < \delta\}$ (the ball in Y).

Let $M \subseteq Y$, $y \in Y$, $\delta > 0$. Put

$$\gamma(y, \delta, M) = \sup\{t > 0 : \exists z \in B(y, \delta) : [B(z, t) \subseteq B(y, \delta)] \wedge [B(z, t) \cap M = \emptyset]\}.$$

If such a $t > 0$ does not exist, then we put $\gamma(y, \delta, M) = 0$. Further let $\gamma_h(y, \delta, M) = h(\gamma(y, \delta, M))$.

We define

$$\underline{p}_h(y, M) = \liminf_{\delta \to 0+} \frac{\gamma_h(y, \delta, M)}{h(\delta)}$$

(the *lower h-porosity* of M at y),

$$\overline{p}_h(y, M) = \limsup_{\delta \to 0+} \frac{\gamma_h(y, \delta, M)}{h(\delta)}$$

(the *upper h-porosity* of M at y). If $\underline{p}_h(y, M) = \overline{p}_h(y, M)$ $(= p_h(y, M))$, then $p_h(y, M)$ is said to be the *h-porosity* of M at y.

Obviously $\underline{p}_h(y, M)$, $\overline{p}_h(y, M)$, $p_h(y, M) \in [0, 1]$.

A set $M \subseteq Y$ is said to be *h-porous* and *c-h-porous* at y provided that $\overline{p}_h(y, M) > 0$ and $\overline{p}_h(y, M) \geq c > 0$, respectively.

A set $M \subseteq Y$ is said to be *very h-porous* and *very c-h-porous* at y provided that $\underline{p}_h(y, M) > 0$ and $\underline{p}_h(y, M) \geq c > 0$, respectively.

A set $M \subseteq Y$ is said to be *uniformly σ-h-porous*, *uniformly σ-c-h-porous*, *uniformly σ-very h-porous* and *uniformly σ-very c-h-porous* in $Y_0 \subseteq Y$ if $M =$

$\bigcup\limits_{n=1}^{\infty} M_n$ and each M_n $(n = 1, 2, \dots)$ is h-porous, c-h-porous, very h-porous and very c-h-porous at each $y \in Y_0$, respectively.

It seems to be convenient to define the *uniform σ-h-porosity* of a set $M \subseteq Y$ in a set $Y_0 \subseteq Y$. This means that the expression $M = \bigcup\limits_{n=1}^{\infty} M_n$ is the same for all points of Y_0. In the opposite case, if we admit distinct expression for distinct points of Y_0, we can easily show that even the whole space Y is σ-h-porous in the set Y^d of its accumulation points. Indeed, let $y \in Y^d$. Put $M_1 = \{y\}$, $M_2 = Y \backslash B(y, 1)$, $M_n = B\big(y, \frac{1}{n-2}\big) \backslash B\big(y, \frac{1}{n-1}\big)$ $(n \geq 3)$. Then the decomposition $Y = \bigcup\limits_{n=1}^{\infty} M_n$ of Y shows that Y is σ-h-porous at y.

Obviously, if a set is h-porous at each $y \in Y$, then it is non-dense in Y and if it is σ-h-porous at each $y \in Y$, then it is of the first Baire category in Y.

If $h(x) = x$ for $x \in [0, +\infty)$, then h-porosity coincides with the usual porosity and we write p instead of p_h.

1. Porosity of the set C^0_{stat} in m

In [11] it is shown that the set C^0_{stat} is a closed nowhere dense set in the space m of all bounded sequences endowed by the metric τ,

$$\tau(x, y) = \sup_{n=1,2,\dots} |x_n - y_n|,$$

$x = (x_n)_{n=1}^{\infty} \in m$, $y = (y_n)_{n=1}^{\infty} \in m$ (see Theorem 2.1 in [11]). We complete this result by stating the porosity of C^0_{stat} at points of m.

Theorem 1.1.

(1) *If $y \in m \setminus C^0_{\mathrm{stat}}$ then $p\big(y, C^0_{\mathrm{stat}}\big) = 1$.*

(2) *If $y \in C^0_{\mathrm{stat}}$ then $p\big(y, C^0_{\mathrm{stat}}\big) = \frac{1}{2}$.*

Proof.

(1) This part of Theorem 1.1 is a straightforward consequence of the definition of $p\big(y, C^0_{\mathrm{stat}}\big)$ and Theorem 2.1 from [11].

(2) Let $y \in C^0_{\mathrm{stat}}$. Since $y = (y_n)_{n=1}^{\infty}$ is a bounded sequence, it has a cluster point $\eta \in \mathbb{R}$ (cf. [6]), i.e., there exists an $\eta \in \mathbb{R}$ such that for every $\varepsilon > 0$ we have

$$\overline{d}\big(\{n : |y_n - \eta| < \varepsilon\}\big) > 0 \tag{1}$$

Let $B(y, \delta)$ be a ball with centre y, let $\frac{1}{2} < \alpha < 1$. Put $\varepsilon_0 = (2\alpha - 1)\delta > 0$. Put $\varepsilon = \frac{\varepsilon_0}{2}$ in (1) and $H = \{n : |y_n - \eta| < \varepsilon\} = \{n_1 < n_2 < \dots\}$.

Then by (1) we have

$$|y_{n_k} - y_{n_{k+1}}| < \varepsilon_0 \qquad (k = 1, 2, \dots) \tag{1'}$$

We shall construct the sequence $z = (z_n)_{n=1}^{\infty}$ as follows:

$$z_{n_k} = y_{n_k} + \alpha\delta \quad \text{for } k \text{ even}, \qquad z_{n_k} = y_{n_k} - \alpha\delta \quad \text{for } k \text{ odd}$$

and

$$z_j = y_j \quad \text{for } j \in \mathbb{N} \setminus H. \tag{2}$$

Obviously $z = (z_n)_{n=1}^{\infty} \in B(y, \delta)$. Construct the ball $B(z, (1-\alpha)\delta)$. If $t = (t_n)_{n=1}^{\infty} \in B(z, (1-\alpha)\delta)$, then by triangular inequality

$$\tau(t, y) \le \tau(t, z) + \tau(z, y) < (1-\alpha)\delta + \alpha\delta = \delta.$$

Hence

$$B(z, (1-\alpha)\delta) \subseteq B(y, \delta). \tag{3}$$

Let $t = (t_n)_{n=1}^{\infty} \in B(z, (1-\alpha)\delta)$. We shall estimate the numbers t_{n_k} $(k = 1, 2, \dots)$. For even k we have by definition of the metric τ

$$t_{n_k} \ge z_{n_k} - (1-\alpha)\delta = y_{n_k} + \alpha\delta - \delta + \alpha\delta = y_{n_k} + (2\alpha - 1)\delta = y_{n_k} + \varepsilon_0$$

and for k odd

$$t_{n_k} \le z_{n_k} + (1-\alpha)\delta = y_{n_k} - \alpha\delta + \delta - \alpha\delta = y_{n_k} - (2\alpha - 1)\delta = y_{n_k} - \varepsilon_0.$$

Hence for k even we have

$$t_{n_k} \ge y_{n_k} + \varepsilon_0, \tag{4}$$

$$t_{n_{k+1}} \le y_{n_{k+1}} - \varepsilon_0. \tag{4'}$$

From (4), (4') using (1') we get

$$t_{n_k} - t_{n_{k+1}} \ge y_{n_k} + \varepsilon_0 - (y_{n_{k+1}} - \varepsilon_0) \ge (y_{n_k} - y_{n_{k+1}}) + 2\varepsilon_0 \ge \varepsilon_0.$$

Similarly for k odd we get

$$t_{n_k} - t_{n_{k+1}} \le -\varepsilon_0.$$

Thus

$$|t_{n_k} - t_{n_{k+1}}| \ge \varepsilon_0 \qquad (k = 1, 2, \dots). \tag{5}$$

The inequalities (5) enable us to show that

$$B(z, (1-\alpha)\delta) \cap C^0_{\mathrm{stat}} = \emptyset. \tag{6}$$

For this we shall use the following criterion for statistical convergence (cf [5]):

(K) A sequence $x = (x_n)_{n=1}^{\infty}$ is statistically convergent if and only if for every $a > 0$ there exists a $g = g(a) \in \mathbb{N}$ such that $d(\{k\colon |x_k - x_{g(a)}| < a\}) = 1$.

Therefore it suffices to show that for each $t = (t_n)_{n=1}^{\infty} \in B\big(z, (1-\alpha)\delta\big)$ the following statement

$$\exists_{a>0} \forall_g \ \overline{d}\big(\{j \colon |t_j - t_g| \ge a\}\big) > 0 \tag{K'}$$

holds.

Let $g \in \mathbb{N}$ be arbitrarily chosen. Then by (5)

$$\varepsilon_0 \le \big|t_{n_k} - t_{n_{k+1}}\big| \le \big|t_{n_k} - t_g\big| + \big|t_g - t_{n_{k+1}}\big| .$$

At least one of the summands on the right-hand side is $\ge \frac{\varepsilon_0}{2}$, i.e., at least one of the numbers $t_{n_k}, t_{n_{k+1}}$ satisfies the inequality

$$|t_{n_s} - t_g| \ge \frac{\varepsilon_0}{2} \qquad (s = k \ \text{ or } \ s = k+1) .$$

If we apply this fact for each of the pairs

$$1,\, 2\colon\ 3,\, 4\colon\ \dots;\ 2k-1,\ 2k\colon\ \dots$$

we see that upper density of such n_j for which $|t_{n_j} - t_g| \ge \varepsilon_0$ is positive.

Thus (K') holds with $a = \varepsilon_0$.

From this we obtain (6) and from (3), (6) the inequality

$$\gamma\big(y, \delta, C^0_{\mathrm{stat}}\big) \ge (1-\alpha)\delta$$

follows.

So we have $\underline{p}\big(y, C^0_{\mathrm{stat}}\big) \ge 1 - \alpha$. If $\alpha \to \frac{1}{2}+$ then

$$\underline{p}\big(y, C^0_{\mathrm{stat}}\big) \ge \frac{1}{2} . \tag{7}$$

We now show that

$$\overline{p}\big(y, C^0_{\mathrm{stat}}\big) \le \frac{1}{2} . \tag{8}$$

Suppose that $\overline{p}\big(y, C^0_{\mathrm{stat}}\big) > \frac{1}{2}$ $(y \in C^0_{\mathrm{stat}})$. Then there is a $\delta > 0$ such that

$$\gamma\big(y, \delta, C^0_{\mathrm{stat}}\big) > \frac{\delta}{2}$$

and again (by definition of $\gamma\big(y, \delta, C^0_{\mathrm{stat}}\big)$) there is a $t > \frac{\delta}{2}$ such that for a suitable $z \in B(y, \delta)$, $z = (z_n)_{n=1}^{\infty}$, we have

a) $B(z, t) \subseteq B(y, \delta)$,

b) $B(z, t) \cap C^0_{\mathrm{stat}} = \emptyset$.

The definition of the metric τ yields $z_j \in (y_j - \delta,\ y_j + \delta)$ $(j = 1, 2, \dots)$.

If $|z_j - y_j| < \frac{\delta}{2}$ $(j = 1, 2, \dots)$ then $\tau(z, y) \le \frac{\delta}{2}$, hence $y \in B(z, t)$ which contradicts b). Therefore there is a $j_0 \in \mathbb{N}$ such that $\left|z_{j_0} - y_{j_0}\right| \ge \frac{\delta}{2}$.

There are two possibilities:

$$1)\ z_{j_0} \ge y_{j_0} + \frac{\delta}{2}, \qquad 2)\ z_{j_0} \le y_{j_0} - \frac{\delta}{2}.$$

In the case 1) we have obviously $z_{j_0} + t > y_{j_0} + \delta$ and so $B(z, t)$ is not a subset of $B(y, \delta)$ — a contradiction to a).

In the case 2) we have $z_{j_0} - t < y_{j_0} - \delta$ and again $B(z, t) \not\subseteq B(y, \delta)$ — a contradiction to a).

Hence (8) holds and from (7), (8) we get $p\big(y, C^0_{\mathrm{stat}}\big) = \frac{1}{2}$ at each $y \in C^0_{\mathrm{stat}}$. □

2. Porosity of C_{stat} in s

The space s of all real sequences is endowed by the Fréchet's metric ρ,

$$\rho(x, y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

$x = (x_k)_{k=1}^{\infty} \in s$, $y = (y_k)_{k=1}^{\infty} \in s$.

In [11] (Theorem 3.1 (ii)) it is established that the set C_{stat} is a set of the first Baire category. Therefore it seems to be convenient to investigate the porosity of this set in s. We shall do it from the point of view of h-porosity, where $h\colon [0, \infty) \to \mathbb{R}$ is defined as follows: $h(0) = 0$, $h(x) = |\ln x|^{-1}$ for $0 < x < e^{-1}$, $h(x) = (x - e^{-1}) + 1$ for $x \ge e^{-1}$. It is easy to check that this function is continuous and increasing on $[0, \infty)$, $h(x) > 0$ for $x > 0$.

Next we shall use the following criterion on statistical convergence (cf [12]):

(C) A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers converges statistically to a $\xi \in \mathbb{R}$ if and only if for each $t \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{itx_k} = e^{it\xi}$$

holds.

The idea of our investigation of h-porosity of C_{stat} will be based on majorizing the set C_{stat} by the following set H, definition of which goes out from (C).

Put (for $m, k \in \mathbb{N}$):

$$H(m,k) = \bigcap_{n=m}^{\infty} \left\{ x = (x_j)_{j=1}^{\infty} \in s \colon \left| \frac{1}{n} \sum_{l=1}^{n} e^{ix_l} \right| \in \left(1 - \tfrac{1}{k}\,,\ 1 + \tfrac{1}{k}\right) \right\},$$

$$H = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} H(m,k)\,.$$

Then by (C) we have

$$C_{\mathrm{stat}} \subseteq H \subseteq \bigcup_{m=1}^{\infty} H(m,k)\,, \qquad (k = 1, 2, \dots)\,, \tag{9}$$

$$C_{\mathrm{stat}} \subseteq \bigcup_{m=1}^{\infty} H(m,k)\,, \qquad (k = 1, 2, \dots)\,. \tag{10}$$

We introduce the main result

Theorem 2.1. *Let $0 < \varepsilon < 1$ be arbitrary. Then the set C_{stat} is uniformly σ-very $(1-\varepsilon)$-h-porous in s.*

The proof of Theorem 2.1 will be based on the following lemmas.

Lemma 2.1. *If $u, v \in \mathbb{R}$ then*

$$|e^{iu} - e^{iv}| \le |u - v|\,.$$

Proof. (cf [9], p. 15). □

Lemma 2.2. *Let $r, t \in \mathbb{N}$, $r < t$, $y = (y_n)_{n=1}^{\infty} \in s$. Let $z_j = y_j$ $(j = 1, 2, \dots, t)$, $z_{t+j} = \pi + y_j$ $(j = 1, 2, \dots, r)$. Then*

$$\left| \frac{1}{t+r} \sum_{j=1}^{t+r} e^{iz_j} \right| \le \frac{t-r}{t+r}\,. \tag{11}$$

Proof. A simple arrangement of the sum on the left-hand side yields

$$\frac{1}{t+r} \sum_{j=1}^{t+r} e^{iz_j} = \frac{1}{t+r} \left(\sum_{l=1}^{r} e^{iy_l} + \sum_{j=1}^{r} e^{iz_{t+j}} \right) + \frac{1}{t+r} \sum_{l=r+1}^{t} e^{iz_l}\,.$$

Note that for $v \in \mathbb{R}$ we have

$$e^{iv} + e^{i(\pi+v)} = 0\,.$$

Thus by definition of z_j $(j = 1, 2, \dots, t+r)$ the first summand on the right-hand side is zero and a trivial estimation gives the assertion. □

Lemma 2.3. *Let $k > 2$, $r = \lfloor \frac{t}{k-1} \rfloor + 1$. Then*

$$\frac{t-r}{t+r} \le 1 - \frac{2}{k}\,. \tag{12}$$

Proof. The inequality (12) can be obtained by a simple calculation. ☐

Proof of Theorem 2.1. First of all we shall deal with porosity of sets $H(m,k)$ $(m,k \in \mathbb{N})$.

Let m,k be fixed, $k > 2$, let $y \in s$, $\delta > 0$. We can assume that $\delta < 2^{-m}$, $\delta < e^{-1}$. Then there is a $t \in \mathbb{N}$ such that $2^{-t} \le \delta < 2^{-t+1}$ (hence $t > m$). From this $B(y, 2^{-t}) \subseteq B(y,\delta)$ follows.

Put $r = \lfloor \frac{t}{k-1} \rfloor + 1$. Define $z = (z_j)_{j=1}^{\infty}$ as follows:

$$z_j = y_j \quad (j = 1,2,\dots,t), \qquad z_{t+j} = \pi + y_j \quad (j = 1,2,\dots,r),$$
$$z_{t+r+j} = y_{t+r+j} \quad (j = 1,2,\dots).$$

Then we have

$$\rho(y,z) = \sum_{j=t+1}^{t+r} 2^{-j} \frac{|y_j - z_j|}{1 + |y_j - z_j|} < 2^{-t+1} + \dots + 2^{-(t+r)} < 2^{-t}.$$

Thus $z \in B(y, 2^{-t})$.

Construct the ball $B\big(z, \frac{1}{k+1} 2^{-(t+r)}\big)$ in s. We show that:

a) $B\big(z, \frac{1}{k+1} 2^{-(t+r)}\big) \subseteq B(y, 2^{-t})$,

b) $B\big(z, \frac{1}{k+1} 2^{-(t+r)}\big) \cap H(m,k) = \emptyset$.

a) If $x \in B\big(z, \frac{1}{k+1} 2^{-(t+r)}\big)$ then by triangular inequality we get

$$\rho(x,y) \le \rho(x,z) + \rho(z,y) < 2^{-(t+1)} + \dots + 2^{-(t+r)} + \frac{1}{k+1} 2^{-(t+r)} < 2^{-t}.$$

b) It suffices to show that if x belongs to left-hand side of the equality b), then

$$\left| \frac{1}{t+r} \sum_{l=1}^{t+r} e^{ix_l} \right| \le 1 - \frac{1}{k}\,. \tag{13}$$

For proving this fact we consider in the first place the similar sum for z. We get by Lemma 2.2 and 2.3

$$\left| \frac{1}{t+r} \sum_{l=1}^{t+r} e^{iz_l} \right| \le \frac{t-r}{t+r} \le 1 - \frac{2}{k}\,. \tag{14}$$

Now we estimate using Lemma 2.1 the difference of the sums (13) and (14).

$$\left|\frac{1}{t+r}\sum_{l=1}^{t+r} e^{ix_l} - \frac{1}{t+r}\sum_{l=1}^{t+r} e^{iz_l}\right| \le \frac{1}{t+r}\sum_{l=1}^{t+r} |x_l - z_l| . \tag{15}$$

Since $x = (x_j)_{j=1}^{\infty}$ belongs to $B\left(z, \frac{1}{k+1}2^{-(t+r)}\right)$, the definition of the metric ρ gives:

$$\frac{|x_l - z_l|}{1+|x_l - z_l|} < \frac{1}{k+1} \qquad (l = 1, 2, \dots, t+r),$$

so

$$|x_l - z_l| < \frac{1}{k} \qquad (l = 1, 2, \dots, t+r).$$

This together with (15) yields

$$\left|\frac{1}{t+r}\sum_{l=1}^{t+r} e^{ix_l} - \frac{1}{t+r}\sum_{l=1}^{t+r} e^{iz_l}\right| < \frac{1}{k} . \tag{16}$$

From (14), (16) the inequality (13) follows.

According to b), we have

$$\gamma\big(y, \delta, H(m,k)\big) \ge \frac{1}{k+1}2^{-(t+r)}$$

and so

$$\gamma_h\big(y, \delta, H(m,k)\big) \ge h\Big(\frac{1}{k+1}2^{-(t+r)}\Big) = \frac{1}{\ln(k+1) + (t+r)\ln 2} . \tag{17}$$

Further by the choice of t we have

$$h(\delta) \le \left|\ln 2^{-(t-1)}\right|^{-1} . \tag{17'}$$

Inequalities (17), (17') give

$$\frac{\gamma_h\big(y, \delta, H(m,k)\big)}{h(\delta)} \ge \frac{(t-1)\ln 2}{\ln(k+1) + \big(t + \lfloor \frac{t}{k-1} \rfloor + 1\big)\ln 2} .$$

If $\delta \to 0+$ then $t \to \infty$ and the limit of the right-hand side is $\frac{k-1}{k}$. So we get

$$\underline{p}_h\big(y, H(m,k)\big) \ge \frac{k-1}{k}$$

for arbitrary m and $k > 2$. This together with (10) shows that the set C_{stat} is for every $k > 2$ uniformly σ-very $\frac{k-1}{k}$-h-porous in s.

If $0 < \varepsilon < 1$ we can choose a $k > 2$ such that $\frac{k-1}{k} > 1-\varepsilon$. Then the assertion of theorem follows immediately. □

In papers concerning statistical convergence only finite statistical limits occur (cf. [2]–[5], [11], [12]). But it is possible to introduce the concept of statistical convergence of a sequence to $+\infty$ or $-\infty$.

Definition 2.1.

a) A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to have *statistical limit* $+\infty$ ($\text{lim-stat}\, x_n = +\infty$) provided that for every $K \in \mathbb{R}$ we have $d(\{n : x_n > K\}) = 1$ (equivalently: $d(\{n : x_n \le K\}) = 0$).

b) A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to have *statistical limit* $-\infty$ ($\text{lim-stat}\, x_n = -\infty$) provided that for every $K \in \mathbb{R}$ we have $d(\{n : x_n < K\}) = 1$ (equivalently: $d(\{n : x_n \ge K\}) = 0$).

Put

$$C_{\text{stat}}^{\infty} = \left\{x = (x_n)_{n=1}^{\infty} \in s :\ \text{lim-stat}\, x_n = +\infty\right\},$$
$$C_{\text{stat}}^{-\infty} = \left\{x = (x_n)_{n=1}^{\infty} \in s :\ \text{lim-stat}\, x_n = -\infty\right\}.$$

We shall investigate the porosity of these sets in the space s. Their porosity character is similar to that of C_{stat}.

The definition of C_{stat}^{∞} yields the following statement:

$$\text{lim-stat}\, x_n = +\infty \iff$$
$$\forall_K \forall_k \exists_m \forall_{n \ge m}\ \frac{1}{n}\big|\{j \le n : x_j \le K\}\big| < \frac{1}{k}. \tag{18}$$

($|M|$ denotes cardinality of M).

Hence

$$C_{\text{stat}}^{\infty} = \bigcap_{K=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} A(K,k,m), \tag{18'}$$

where

$$A(K,k,m) = \bigcap_{n=m}^{\infty} \left\{x = (x_n)_{n=1}^{\infty} \in s :\ \frac{1}{n}\big|\{j \le n : x_j \le K\}\big| < \frac{1}{k}\right\}.$$

Especially from $(18')$ we get

$$C_{\text{stat}}^{\infty} \subseteq \bigcup_{m=1}^{\infty} A(K,k,m) \tag{19}$$

for every $K \ge 1$, $k \ge 1$.

Lemma 2.4. *Let $K \ge 1$, $k > 1$, $m \ge 1$ be fixed, then the set $A(K,k,m)$ is very $\left(1 - \frac{1}{k}\right)$-$h$-porous at each $y \in s$.*

P r o o f. Let $y = (y_j)_{j=1}^{\infty} \in s$, $0 < \delta < e^{-1}$, $\delta < 2^{-m}$. Then there is a $p \in \mathbb{N}$ such that

$$2^{-p} < \delta \le 2^{-p+1}. \tag{20}$$

Hence $p > m$.

Construct the sequence $z = (z_j)_{j=1}^{\infty}$ as follows:

$$z_j = y_j \quad (j = 1, 2, \dots, p+1), \qquad z_j = 0 \quad \text{for } j > p+1\,.$$

Then $\rho(z, y) \le 2^{-(p+1)} < \delta$.

Put for brevity $i = \lfloor \frac{p+2k}{k-1} \rfloor + 1$. Then $i > 2$. We show that

a) $B\big(z, 2^{-(p+i)}\big) \subseteq B(y, \delta)$,

b) $B\big(z, 2^{-(p+i)}\big) \cap A(K, k, m) = \emptyset$.

a) Let $x \in B\big(z, 2^{-(p+i)}\big)$. By triangle inequality we get $\rho(x, y) \le \rho(x, z) + \rho(z, y) < 2^{-(p+1)} + 2^{-(p+i)} < 2^{-p} < \delta$. Thus a) follows.

b) Suppose that $x = (x_j)_{j=1}^{\infty} \in B(z, 2^{-(p+i)})$. If

$$p + 1 < n \le p + i - 1 \tag{21}$$

then by definition of the metric ρ and sequence z we get

$$2^{-(p+i)} > \rho(x, z) > 2^{-n} \frac{|x_n|}{1 + |x_n|}\,.$$

So $|x_n| < 1 \le K$.

The number of all n's satisfying (21) is

$$i - 2 > \frac{p + 2k}{k - 1} - 2 = \frac{p + 2}{k - 1}\,.$$

A simple calculation shows that

$$i > \frac{2k + p + i}{k}, \qquad i - 2 > \frac{p + i}{k}\,.$$

A simple estimation yields

$$\big|\{n \le p + i \colon x_n < K\}\big| \ge \frac{p + i}{k}\,. \tag{22}$$

Thus b) follows.

From a), b) we conclude

$$\gamma_h\big(y, \delta, A(K, k, m)\big) \ge h\big(2^{-(p+i)}\big)\,.$$

Further from (20) we get $h(\delta) \le h(2^{-p+1})$. So we have

$$\frac{\gamma_h\big(y, \delta, A(K, k, m)\big)}{h(\delta)} \ge \frac{|\ln 2^{-p+1}|}{|\ln 2^{-p-i}|} = \frac{p - 1}{p + i}\,, \qquad i = \left\lfloor \frac{p + 2k}{k - 1} \right\rfloor + 1\,. \tag{23}$$

If $\delta \to 0+$, then $p \to \infty$ and from (23) we get

$$\underline{p}_h\big(y, A(K, k, m)\big) \ge 1 - \frac{1}{k}\,.$$

$\square$

THEOREM 2.2. *Let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary. Then each of the sets $C^{\infty}_{\mathrm{stat}}$, $C^{-\infty}_{\mathrm{stat}}$ is uniformly σ-very $(1-\varepsilon)$-h-porous in s.*

Proof. We can choose a $k \in \mathbb{N}$, $k > 1$, such that $k^{-1} < \varepsilon$. Then $1 - \frac{1}{k} > 1 - \varepsilon$ and the assertion follows from Lemma 2.4 according to (19).

The part of the assertion concerning $C^{-\infty}_{\mathrm{stat}}$ can be obtained in an analogous way. $\square$

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Received September 22, 1998

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