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NOTE ON INVARIANT AND ALMOST INVARIANT MEASURABLE SETS

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ABSTRACT. Let $\mathscr{A}_j$ stand for σ-algebras of subsets of non-empty sets Ω_j and G_j for groups of transformations together with a σ-algebra $\mathscr{B}(G_j)$ of subsets of G_j admitting some left-invariant probability measure such that the mapping $(g_j, \omega_j) \to g_j(\omega_j)$, $g_j \in G_j$, $\omega_j \in \Omega_j$, is $(\mathscr{B}(G_j) \otimes \mathscr{A}_j, \mathscr{A}_j)$-measurable and the mappings $g_j \to g \circ g_j$, $g_j \in G_j$, $g \in G_j$ fixed, are $\big(\mathscr{B}(G_j), \mathscr{B}(G_j)\big)$-measurable, $j = 1, 2$. Then it is shown that $\mathscr{B}(G_1 \times G_2, \mathscr{A}_1 \otimes \mathscr{A}_2) = \mathscr{B}(G_1, \mathscr{A}_1) \otimes \mathscr{B}(G_2, \mathscr{A}_2)$ for compact groups G_j with $\mathscr{L}(G_j)$ as the Borel σ-algebra of G_j, $j = 1, 2$, and $\mathscr{B}(G_1 \times G_2, \mathscr{A}_1 \otimes \mathscr{A}_2, Q_1 \otimes Q_2) = \mathscr{B}(G_1, \mathscr{A}_1, Q_1) \otimes \mathscr{B}(G_2, \mathscr{A}_2, Q_2)$ $[Q_1 \otimes Q_2]$ is valid. Here $\mathscr{B}(G_j, \mathscr{A}_j)$ respectively $\mathscr{B}(G_j, \mathscr{A}_j, Q_j)$ denotes the sub-σ-algebra of $\mathscr{A}_j$ consisting of all G_j-invariant respectively Q_j-almost G_j-invariant sets $A_j \in \mathscr{A}_j$, i.e., $A_j = g_j(A_j)$, $g_j \in G_j$, respectively, $Q_j\big(A_j \Delta g_j(A_j)\big) = 0$, $g_j \in G_j$, holds true, where Q_j is some G_j-invariant probability measure on $\mathscr{A}_j$, $j = 1, 2$. Moreover, in the almost invariant case one might replace countable additivity of the invariant probability measures on $\mathscr{B}(G_j)$, $j = 1, 2$, by finite additivity.

1. Introduction and auxiliary results

The study of the product structure of measurable and almost invariant sets will be based on a discrete approximation procedure for probability measures, which is in the non-invariant case essentially known and might be proved by a basic separation theorem for convex sets (cf. [2], V.2.7) and a characterization of the dual space consisting of all bounded and measurable functions (cf. [2], IV.5.1). Here an elementary approach is presented allowing to include also the invariant case. For this purpose one introduces the notion of a $\mathscr{F}$-invariant probability measure P on a σ-algebra $\mathscr{A}$ of subsets of some non-empty set Ω by the property $P = P^f$, $f \in \mathscr{F}$, where $\mathscr{F}$ denotes a set consisting of $(\mathscr{A}, \mathscr{A})$-measurable functions $f \colon \Omega \to \Omega$. The notion of a $\mathscr{F}$-invariant probability measure might be carried over to the case, where countable additivity of P is replaced by finite additivity. In the sequel non-negative, normed, and finitely

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additive set functions on a σ-algebra of subsets of some non-empty set will be called finitely additive probabilities. Now everything is prepared for

Lemma 1. *Let P stand for a finitely additive probability on a σ-algebra $\mathscr{A}$ of subsets of some non-empty set Ω and $\mathscr{F}$ for a non-empty set consisting of $(\mathscr{A},\mathscr{A})$-measurable functions $f\colon \Omega \to \Omega$. Then the following assertions hold true:*

(i) *There exists a net $(P_d)_{d\in D}$ of probability measures on $\mathscr{A}$ consisting of finite linear combinations of one-point masses restricted to $\mathscr{A}$ with non-negative coefficients converging setwise on $\mathscr{A}$ to P. Moreover, the net $(P_d)_{d\in D}$ might be replaced by a sequence if and only if P is a countable linear combination of one-point masses restricted to $\mathscr{A}$ with non-negative coefficients.*

(ii) *P is some $\mathscr{F}$-invariant probability on $\mathscr{A}$ if and only if there exists a net $(P_d)_{d\in D}$ consisting of finite linear combinations of one-point masses restricted to $\mathscr{A}$ with non-negative coefficients converging setwise on $\mathscr{A}$ to P with the following property: For any $A_0 \in \mathscr{A}$ there exists some $d_0 \in D$ such that $P_d(A_0) = P_d^f(A_0)$, $f \in \mathscr{F}$, $d \in D$ and $d \geq d_0$ is valid, where $\geq$ stands for the partial order of the increasingly directed set D.*

P r o o f . The index set D will be introduced by the system of all finite subsets $\{A_1,\dots,A_n\}$ of $\mathscr{A}$ being increasingly directed by inclusion. Here $P_{\{A_1,\dots,A_n\}}$ stands for an extension of the restriction $P|\sigma(\{A_1,\dots,A_n\})$ of P to the σ-algebra generated by $A_1,\dots,A_n$. Therefore, $P_{\{A_1,\dots,A_n\}}$ might be chosen as a finite linear combination of one-point masses with non-negative coefficients, since $P|\sigma(\{A_1,\dots,A_n\})$ might be represented as $P|\sigma(\{A_1,\dots,A_n\}) = \sum_{j=1}^{m} P(B_j)\delta_{\omega_j}$, where δ_{ω_j} stands for the one-point mass at ω_j and where ω_j should belong to B_j, $j = 1,\dots,m$, being the atoms of $\sigma(\{A_1,\dots,A_n\})$. Moreover, the corresponding net $(P_d)_{d\in D}$ converges setwise on $\mathscr{A}$ to P. In particular, P is a countable linear combination of one-point masses restricted to $\mathscr{A}$ with non-negative coefficients if there already exists a sequence $(P_n)_{n\in\mathbb{N}}$ consisting of finite linear combinations of one-point masses restricted to $\mathscr{A}$ with non-negative coefficients converging setwise on $\mathscr{A}$ to P. In the special case, where all singletons $\{\omega\}$, Ω, belong to $\mathscr{A}$, this last assertion is obviously true, since according to a theorem of N i k o d y m (cf. [2], III.7.4) P is already countably additive. In the general case one should take into consideration that P is absolutely continuous with respect to the restriction $Q|\mathscr{A}$ of the probability measure Q defined on the system $\mathscr{P}(\Omega)$ of all subsets of Ω according to $Q = \sum_{n=1}^{\infty} \frac{1}{2^n} Q_n$, where Q_n

is a finite linear combination of one-point masses on $\mathscr{P}(\Omega)$, $n \in \mathbb{N}$, satisfying $\lim_{n\to\infty} Q_n(A) = P(A)$, $A \in \mathscr{A}$. Moreover, $\bar{P}$ defined by $\bar{P}(B) = \int_B \frac{dP}{d(Q\,|\,\mathscr{A})}\,dQ$, $B \in \mathscr{P}(\Omega)$, is a probability measure on $\mathscr{P}(\Omega)$ whose restriction to $\mathscr{A}$ coincide with P, i.e., P might be represented as a countable linear combination of one-point masses restricted to $\mathscr{A}$. Here $\frac{dP}{d(Q\,|\,\mathscr{A})}$ denotes some version of the Radon-Nikodym derivative of P with respect to $Q\,|\,\mathscr{A}$.

For the proof of part (ii) one has only to take into consideration that by the construction of the net $(P_d)_{d\in D}$ in connection with the proof of part (i) the net $(P_d)_{d\in D}$ has the property of asymptotic $\mathscr{F}$-invariance described in (ii), if P is some $\mathscr{F}$-invariant probability. Furthermore, any net $(P_d)_{d\in D}$ of probability measures on $\mathscr{A}$ converging setwise on $\mathscr{A}$ to the probability P on $\mathscr{A}$ and being asymptotically $\mathscr{F}$-invariant in the sense of (ii) implies that P is already $\mathscr{F}$-invariant. $\square$

The next auxiliary result is a version of a result of [5], Lemma 10.1, p. 78, in the language of sufficient sub-σ-algebras with a different proof based on a discussion of the equality sign occurring in the inequality of Cauchy-Schwarz. Here a sub-σ-algebra $\mathscr{B}$ of a σ-algebra $\mathscr{A}$ is sufficient for a set $\mathscr{P}$ of probability measures P on $\mathscr{A}$ if there exists a common version of the conditional probabilities $P(A\,|\,\mathscr{B})$, $P \in \mathscr{P}$, $A \in \mathscr{A}$ fixed, independent of $P \in \mathscr{P}$.

Lemma 2. *Let $\mathscr{A}$ denote a σ-algebra of subsets of some non-empty set Ω, $\mathscr{F}$ a non-empty set consisting of $(\mathscr{A},\mathscr{A})$-measurable functions $f\colon \Omega \to \Omega$, and $\mathscr{B}$ the sub-σ-algebra of $\mathscr{A}$ consisting of all Q-almost $\mathscr{F}$-invariant sets $A \in \mathscr{A}$, i.e., $Q\big(A \,\Delta\, f^{-1}(A)\big) = 0$, $f \in \mathscr{F}$, is valid, where Q is some $\mathscr{F}$-invariant probability measure on $\mathscr{A}$. Then $\mathscr{B}$ is sufficient for the set $\mathscr{P}$ consisting of all $\mathscr{F}$-invariant probability measures on $\mathscr{A}$ being absolutely continuous with respect to Q.*

P r o o f. It will be shown that there exists some $\mathscr{B}$-measurable version of the Radon-Nikodym $\frac{dP}{dQ}$ for any $P \in \mathscr{P}$ from which sufficiency of $\mathscr{B}$ for $\mathscr{P}$ follows. For this purpose let g stand for some version of $\frac{dP}{d\frac{P+Q}{2}}$ for $P \in \mathscr{P}$ fixed. Then the property of P and Q to be $\mathscr{F}$-invariant implies $\int_{f^{-1}(A)} g \circ f\, d\big(\frac{P+Q}{2}\big) = \int_{f^{-1}(A)} g\, d\big(\frac{P+Q}{2}\big)$, $A \in \mathscr{A}$, $f \in \mathscr{F}$, from which $E_{\frac{P+Q}{2}}\big(g\,|\,f^{-1}(\mathscr{A})\big) = g \circ f\ \frac{Q+P}{2}$-a.e., $f \in \mathscr{F}$, and, therefore, also Q-a.e. follows. In particular, $E_{\frac{P+Q}{2}}(g \circ f \cdot g) = E_{\frac{P+Q}{2}}\big((g\circ f)^2\big)$, $f \in \mathscr{F}$, is valid, from which one arrives by the inequality of Cauchy-Schwarz at $E_{\frac{P+Q}{2}}\big((g\circ f)^2\big)\cdot E_{\frac{P+Q}{2}}(g^2)\ \Big(= E^2_{\frac{P+Q}{2}}(g^2)\Big) \geq E^2_{\frac{P+Q}{2}}\big((g\circ f)^2\big)$ $\Big(= E^2_{\frac{P+Q}{2}}(g^2)\Big)$, $f \in \mathscr{F}$, i.e., the equality sign holds true for the corresponding

Cauchy-Schwarz inequality. Hence, $g = a\, g \circ f\, \frac{Q+P}{2}$-a.e. $f \in \mathscr{F}$, and, therefore, Q-a.e. is valid for some $a \in \mathbb{R}$, since $g = 0$ Q-a.e. does not hold true. Finally, $g = a\, g \circ f$ Q-a.e., $f \in \mathscr{F}$, implies $g = g \circ f$ Q-a.e., $f \in \mathscr{F}$, since Q is $\mathscr{F}$-invariant, i.e., there exists some $\mathscr{B}$-measurable version of $\frac{dP}{d(P+Q)}$, and, therefore, also of $\frac{dP}{dQ}$, $P \in \mathscr{P}$, because of $\frac{dP}{dQ} = \frac{\frac{dP}{d(P+Q)}}{1-\frac{dP}{d(P+Q)}} I_{\{\frac{dP}{d(P+Q)}<1\}}$ Q-a.e., $P \in \mathscr{P}$. $\square$

The next auxiliary result refers to the notion of bounded completeness. Here a sub-σ-algebra $\mathscr{B}$ of a σ-algebra $\mathscr{A}$ is boundedly complete for a set $\mathscr{P}$ consisting of probability measures P on $\mathscr{A}$ if $h = 0$ P-a.e., $P \in \mathscr{P}$, whenever h is a bounded and $\mathscr{B}$-measurable function such that $\int h\, dP = 0$, $P \in \mathscr{P}$.

Lemma 3. *The sub-σ-algebra $\mathscr{B}$ of $\mathscr{A}$ consisting of all Q-almost $\mathscr{F}$-invariant sets belonging to $\mathscr{A}$ is boundedly complete for the set $\mathscr{P}$ consisting of all probability measures P on $\mathscr{A}$ being $\mathscr{F}$-invariant and absolutely continuous with respect to the $\mathscr{F}$-invariant probability measure Q on $\mathscr{A}$.*

P r o o f . One might replace $P \in \mathscr{P}$ occurring in $\int h\, dP = 0$, $P \in \mathscr{P}$, with h being bounded and $\mathscr{B}$-measurable, by the elementary conditional probability $P(\ |B)$ for any $B \in \mathscr{B}$ with $P(B) > 0$. Thus one arrives at $\int_B h\, dP = 0$, $P \in \mathscr{P}$, for all $B \in \mathscr{B}$, from which $h = 0$ P-a.e., $P \in \mathscr{P}$, follows. $\square$

The meaning of sufficiency together with bounded completeness illustrates the next auxiliary result, which is well-known (cf. [4]). For the sake of completeness, a sketch of the proof is included. The corresponding auxiliary result refers to the following notion of inclusion for sub-σ-algebras $\mathscr{B}$ and $\mathscr{C}$ of a σ-algebra $\mathscr{A}$ with respect to some set $\mathscr{P}$ consisting of probability measures P on $\mathscr{A}$, namely, $\mathscr{B} \subset \mathscr{C}[\mathscr{P}]$ if for any $B \in \mathscr{B}$ there exists some $C \in \mathscr{C}$ with $P(B \Delta C) = 0$, $P \in \mathscr{P}$. In particular, $\mathscr{B} = \mathscr{C}[\mathscr{P}]$ means $\mathscr{B} \subset \mathscr{C}[\mathscr{P}]$ and $\mathscr{C} \subset \mathscr{B}[\mathscr{P}]$. In the special case, where $\mathscr{P}$ consists only of one probability measure P on $\mathscr{A}$, one writes $\mathscr{B} = \mathscr{C}[P]$ instead of $\mathscr{B} = \mathscr{C}\big[\{P\}\big]$.

Lemma 4. *The following assertions hold true:*

(i) *Let $\mathscr{B}, \mathscr{C}$ denote sub-σ-algebras of a σ-algebra $\mathscr{A}$, where $\mathscr{B}$ is sufficient and boundedly complete for a set $\mathscr{P}$ consisting of probability measures P on $\mathscr{A}$. Then $\mathscr{C}$ shares these properties with $\mathscr{B}$ if and only if $\mathscr{B} = \mathscr{C}[\mathscr{P}]$ is valid.*

(ii) *Let $\mathscr{B}_j$ stand for sub-σ-algebras of σ-algebras $\mathscr{A}_j$ being sufficient and boundedly complete for sets $\mathscr{P}_j$ consisting of probability measures P_j on $\mathscr{A}_j$, $j = 1, 2$. Then $\mathscr{B}_1 \otimes \mathscr{B}_2$ is sufficient and boundedly complete for $\mathscr{P}_1 \otimes \mathscr{P}_2$ defined as the set consisting of all probability measures $P_1 \otimes P_2$, $P_j \in \mathscr{P}_j$, $j = 1, 2$, on $\mathscr{A}_1 \otimes \mathscr{A}_2$.*

Proof. Part (i) is a rather straightforward consequence of the inequality for the variances $\operatorname{Var}_P(P(A|\mathscr{B})) \le \operatorname{Var}_P(I_A)$ for $P \in \mathscr{P}$ and $A \in \mathscr{A}$ fixed, where I_A stands for the indicator function of A and where the equality sign holds true if and only if $P(A|\mathscr{B}) = I_A$ P-a.e. is valid.

A proof of (ii) concerning sufficiency of $\mathscr{B}_1 \otimes \mathscr{B}_2$ for $\mathscr{P}_1 \otimes \mathscr{P}_2$ might be based on a Dynkin system argument starting from the observation that one yields already some common version of the conditional probabilities $(P_1 \otimes P_2)(A_1 \times A_2 | \mathscr{B}_1 \otimes \mathscr{B}_2)$, $P_j \in \mathscr{P}_j$, $A_j \in \mathscr{A}_j$, $j = 1, 2$, independent of $P_1 \otimes P_2 \in \mathscr{P}_1 \otimes \mathscr{P}_2$.

The proof of (ii) in connection with bounded completeness relies on a Fubini type argument yielding $\int_{A_1 \times A_2} h \, d(P_1 \otimes P_2) = 0$, $P_j \in \mathscr{P}_j$, $A_j \in \mathscr{B}_j$, if h is a bounded and $\mathscr{B}_1 \otimes \mathscr{B}_2$-measurable function such that $\int h \, d(P_1 \otimes P_2) = 0$, $P_j \in \mathscr{P}_j$, $j = 1, 2$. An argument based on Dynkin systems leads to $\int_A h \, d(P_1 \otimes P_2) = 0$, $P_j \in \mathscr{P}_j$, $j = 1, 2$, for all $A \in \mathscr{B}_1 \otimes \mathscr{B}_2$, i.e., $h = 0 (P_1 \otimes P_2)$-a.e., $P_j \in \mathscr{P}_j$, $j = 1, 2$, is valid. □

Combining Lemma 2, 3, and 4, one arrives at

Lemma 5. *Let $\mathscr{A}_j$ stand for σ-algebras of subsets of non-empty sets Ω_j, $\mathscr{F}_j$ for non-empty sets of $(\mathscr{A}_j, \mathscr{A}_j)$-measurable functions $f_j \colon \Omega_j \to \Omega_j$, Q_j for some $\mathscr{F}_j$-invariant probability measure on $\mathscr{A}_j$, and $\mathscr{B}_j$ the sub-σ-algebra of $\mathscr{A}_j$ consisting of all Q_j-almost $\mathscr{F}_j$-invariant sets belonging to $\mathscr{A}_j$, $j = 1, 2$. Then $\mathscr{B}_1 \otimes \mathscr{B}_2 = \mathscr{B}[Q_1 \otimes Q_2]$ is valid for the sub-σ-algebra $\mathscr{B}$ of $\mathscr{A}_1 \otimes \mathscr{A}_2$ consisting of all $(Q_1 \otimes Q_2)$-almost $(\mathscr{F}_1 \times \mathscr{F}_2)$-invariant sets belonging to $\mathscr{A}_1 \otimes \mathscr{A}_2$ if and only if $\mathscr{B}$ is boundedly complete for $\mathscr{P}_1 \otimes \mathscr{P}_2$, where $\mathscr{P}_j$ stands for the set consisting of all probability measures P_j on $\mathscr{A}_j$ being $\mathscr{F}_j$-invariant and absolutely continuous with respect to Q_j, $j = 1, 2$.*

Proof. The property of sufficiency of the sub-σ-algebra $\mathscr{B}$ for the set consisting of all probability measures on $\mathscr{A}_1 \otimes \mathscr{A}_2$ being $(\mathscr{F}_1 \times \mathscr{F}_2)$-invariant and absolutely continuous with respect to $Q_1 \otimes Q_2$ according to Lemma 3 is obviously preserved with respect to the subset $\mathscr{P}_1 \otimes \mathscr{P}_2$. Hence Lemma 5 follows from Lemma 2, 3, and 4. □

2. Main results

In the sequel $\mathscr{F}_j$ will stand for groups of transformations $g_j \colon \Omega_j \to \Omega_j$, i.e., g_j is one-to-one and onto, $j = 1, 2$. Furthermore, it will be assumed that G_j acts measurably in the sense described in the following

Theorem. *Let $\mathscr{A}_j$ denote σ-algebras of subsets of non-empty sets Ω_j, G_j groups of transformations $g_j\colon \Omega_j \to \Omega_j$ together with a σ-algebra $\mathscr{B}(G_j)$ of subsets of G_j such that the mapping $(g_j,\omega_j) \to g_j(\omega_j)$ is $\big(\mathscr{B}(G_j) \otimes \mathscr{A}_j, \mathscr{A}_j\big)$-measurable, and the mappings $g_j \to g \circ g_j$, $g_j \in G_j$, $g \in G_j$ fixed, are $\big(\mathscr{B}(G_j),\ \mathscr{B}(G_2)\big)$-measurable, where $\mathscr{B}(G_j)$ admits some G_j-invariant, finitely additive probability, $j = 1,2$. Moreover, $\mathscr{B}(G_j, \mathscr{A}_j, Q_j)$ stands for the sub-σ-algebra of $\mathscr{A}_j$ consisting of all Q_j-almost G_j-invariant sets $A_j \in \mathscr{A}_j$, i.e., $Q_j\big(A_j \Delta g_j(A_j)\big) = 0$, $g_j \in G_j$, is valid, where Q_j is some G_j-invariant probability measure on $\mathscr{A}_j$, $j = 1,2$. Then $\mathscr{B}(G_1 \times G_2,\ \mathscr{A}_1 \otimes \mathscr{A}_2,\ Q_1 \otimes Q_2) = \mathscr{B}(G_1, \mathscr{A}_1, Q_1) \otimes \mathscr{B}(G_2, \mathscr{A}_2, Q_2)[Q_1 \otimes Q_2]$ holds true for the sub-σ-algebra $\mathscr{B}(G_1 \times G_2,\ \mathscr{A}_1 \otimes \mathscr{A}_2,\ Q_1 \otimes Q_2)$ of $\mathscr{A}_1 \otimes \mathscr{A}_2$ consisting of all $(Q_1 \otimes Q_2)$-almost invariant sets belonging to $\mathscr{A}_1 \otimes \mathscr{A}_2$.*

P r o o f . According to Lemma 5 it has to be shown that $\mathscr{B}(G_1 \times G_2,\ \mathscr{A}_1 \otimes \mathscr{A}_2,\ Q_1 \otimes Q_2)$ is boundedly complete for $\mathscr{P}_1 \otimes \mathscr{P}_2$, where $\mathscr{P}_j$ stands for the set consisting of all G_j-invariant probability measures on $\mathscr{A}_j$ being absolutely continuous with respect to Q_j, i.e., one has to conclude that, if h is a bounded and $\mathscr{B}(G_1 \times G_2,\ \mathscr{A}_1 \otimes \mathscr{A}_2,\ Q_1 \otimes Q_2)$-measurable function such that $\int h\, d(P_1 \otimes P_2) = 0$, $P_j \in \mathscr{P}_j$, $j = 1,2$, then $h = 0$ $(P_1 \otimes P_2)$-a.e. for all $P_j \in \mathscr{P}_j$, $j = 1,2$. It will be shown that one might replace $P_j \in \mathscr{P}_j$ by any probability measure P_j on $\mathscr{A}_j$ satisfying $P_j \le c_j Q_j$ for some $c_j > 0$, $j = 1,2$, which yields $\int\limits_{A_1 \times A_2} h\, d(Q_1 \otimes Q_2) = 0$ for all $A_j \in \mathscr{A}_j$, $j = 1,2$. An argument based on Dynkin systems results in $\int\limits_A h\, d(Q_1 \otimes Q_2) = 0$, $A \in \mathscr{A}_1 \otimes \mathscr{A}_2$, i.e., $h = 0$ $(Q_1 \otimes Q_2)$-a.e. is valid, which proves that $\mathscr{B}(G_1 \times G_2,\ \mathscr{A}_1 \otimes \mathscr{A}_2,\ Q_1 \otimes Q_2)$ is boundedly complete for $\mathscr{P}_1 \otimes \mathscr{P}_2$, since $P_j \in \mathscr{P}_j$ occurring in the equation $\int h\, d(P_1 \otimes P_2) = 0$ might be replaced by a probability measure P_j on $\mathscr{A}_j$ satisfying $P_j \le c_j Q_j$ for some $c_j > 0$, $j = 1,2$. For this purpose one introduces the corresponding probability measures $\bar{P}_j$ on $\mathscr{A}_j$ defined by $\int P_j^{g_j}(A_j)\, d\mu_j(g_j)$, $A_j \in \mathscr{A}_j$, where μ_j is some G_j-invariant, finitely additive probability on $\mathscr{B}(G_j)$, $j = 1,2$ (cf. [2], III.2.17 for the corresponding integration theory in the finitely additive case). Clearly, $\bar{P}_j$ is G_j-invariant and the property of $\bar{P}_j$ to be countably additive follows from the property that $P_j^{g_j}$, $g_j \in G_j$, are uniformly absolutely continuous with respect to Q_j on account of $P_j \le c_j Q_j$ for some $c_j > 0$, $j = 1,2$. One might also show that $\bar{P}_j$, $j = 1,2$, is countably additive as follows: Let $(\mu_{1d_1})_{d_1 \in D_1}$ respectively $(\mu_{2d_2})_{d_2 \in D_2}$ denote nets of probability measures on $\mathscr{B}(G_j)$ being finite linear combinations of one-point masses restricted to $\mathscr{B}(G_j)$ with non-negative coefficients converging setwise on $\mathscr{B}(G_j)$ to μ_j, which have been already introduced as G_j-invariant finitely additive probabilities on $\mathscr{B}(G_j)$,

$j = 1, 2$. Then, there exist subnets $\left(\int \frac{dP_j^{g_j}}{dQ_j} d\mu_{j d'_j}(g_j)\right)_{d'_j \in D'_j}$, $j = 1, 2$, satisfying $\lim_{d'_j} \int_{A_j} \left(\int \frac{dP_j^{g_j}}{dQ_j} d\mu_{j d'_j}(g_j)\right) dQ_j = \int_{A_j} f_j \, dQ_j$ for some non-negative function f_j belonging to $\mathscr{L}_1(\Omega_j, \mathscr{A}_j, Q_j)$, $j = 1, 2$ (cf. [2], IV.8.3 and V.4.2), i.e., $\bar{P}_j(A_j) = \int_{A_j} f_j \, dQ_j$, $A_j \in \mathscr{A}_j$, $j = 1, 2$, holds true.

Now an argument based on the theorem of Fubini applied to $\lim_{d_1} \lim_{d_2} \int \int h \, d(P_1^{g_1} \otimes P_2^{g_2}) \, d(\mu_{1 d_1} \otimes \mu_{2 d_2})(g_1, g_2)$, together with the property of the bounded function h to be $\mathscr{B}(G_1 \times G_2, \mathscr{A}_1 \otimes \mathscr{A}_2, Q_1 \otimes Q_2)$-measurable, shows that $\int h \, d\bar{P}_1 \otimes \bar{P}_2 = \int h \, dP_1 \otimes P_2$ holds true for any probability measures P_j on $\mathscr{A}_j$ satisfying $P_j \leq c_j Q_j$ for some $c_j > 0$, $j = 1, 2$, where the theorem of Fubini might be applied, since the mapping $(g_1, g_2, \omega_1, \omega_2) \to \big(g_1(\omega_1), g_2(\omega_2)\big)$, $g_j \in G_j$, $\omega_j \in \Omega_j$, $j = 1, 2$, is $\big(\mathscr{B}(G_1) \otimes \mathscr{B}(G_2) \otimes \mathscr{A}_1 \otimes \mathscr{A}_2, \mathscr{A}_1 \otimes \mathscr{A}_2\big)$-measurable. Therefore, bounded completeness of $\mathscr{B}(G_1 \times G_2, \mathscr{A}_1 \otimes \mathscr{A}_2, Q_1 \otimes Q_2)$ for $\mathscr{P}_1 \otimes \mathscr{P}_2$ has been proved. $\square$

The second approach for the proof of the property of countable additivity of $\bar{P}_j$ occurring in the proof of the preceding theorem based on the theorem of Alaoglu and the representation of the dual space of $\mathscr{L}_1(\Omega_j, \mathscr{A}_j, Q_j)$ by $\mathscr{L}_\infty(\Omega_j, \mathscr{A}_j, Q_j)$, $j = 1, 2$, might be applied by using the representation of the dual space of $\mathscr{L}_p(\Omega_j, \mathscr{A}_j, P_{0j})$ by $\mathscr{L}_q(\Omega_j, \mathscr{A}_j, P_{0j})$, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ (cf. [2], IV.8.1), where P_{0j} are probability measures on $\mathscr{A}_j$, to characterize the existence of G_j-invariant probability measure Q_j, $j = 1, 2$, as follows: There exists some G_j-invariant probability measure Q_j on $\mathscr{A}_j$ being absolutely continuous with respect to P_{0j} with $\frac{dQ_j}{dP_{0j}} \in \mathscr{L}_p(\Omega, \mathscr{A}_j, P_{0j})$, $j = 1, 2$, for some $p > 1$ if and only if there exists some probability measure P_j on $\mathscr{A}_j$ such that $P_j^{g_j}$, is absolutely continuous with respect to P_{0j} and $\frac{dP_j^{g_j}}{dP_{0j}} \in \mathscr{L}_p(\Omega_j, \mathscr{A}_j, P_{0j})$, $g_j \in G_j$, $j = 1, 2$, is fulfilled.

Moreover, one might consult [3] in connection with the problem whether locally compact groups admit invariant, finitely additive probabilities on the set of all subsets of the underlying groups. Furthermore, [3] contains also results concerning uniqueness of invariant, finitely additive probabilities defined on the set consisting of all subsets of a locally compact group based on probabilistic methods. Finally, in [1] one might find a recent characterization of the existence of invariant Borel probability measures based on paradoxical decompositions in connection with Polish group actions.

In the case of compact groups G_j, $j = 1,2$, of transformations acting measurably in the sense of the preceding theorem, the multiplicative structure of measurable and invariant sets without exceptional zero sets can be proved, namely

Corollary 1. *Let $\mathscr{A}_j$ stand for σ-algebras of subsets of non-empty sets Ω_j and G_j for compact groups of transformations with corresponding Borel σ-algebras $\mathscr{B}(G_j)$ such that the mapping $(g_j, \omega_j) \to g_j(\omega_j)$ is $\big(\mathscr{B}(G_j) \otimes \mathscr{A}_j, \mathscr{A}_j\big)$-measurable, $j = 1,2$. Moreover, $\mathscr{B}(G_j, \mathscr{A}_j)$ denotes the sub-σ-algebra of $\mathscr{A}_j$ consisting of all G_j-invariant sets $A_j \in \mathscr{A}_j$, i.e., $A_j = g_j(A_j)$, $g_j \in G_j$, $j = 1,2$, is valid. Then $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2) = \mathscr{B}(G_1, \mathscr{A}_1) \otimes \mathscr{B}(G_2, \mathscr{A}_2)$ holds true for the sub-σ-algebra $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$ of $\mathscr{A}_1 \otimes \mathscr{A}_2$ consisting of all $(G_1 \times G_2)$-invariant sets belonging to $\mathscr{A}_1 \otimes \mathscr{A}_2$.*

Proof. At first it will be shown that $\mathscr{B}(G_j, \mathscr{A}_j)$ is sufficient and boundedly complete for the set $\mathscr{P}_j$ consisting of all G_j-invariant probability measures P_j on $\mathscr{A}_j$, $j = 1,2$. Sufficiency of $\mathscr{B}(G_j, \mathscr{A}_j)$ for $\mathscr{P}_j$ follows from the observation that $\int I_{A_j} \circ g_j \, d\mu_j(g_j)$ with μ_j as the left-invariant, normed Haar measure of G_j on $\mathscr{B}(G_j)$ is a common version of the conditional probabilities $P_j\big(A_j \,|\, \mathscr{B}(G_j, \mathscr{A}_j)\big)$, $P_j \in \mathscr{P}_j$, $A_j \in \mathscr{A}_j$ fixed, independent of $P_j \in \mathscr{P}_j$, $j = 1,2$. Bounded completeness of $\mathscr{B}(G_j, \mathscr{A}_j)$ for $\mathscr{P}_j$ might obviously be derived from the observation that, if h_j are bounded $\mathscr{B}(G_j, \mathscr{A}_j)$-measurable functions h_j such that $\int h_j \, dP_j = 0$, $P_j \in \mathscr{P}_j$, then $\int_{B_j} h_j \, dP_j = 0$, $B_j \in \mathscr{B}(G_j, \mathscr{A}_j)$, $P_j \in \mathscr{P}_j$, $j = 1,2$, since P_j might be replaced by the elementary conditional probabilities $P_j(\ \,|B_j)$, $P_j \in \mathscr{P}_j$, $B_j \in \mathscr{B}_j$, in the case $P_j(B_j) > 0$, $j = 1,2$. In particular, Lemma 3 together with Lemma 4 (ii) shows that $\mathscr{B}(G_1, \mathscr{A}_1) \otimes \mathscr{B}(G_2, \mathscr{A}_2)$ is sufficient and boundedly complete for $\mathscr{P}_1 \otimes \mathscr{P}_2$. It will now be shown that $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$ is sufficient and boundedly complete for $\mathscr{P}_1 \otimes \mathscr{P}_2$. Sufficiency of $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$ for $\mathscr{P}_1 \otimes \mathscr{P}_2$ follows from the fact that, according to the preceding considerations concerning sufficiency, $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$ is sufficient for the set consisting of all $(G_1 \times G_2)$-invariant probability measures on $\mathscr{A}_1 \otimes \mathscr{A}_2$. For the proof of bounded completeness of $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$ for $\mathscr{P}_1 \otimes \mathscr{P}_2$, let h be a bounded and $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$-measurable function such that $\int h \, d(P_1 \otimes P_2) = 0$, $P_j \in \mathscr{P}_j$. One can replace $P_j \in \mathscr{P}_j$ occurring in the equation $\int h \, d(P_1 \otimes P_2) = 0$, $P_j \in \mathscr{P}_j$, $j = 1,2$, by $\bar{P}_j$ defined according to $\bar{P}_j(A_j) = \int P_j^{g_j}(A_j) \, d\mu_j(g_j)$, $A_j \in \mathscr{A}_j$, for any probability measure P_j on $\mathscr{A}_j$, $j = 1,2$. An argument based on the theorem of Fubini, which might be applied here, since the mapping $(g_1, g_2, \omega_1, \omega_2) \to \big(g_1(\omega_1), g_2(\omega_2)\big)$, $g_j \in G_j$, $\omega_j \in \Omega_j$, $j = 1,2$, is $\big(\mathscr{B}(G_1) \otimes \mathscr{B}(G_2) \otimes \mathscr{A}_1 \otimes \mathscr{A}_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2\big)$-measurable, together with the property of h to be $\mathscr{B}(G_1 \times G_2,\, \mathscr{A}_1 \otimes \mathscr{A}_2)$-measurable, yields $\int h \, d(P_1 \otimes P_2) = 0$ for all probability measures P_j on $\mathscr{A}_j$, $j = 1,2$. Choosing for P_j the one-point masses at $\omega_j \in \Omega_j$ restricted to $\mathscr{A}_j$, one arrives at $h(\omega_1, \omega_2) = 0$. In particular, a set $B \in \mathscr{B}(G_1 \times G_2, \mathscr{A}_1 \otimes \mathscr{A}_2)$ with $(P_1 \otimes P_2)(B) = 0$, $P_j \in \mathscr{P}_j$, $j = 1, 2$, must be already empty. Finally, Lemma 5 finishes the proof of $\mathscr{B}(G_1 \times G_2, \mathscr{A}_1 \otimes \mathscr{A}_2) = \mathscr{B}(G_1, \mathscr{A}_1) \otimes \mathscr{B}(G_2, \mathscr{A}_2)$. $\square$

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