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ON EXTENSIONS OF CONVERGENCES

JAN POCHCIAŁ

ABSTRACT. The relation of inclusion between **FLUSH** convergences in a linear space is considered. Properties of outer sequences defined in the paper and convergences related to them are studied. Some examples and counterexamples are shown.

1. Introduction

Assume that X is a linear space over the field $\mathbb{R}$ of real numbers. By a *convergence* $\mathcal{G}$ in X we mean an arbitrary subset of $X^{\mathbb{N}} \times X$. If $\langle (x_n), x \rangle \in \mathcal{G}$ for $(x_n) \in X^{\mathbb{N}}$ and $x \in X$, then we say that the sequence (x_n) *is convergent to* x in $\mathcal{G}$ and write $x_n \xrightarrow{\mathcal{G}} x$. In the case of the convergence of real numbers, we shall simply write $\alpha_n \to \alpha$.

The following conditions are usually imposed on convergences $\mathcal{G}$ in linear spaces:

F. If $x_n \xrightarrow{\mathcal{G}} x$, then $x_{p_n} \xrightarrow{\mathcal{G}} x$ for every subsequence (x_{p_n}) of (x_n);

U. If $x_n \overset{\mathcal{G}}{\not\to} x$, then there is a subsequence (x_{p_n}) of (x_n) such that $x_{q_n} \overset{\mathcal{G}}{\not\to} x(\mathcal{G})$ for every subsequence (x_{q_n}) of (x_{p_n});

S. If $x_n = x$ for each $n \in \mathbb{N}$, then $x_n \xrightarrow{\mathcal{G}} x$;

H. If $x_n \xrightarrow{\mathcal{G}} x$ and $x_n \xrightarrow{\mathcal{G}} y$, then $x = y$;

L. If $x_n \xrightarrow{\mathcal{G}} x$, $y_n \xrightarrow{\mathcal{G}} y$, $\alpha_n \to \alpha$ and $\beta_n \to \beta$, then $\alpha_n x_n + \beta_n y_n \xrightarrow{\mathcal{G}} \alpha x + \beta y$.

It is easy to check that the convergence in a linear topological space satisfies all the conditions **F**, **U**, **S**, **H**, **L**. On the other side, there exist **FLUSH**-

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convergences which are non-topological, i.e., not generated by any linear topology (see, e.g., [8, 1, 9, 10]).

We shall say that a convergence $\hat{\mathcal{G}}$ is an extension of a convergence $\mathcal{G}$ if $\mathcal{G} \subset \hat{\mathcal{G}}$, i.e., every sequence convergent in $\mathcal{G}$ is convergent in $\hat{\mathcal{G}}$ to the same limit.

An important case of extension is the so called Urysohn modification of $\mathcal{G}$ usually denoted by $\mathcal{G}^*$ and defined as follows (for applications see, e.g., [11]):

$x_n \xrightarrow{\mathcal{G}^*} x$ if for every subsequence (x_{p_n}) of (x_n) there is a subsequence (x_{q_n}) of (x_{p_n}) such that $x_{q_n} \xrightarrow{\mathcal{G}} x$.

It is easy to prove that if $\mathcal{G}$ satisfies condition **F**, then $\mathcal{G}^*$ is the smallest extension of $\mathcal{G}$ satisfying condition **U**. Moreover, if $\mathcal{G}$ satisfies conditions **S**, **H**, **L**, then also $\mathcal{G}^*$ does.

In the paper, we shall deal with such extensions that both convergences $\mathcal{G}$ and its extension $\hat{\mathcal{G}}$ satisfy all the conditions **F**, **U**, **S**, **H**, **L**. Questions concerning the possibility of constructing such extensions are strictly connected with the maximality of **FLUSH**-convergences (see, e.g., [5] for linear spaces and [4, 2] for groups) as well as with the bases of convergences (see, e.g., [3, 6, 7]).

In the paper, the notions of outer, essentially outer and strongly outer sequences are introduced. Properties of these sequences and convergences generated by these sequences are studied.

2. Preliminaries

Let $(\bar{x}_n)$ be a fixed sequence in X.

Define the sets $C[\bar{x}_n]$ and $B[\bar{x}_n]$ of sequences as follows:

$$(y_n) \in C[\bar{x}_n] \quad \text{if} \quad y_n = \sum_{i=1}^{k} \alpha_{n,i} \bar{x}_{p_{n,i}},$$

where $k \in \mathbb{N}$, $(\bar{x}_{p_{n,i}})$ are subsequences of $(\bar{x}_n)$ for $i = 1, 2, \ldots, k$ and $(\alpha_{n,i})$ are convergent sequences of real numbers for $i = 1, 2, \ldots, k$;

$$(y_n) \in B[\bar{x}_n] \quad \text{if} \quad y_n = \sum_{i=1}^{k} \alpha_{n,i} \bar{x}_{p_{n,i}},$$

where $k \in \mathbb{N}$, $(\bar{x}_{p_{n,i}})$ are subsequences of $(\bar{x}_n)$ for $i = 1, 2, \ldots, k$ and the set of real numbers $\{\alpha_{n,i} \colon n \in \mathbb{N},\ i = 1, 2, \ldots, k\}$ is bounded.

Remark 1. Notice that $(y_n) \in B[\bar{x}_n]$ iff for every subsequence (y_{p_n}) of (y_n) there is a subsequence (y_{q_n}) of (y_{p_n}) such that $(y_{q_n}) \in C[\bar{x}_n]$.

Remark 2. If a convergence $\mathcal{G}$ satisfies conditions **F**, **L** and $\bar{x}_n \xrightarrow{\mathcal{G}} 0$, then $y_n \xrightarrow{\mathcal{G}} 0$ for every $(y_n) \in C[\bar{x}_n]$. If $\mathcal{G}$ satisfies additionally condition **U**, then $y_n \xrightarrow{\mathcal{G}} 0$ for every $(y_n) \in B[\bar{x}_n]$.

Given a sequence $(\bar{x}_n)$ and a convergence $\mathcal{G}$ we can define convergences $\mathcal{G}_C$ and $\mathcal{G}_B$ as follows:

$x_n \xrightarrow{\mathcal{G}_I} 0$ if there are sequences (y_n) and (z_n) such that $x_n = y_n + z_n$ for each $n \in \mathbb{N}$, $(y_n) \in I[\bar{x}_n]$ and $z_n \xrightarrow{\mathcal{G}} 0$;

$x_n \xrightarrow{\mathcal{G}_I} x$ if $x_n - x \xrightarrow{\mathcal{G}_I} 0$, where the character I stands for C or B.

One can easily prove the following proposition (see also [12]):

Proposition 1. *If a convergence $\mathcal{G}$ satisfies conditions* **F**, **S**, **L**, *then the convergences $\mathcal{G}_C$ and $\mathcal{G}_B$ also satisfy conditions* **F**, **S**, **L** *and, consequently, their Urysohn modifications $\mathcal{G}_C^*$ and $\mathcal{G}_B^*$ satisfy conditions* **F**, **U**, **S**, **L**. *Moreover,*

$$\mathcal{G} \subset \mathcal{G}_C^* = \mathcal{G}_B^* .$$

Note that the extensions of a convergence $\mathcal{G}$ constructed above need not satisfy condition **H** even if the convergence $\mathcal{G}$ satisfies all the conditions $\mathbf{F}, \mathbf{U}, \mathbf{S}, \mathbf{H}, \mathbf{L}$.

The convergence $\mathcal{G}_C^* (= \mathcal{G}_B^*)$ depends only on the sequence $(\bar{x}_n)$ and the convergence $\mathcal{G}$. We shall denote it by $\mathcal{G}_{(\bar{x}_n)}$, i.e,

$$\mathcal{G}_{(\bar{x}_n)} := \mathcal{G}_C^* .$$

In the sequel, we shall examine only extensions of convergences that satisfy all the conditions **F**, **U**, **S**, **H**, **L** and we require that the obtained extensions satisfy all the conditions **F**, **U**, **S**, **H**, **L** as well.

3. Outer sequences

Definition 1. A sequence $(\bar{x}_n)$ will be called *outer* for a convergence $\mathcal{G}$ if it satisfies the following conditions:

(a) $\bar{x}_n \overset{\mathcal{G}}{\not\to} 0$,

(b) $y_n \overset{\mathcal{G}}{\not\to} x$ whenever $x \neq 0$ and $(y_n) \in B[\bar{x}_n]$.

Conditions (a) and (b) were first used in [5] (for the case of groups see also [4, 2]).

Remark 3. If $(\bar{x}_n)$ is outer for $\mathcal{G}$ and $\breve{\mathcal{G}} \subset \mathcal{G}$, then $(\bar{x}_n)$ is outer for $\breve{\mathcal{G}}$.

Remark 4. If $(\bar{x}_n)$ is outer for $\mathcal{G}$, then $\dim \operatorname{span}\{\bar{x}_n \colon n \in \mathbb{N}\} = \infty$.

Corollary 1. *If $\mathcal{G}$ and $\tilde{\mathcal{G}}$ are* **FLUSH***-convergences in X and $Y \subset X$ is a space of finite dimension, then $\mathcal{G} = \tilde{\mathcal{G}}$ in Y.*

Example 1. Let (e_i) for $i \in \mathbb{N}$ be a base of the space X or, in other words, let X be the space of real sequences with finite support, i.e., $x \in X$ if $x = (t_1, t_2, \ldots)$, $t_i \in \mathbb{R}$ and $\operatorname{supp} x$ is finite (t_i are coordinates of x in the base (e_i)).

Define a number of convergences in X:

0^0 $x_n \xrightarrow{\mathcal{G}_0} 0$ if $t_{in} \to 0$ (in $\mathbb{R}$), for each $i \in \mathbb{N}$
(where $x_n = (t_{1n}, t_{2n}, \ldots)$);

1^0 $x_n \xrightarrow{\mathcal{G}_1} 0$ if $x_n \xrightarrow{\mathcal{G}_0} 0$ and
$\operatorname{card} \bigcup\limits_{n=1}^{\infty} \operatorname{supp}(x_n)$ is finite;

2^0 $x_n \xrightarrow{\mathcal{G}_2} 0$ if $x_n \xrightarrow{\mathcal{G}_0} 0$ and
there is a K such that $\operatorname{card} \operatorname{supp} x_n < K$ for each $n \in \mathbb{N}$;

3^0 $x_n \xrightarrow{\mathcal{G}_3} 0$ if $x_n \xrightarrow{\mathcal{G}_0} 0$ and
$\sum\limits_{i=1}^{\infty} t_{in} \to 0$ (in $\mathbb{R}$);

4^0 $x_n \xrightarrow{\mathcal{G}_4} 0$ if $x_n \xrightarrow{\mathcal{G}_0} 0$ and
there is an M such that $|t_{in}| < M$ for each $i, n \in \mathbb{N}$.

Moreover, $x_n \xrightarrow{\mathcal{G}_i} x$ if $x_n - x \xrightarrow{\mathcal{G}_i} 0$, where $i = 0, 1, 2, 3, 4$.

It is easy to check that all the convergences $\mathcal{G}_i$ satisfy conditions **F**, **U**, **S**, **H**, **L** and the following inclusions hold:

$$\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{G}_0, \qquad \mathcal{G}_1 \subset \mathcal{G}_3 \subset \mathcal{G}_0, \qquad \mathcal{G}_1 \subset \mathcal{G}_4 \subset \mathcal{G}_0.$$

Evidently, $e_n \xrightarrow{\mathcal{G}_2} 0$ and $e_n \xrightarrow{\mathcal{G}_4} 0$ (hence $e_n \to 0(\mathcal{G}_0)$) and (e_n) is outer for $(\mathcal{G}_3)$ (hence for $\mathcal{G}_1$).

Let $f_n = \sum\limits_{i=1}^{n} e_i$. We shall show that (f_n) is outer for $\mathcal{G}_0$, (hence for $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4$).

Indeed, since each coordinate of (f_n) is convergent to 1 (in $\mathbb{R}$), we get $f_n \overset{\mathcal{G}_0}{\not\longrightarrow} 0$ which proves (a).

To prove (b) assume that $y_n \xrightarrow{\mathcal{G}_0} x$, where $y_n = (t_{1n}, t_{2n}, \ldots) = \sum\limits_{i=1}^{k} \alpha_{n,i} f_{p_{n,i}}$.

Then $t_{jn} = \sum\limits_{i=1}^{k} \alpha_{n,i}$ for $j \leq \min\limits_i p_{n,i}$. Therefore, $x = (t, t, \ldots)$ and, consequently, $x = 0$.

EXAMPLE 2. Let $\mathcal{G}$ denote the norm convergence in a normed space X of infinite dimension.

Then by Riesz's Lemma there is a sequence $(\bar{x}_n)$ in X which satisfies the following conditions:

(i) $\| \bar{x}_n \| = 1$;

(ii) $\operatorname{dist}(\bar{x}_n, \operatorname{span}\{\bar{x}_i : 1 \le i < n\}) > \frac{1}{2}$.

It is shown in [12] that the sequence $(\bar{x}_n)$ is outer for $\mathcal{G}$.

PROPOSITION 2. *If a sequence $(\bar{x}_n)$ is outer for $\mathcal{G}$, then $\mathcal{G}_{(\bar{x}_n)}$ is a* **FLUSH**-*extension of $\mathcal{G}$. Moreover, if $\mathcal{G} \subset \hat{\mathcal{G}}$, $\bar{x}_n \xrightarrow{\hat{\mathcal{G}}} 0$ and $\bar{x}_n \overset{\mathcal{G}}{\not\to} 0$, then $(\bar{x}_n)$ is outer for $\mathcal{G}$ and $\mathcal{G}_{(\bar{x}_n)} \subset \hat{\mathcal{G}}$.*

Proof. To prove that $\mathcal{G}_{(\bar{x}_n)}$ is a **FLUSH**-extension of $\mathcal{G}$, it suffices, by Proposition 1, to check condition **H** only. Suppose, on the contrary, that $x_n - x \xrightarrow{\mathcal{G}_{(\bar{x}_n)}} 0$ and $x_n \xrightarrow{\mathcal{G}_{(\bar{x}_n)}} 0$ for some (x_n) and $x \neq 0$. Without loss of generality, we may assume that $x_n - x \xrightarrow{\mathcal{G}_B} 0$ and $x_n \xrightarrow{\mathcal{G}_B} 0$, i.e.,

$$x_n = \sum_{i=1}^{k_1} \alpha_{n,i} \bar{x}_{p_{n,i}} + y_n \qquad \text{and} \qquad x_n - x = \sum_{i=1}^{k_2} \beta_{n,i} \bar{x}_{q_{n,i}} + z_n ,$$

where $\alpha_{n,i}, \beta_{n,i}, \bar{x}_{p_{n,i}}, \bar{x}_{q_{n,i}}, y_n, z_n$ satisfy all the conditions required in the definition of $\mathcal{G}_B$. Hence

$$x = \left(\sum_{i=1}^{k_1} \alpha_{n,i} \bar{x}_{p_{n,i}} - \sum_{i=1}^{k_2} \beta_{n,i} \bar{x}_{q_{n,i}} \right) + (y_n - z_n) .$$

By conditions **L** and (b) of Definition 1, we get $x = 0$. Since the Urysohn modification preserves **H**, the convergence $\mathcal{G}_{(\bar{x}_n)} = \mathcal{G}_B^*$ also satisfies **H**.

Now suppose that $\bar{x}_n \xrightarrow{\hat{\mathcal{G}}} 0$, $\bar{x}_n \overset{\mathcal{G}}{\not\to} 0$, and $(y_n) \in B[\bar{x}_n]$. Then, by Remark 2, we have $y_n \xrightarrow{\hat{\mathcal{G}}} 0$. Therefore, by $\mathcal{G} \subset \hat{\mathcal{G}}$, we get $y_n \overset{\mathcal{G}}{\not\to} x$, for $x \neq 0$, which proves that $(\bar{x}_n)$ is outer for $\mathcal{G}$. The inclusion $\mathcal{G}_{(\bar{x}_n)} \subset \hat{\mathcal{G}}$ follows directly from the definition of $\mathcal{G}_{(\bar{x}_n)}$. $\square$

PROPOSITION 3. *Let $(\bar{x}_n)$ be an outer sequence for $\mathcal{G}$ and (z_n) a sequence such that $z_n \xrightarrow{\mathcal{G}} 0$. Then the sequence $(\bar{x}_n + z_n)$ is outer for $\mathcal{G}$ and*

$$\mathcal{G}_{(\bar{x}_n + z_n)} = \mathcal{G}_{(\bar{x}_n)} .$$

Proof. The assertion follows directly from the definition of $\mathcal{G}_{(\bar{x}_n)}$. $\square$

PROPOSITION 4. *Let $(\bar{x}_n)$ be outer for $\mathcal{G}$ and let (α_n) be a bounded sequence of real numbers. Then either $(\alpha_n \bar{x}_n)$ is outer for $\mathcal{G}$ or $\alpha_n \bar{x}_n \xrightarrow{\mathcal{G}} 0$. In both cases, we have*

$$\mathcal{G}_{(\alpha_n \bar{x}_n)} \subset \mathcal{G}_{(\bar{x}_n)}.$$

If, additionally, the sequence $\left(\frac{1}{\alpha_n}\right)$ is bounded, then $(\alpha_n \bar{x}_n)$ is outer for $\mathcal{G}$ and

$$\mathcal{G}_{(\alpha_n \bar{x}_n)} = \mathcal{G}_{(\bar{x}_n)}.$$

P r o o f . The assertion follows directly from the definitions of outer sequence and the convergence $\mathcal{G}_{(\bar{x}_n)}$. □

The following examples show that the assumptions of boundedness of the sequences (α_n) and $\left(\frac{1}{\alpha_n}\right)$ are essential.

EXAMPLE 3. Consider the convergence $\mathcal{G}_2$ and the sequence (f_n) in Example 1.

Let $u_n = f_n + \frac{e_1}{2^n}$ and $w_n = 2^n f_n + e_1$. By Example 1 and Proposition 3, the sequence (u_n) is outer for $\mathcal{G}_2$. Since $2w_{n-1} - w_n \xrightarrow{\mathcal{G}_2} e_1$ the sequence (w_n) is not outer for $\mathcal{G}_2$.

EXAMPLE 4. Consider the convergence $\mathcal{G}_3$ and the sequence (e_n) in Example 1. For every (α_n) such that $\alpha_n \to 0$, we have $\alpha_n e_n \xrightarrow{\mathcal{G}_3} 0$, hence the sequence $(\alpha_n e_n)$ is not outer for $\mathcal{G}_3$.

In particular, from Proposition 4 we obtain that $\mathcal{G}_{(\bar{x}_{p_n})} \subset \mathcal{G}_{(\bar{x}_n)}$ for every subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$. The following simple example shows that there exists a convergence $\mathcal{G}$ and a sequence $(\bar{x}_n)$ such that for every subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$, we have $\mathcal{G}_{(\bar{x}_{p_n})} = \mathcal{G}_{(\bar{x}_n)}$.

EXAMPLE 5. Consider the convergence $\mathcal{G}_3$ and the sequence (e_n) in Example 1. For every subsequence (p_n) of natural numbers the sequence (e_{p_n}) is outer for $\mathcal{G}_3$. Since $e_n - e_{p_n} \xrightarrow{\mathcal{G}_3} 0$, we get $\mathcal{G}_{(e_n)} = \mathcal{G}_{(e_{p_n})}$.

COROLLARY 2. *Let $\mathcal{G}_5 = \mathcal{G}_2 \bigcap \mathcal{G}_3 \bigcap \mathcal{G}_4$ and $\mathcal{G}_6 = \mathcal{G}_2 \bigcap \mathcal{G}_4$, where $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ are the same as for Example 1. Then $\mathcal{G}_{5(e_n)} = \mathcal{G}_6$ and there is no convergence strictly contained between $\mathcal{G}_5$ and $\mathcal{G}_6$.*

P r o o f . We shall first show that $\mathcal{G}_{5(e_n)} = \mathcal{G}_6$. Since $e_n \xrightarrow{\mathcal{G}_6} 0$ and $e_n \overset{\mathcal{G}_5}{\not\longrightarrow} 0$, it suffices to prove that $\mathcal{G}_6 \subset \mathcal{G}_{5(e_n)}$. Suppose that $x_n \xrightarrow{\mathcal{G}_6} 0$, where $x_n = (t_{1n}, t_{2n}, \ldots)$, i.e., $x_n = \sum_{i=1}^{\infty} t_{in} e_{n,i}$. By the definitions of convergences $\mathcal{G}_2$ and $\mathcal{G}_4$

we get that the sequence (x_n) can be represented in the form $x_n = \sum_{i=1}^{K} \alpha_{n,i} e_{p_{n,i}}$, where $\sum_{i=1}^{K} \alpha_{n,i} < M \cdot K$, for each $n \in \mathbb{N}$.

Therefore we have $x_n - \left(\sum_{i=1}^{K} \alpha_{n,i}\right) e_n \xrightarrow{\mathcal{G}_5} 0$ and, consequently, the sequence $x_n = x_n - \left(\sum_{i=1}^{K} \alpha_{n,i}\right) e_n + \left(\sum_{i=1}^{K} \alpha_{n,i}\right) e_n$ is convergent to 0 in the convergence $\mathcal{G}_{5(e_n)}$.

Now, suppose that $\mathcal{G}_5 \subset \mathcal{G} \subset \mathcal{G}_6$. Let $x_n \xrightarrow{\mathcal{G}} 0$ and $x_n \overset{\mathcal{G}_5}{\not\to} 0$. Since $\mathcal{G} \subset \mathcal{G}_6 = \mathcal{G}_{5(e_n)}$, we may assume without loss of generality that $x_n = y_n + z_n$ for each $n \in \mathbb{N}$, where $y_n \xrightarrow{\mathcal{G}_5} 0$, $z_n = \sum_{i=1}^{K} \alpha_{n,i} e_{p_{n,i}}$, and $\sum_{i=1}^{K} \alpha_{n,i} \to \alpha \neq 0$ (in $\mathbb{R}$).

Then $z_n - \alpha e_n \xrightarrow{\mathcal{G}_5} 0$ and, by condition **L** and the inclusion $\mathcal{G}_5 \subset \mathcal{G}$, we get $e_n = \left(e_n - \frac{z_n}{\alpha}\right) + \frac{z_n}{\alpha} \xrightarrow{\mathcal{G}} 0$. Hence $\mathcal{G} = \mathcal{G}_6$. □

DEFINITION 2. A sequence $(\bar{x}_n)$ will be called *essentially outer* for $\mathcal{G}$ if it is linearly independent and every subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$ is outer for $\mathcal{G}$.

PROPOSITION 5. *Let $(\bar{x}_n)$ be an outer sequence for $\mathcal{G}$. Then there is a subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$ such that $(\bar{x}_{p_n})$ is essentially outer for $\mathcal{G}$.*

P r o o f. By condition **U**, there is a subsequence $(\bar{x}_{q_n})$ of $(\bar{x}_n)$ such that $\bar{x}_{r_n} \overset{\mathcal{G}}{\not\to} 0$ for every subsequence $(\bar{x}_{r_n})$ of $(\bar{x}_{q_n})$. By Remark 4, there is a linearly independent subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_{q_n})$. Evidently, the sequence $(\bar{x}_{p_n})$ has the required properties. □

The following question seems to be a natural one: is it always possible to select from an arbitrary outer sequence $(\bar{x}_n)$ an essentially outer subsequence $(\bar{x}_{p_n})$ such that $\mathcal{G}_{(\bar{x}_n)} = \mathcal{G}_{(\bar{x}_{p_n})}$?

The following simple example gives a negative answer to this question.

EXAMPLE 6. Consider the convergence $\mathcal{G}_1$ for Example 1. Let $(\bar{x}_n)$ consist of elements of the two sequences: (e_n) and (g_n), where $g_n = \sum_{i=2^{n-1}+1}^{2^n} e_i$.

It is easy to see that $(\bar{x}_n)$ is outer for $\mathcal{G}_1$. Moreover, $\mathcal{G}_{(e_n)} \cap \mathcal{G}_{(f_n)} = \mathcal{G}_1$. Now, consider an arbitrary linearly independent subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$. Then either $(\bar{x}_{p_n})$ does not contain any subsequence of (e_n) or $(\bar{x}_{p_n})$ does not contain any subsequence of (g_n). In both cases, we have $\mathcal{G}_{(\bar{x}_{p_n})} \neq \mathcal{G}_{(\bar{x}_n)}$.

DEFINITION 3. An outer sequence $(\bar{x}_n)$ that satisfies the following condition

(c) $y_n \xrightarrow{\mathcal{G}} 0$ and $(y_n) \in B[\bar{x}_n]$ implies $\sum_{i=1}^{k} \alpha_{n,i} \to 0$

will be called *strongly outer* for $\mathcal{G}$.

EXAMPLE 7. Consider the convergences $\mathcal{G}_1, \mathcal{G}_3$ and the sequence (e_n) of Example 1. It is easy to prove that (e_n) is strongly outer for $\mathcal{G}_1$ and $\mathcal{G}_3$, while $(2^n e_n)$ is strongly outer for $\mathcal{G}_1$ but not for $\mathcal{G}_3$.

Remark 5. If $(\bar{x}_n)$ is strongly outer for $\mathcal{G}$, then $\bar{x}_{p_n} \overset{\mathcal{G}}{\not\to} 0$ for every subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$.

PROPOSITION 6. *The following conditions are equivalent:*

(i) *a sequence* $(\bar{x}_n)$ *is strongly outer for* $\mathcal{G}$*;*

(ii) *for every* $(\bar{z}_n)$ *such that* $\bar{z}_n \xrightarrow{\mathcal{G}} z$ *the sequence* $(\bar{x}_n + \bar{z}_n)$ *is strongly outer for* $\mathcal{G}$*;*

(iii) *for every* $(\bar{z}_n)$ *such that* $\bar{z}_n \xrightarrow{\mathcal{G}} z$ *the sequence* $(\bar{x}_n + \bar{z}_n)$ *is outer for* $\mathcal{G}$*;*

(iv) *there is* $(\bar{z}_n)$ *such that* $\bar{z}_n \xrightarrow{\mathcal{G}} z$*, where* $z \neq 0$*, and the sequence* $(\bar{x}_n + \bar{z}_n)$ *is strongly outer for* $\mathcal{G}$*;*

(v) *there is* $(\bar{z}_n)$ *such that* $\bar{z}_n \xrightarrow{\mathcal{G}} z$*, where* $z \neq 0$*, and the sequences* $(\bar{x}_n)$ *and* $(\bar{x}_n + \bar{z}_n)$ *are outer for* $\mathcal{G}$.

Proof. Since the implications (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv), and (iii) $\Rightarrow$ (v) are evident, and (iv) $\Rightarrow$ (v) follows from (i) $\Rightarrow$ (ii), it remains to prove the implications (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii)

Let $(\bar{x}_n)$ be strongly outer for $\mathcal{G}$ and $\bar{z}_n \xrightarrow{\mathcal{G}} z$. We shall show that the sequence $(\bar{x}_n + \bar{z}_n)$ satisfies conditions (a), (b), (c) of Definitions 1 and 3.

To prove condition (a) it suffices to observe that, by (b), $\bar{x}_n \overset{\mathcal{G}}{\not\to} -z$ and this, by **L**, implies $\bar{x}_n + \bar{z}_n \overset{\mathcal{G}}{\not\to} 0$.

To prove (b) suppose that

$$\sum_{i=1}^{k} \alpha_{n,i}(\bar{x}_{p_{n,i}} + \bar{z}_{p_{n,i}}) \xrightarrow{\mathcal{G}} x,$$

for some x. Without loss of generality we may assume that $\alpha_{n,i} \to \alpha_i$ for each $i = 1, 2, \ldots, k$. Then

$$\sum_{i=1}^{k} \alpha_{n,i}\bar{z}_{p_{n,i}} \xrightarrow{\mathcal{G}} \sum_{i=1}^{k} \alpha_i z \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_{n,i}\bar{x}_{p_{n,i}} \xrightarrow{\mathcal{G}} x - \sum_{i=1}^{k} \alpha_i z = y.$$

By (b), we get $y = 0$ and, by (c), $\sum_{i=1}^{k} \alpha_{n,i} \to 0$. Hence $\sum_{i=1}^{k} \alpha_i z = 0$ and, consequently, $x = 0$, which proves condition (b).

Now suppose that

$$\sum_{i=1}^{k} \alpha_{n,i}(\bar{x}_{p_{n,i}} + \bar{z}_{p_{n,i}}) \xrightarrow{\mathcal{G}} 0 .$$

Since $\sum_{i=1}^{k} \alpha_{n,i} \to \alpha \neq 0$ implies $\sum_{i=1}^{k} \alpha_{n,i} x_{p_{n,i}} \xrightarrow{\mathcal{G}} -\alpha z \neq 0$, we get condition (c).

(v) $\Rightarrow$ (i)

Suppose, on the contrary, that $(\bar{x}_n)$ is not strongly outer for $\mathcal{G}$, i.e., there is a sequence (y_n) of the form $y_n = \sum_{i=1}^{k} \alpha_{n,i}\bar{x}_{p_{n,i}}$ such that $y_n \to 0$ and $\sum_{i=1}^{k} \alpha_{n,i} \to \alpha$. Then, by **L**, we have

$$\sum_{i=1}^{k} \alpha_{n,i}(\bar{x}_{p_{n,i}} + z_{p_{n,i}}) \xrightarrow{\mathcal{G}} \alpha z$$

for every $(\bar{z}_n)$ such that $\bar{z}_n \xrightarrow{\mathcal{G}} z$. Hence, if $z \neq 0$, the sequence $(\bar{x}_n + \bar{z}_n)$ does not satisfy condition (b) of Definition 1. We get a contradiction. □

Corollary 3. *If $(\bar{x}_n)$ is strongly outer for $\mathcal{G}$, then for every $x \in X$ there is an extension $\hat{\mathcal{G}}$ of $\mathcal{G}$ such that $\bar{x}_n \xrightarrow{\hat{\mathcal{G}}} x$.*

Remark 6. If $(\bar{x}_n)$ is strongly outer for $\mathcal{G}$, then for every sequence (α_n) of real numbers such that $\alpha_n \to \infty$ and every subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$ the sequence $(\alpha_{p_n}\bar{x}_{p_n})$ is not convergent in $\mathcal{G}_{(\bar{x}_n)}$.

Proof. It suffices to show that $(\alpha_{p_n}\bar{x}_{p_n})$ is not convergent to 0 in $\mathcal{G}_{(\bar{x}_n)}$. Suppose, on the contrary, that $\alpha_{p_n}\bar{x}_{p_n} \xrightarrow{\mathcal{G}_{(\bar{x}_n)}} 0$ for some subsequence $(\bar{x}_{p_n})$ of $(\bar{x}_n)$ and $\alpha_{p_n} \to \infty$. Then there is a subsequence $(\bar{x}_{q_n})$ of $(\bar{x}_{p_n})$ such that $\alpha_{q_n}\bar{x}_{q_n} = y_n + z_n$ for each $n \in \mathbb{N}$, where $y_n \in C[\bar{x}_n]$ and $z_n \xrightarrow{\mathcal{G}} 0$.

Denote

$$\bar{y}_n = \bar{x}_{q_n} - \frac{y_n}{\alpha_{q_n}} = \sum_{i=1}^{k} \beta_{n,i}\bar{x}_{p_{n,i}} .$$

Hence, by condition **L**, we have $\bar{y}_n \xrightarrow{\mathcal{G}} 0$. Since $\bar{y}_n \in C[\bar{x}_n]$ and $\sum_{i=1}^{k} \beta_{n,i} \to 1$, we get a contradiction. □

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Institute of Mathematics
Silesian Technical University
Kaszubska 23
PL–44-100 Gliwice
POLAND

E-mail: pochcial@zeus.polsl.gliwice.pl