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METRIC PRESERVING FUNCTIONS AND INFIMAL CONVOLUTION

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ABSTRACT. The paper [2] contains three theorems each of them we can obtain as a simply consequence of known results and some lemmas which contain some examples of metric preserving functions.

Let us begin by some notions. (See [4]). Let $\mathbb{R}_0 = [0, +\infty)$.

Let (M, d) be a metric space. We call a function $f\colon \mathbb{R}_0 \to \mathbb{R}_0$ metric preserving iff for each metric space (M, d) the function $d_f(x, y) = f\bigl(d(x, y)\bigr)$ for $\forall\, x, y \in M$ is a metric on M.

Denote by $\mathscr{M}$ the class of all metric preserving functions.

Let a, b and c be positive real numbers. We call the triplet (a, b, c) a *triangle triplet* iff $a \le b + c$, $b \le a + c$, and $c \le a + b$.

If $f, g\colon \mathbb{R}_0 \to \mathbb{R}_0$, then their infimal convolute $f_{\square}\, g$ is the following function: $(f_{\square}\, g)(x) = \inf\bigl\{f(y) + g(z)\colon y, z \in \mathbb{R}_0 \text{ and } y + z = x\bigr\}$ for $\forall x \in \mathbb{R}_0$. (See [5].)

Let Φ be a class of all functions $\varphi\colon \mathbb{R}_0 \to \mathbb{R}_0$ such that $\varphi(0) = 0$ and $\varphi(\mathbb{R}^+) \subseteq \mathbb{R}^+$ where $\mathbb{R}^+ = (0, +\infty)$. (See [2].)

The following first theorem from [2] follows immediately from [1, 2.7. Theorem].

Theorem 1. [3] *Let $\varphi \in \Phi$. Then $\varphi \in \mathscr{M}$ iff $\forall x, y \in \mathbb{R}^+$ $\varphi(x) + \varphi(y) \ge \max\bigl\{\varphi(z)\colon |x - y| \le z \le x + y\bigr\}$.*

It is easy to verify that the following proposition is true:

Proposition. *For each $x, y, z \in \mathbb{R}^+$, the following properties are equivalent*

(1) *the triplet (x, y, z) is a triangle triplet;*

(2) $x + y + z \ge 2 \max\{x, y, z\}$;

(3) $|x - y| \le z \le x + y$.

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Some following notions were introduced in [2]. Let $a, b \in \mathbb{R}^+$ and $a \le b \le 2a$, then $\mathscr{M}_{a,b} = \{\varphi \in \mathscr{M} : \varphi(\mathbb{R}^+) \subseteq [a,b]\}$.

Denote by $\mathscr{M}^* = \{\varphi \in \mathscr{M} : \varphi$ is subadditive and nondecreasing$\}$, $\mathscr{M}^*_< = \{\varphi \in \mathscr{M} : \varphi$ is subadditive and increasing (strictly)$\}$, and $\mathscr{M}^*_> = \{\varphi \in \mathscr{M} : \varphi$ is nondecreasing and $\forall x, y \in \mathbb{R}^+ : \varphi(x) + \varphi(y) > \varphi(x+y)\}$.

The main result of the last part of [2] is obtained in the following lemma.

Lemma 1 [2, Lemma 6]. *The classes $\mathscr{M}_{a,b}$, $\mathscr{M}^*$, $\mathscr{M}^*_<$, $\mathscr{M}^*_>$, $\{\min(a,x) : a \in \mathbb{R}^+\}$, $\{\frac{ax}{b+cx} : a,b,c \subset \mathbb{R}^+\}$, and $\{x^a : a \in (0,1)\}$ are closed subclasses of the class $\mathscr{M}$ in the sense of composition.*

Lemma 2. *Let $f, g \in \Phi$ and let f and g be both nondecreasing. Then $h = f_\square g$ is nondecreasing.*

Proof. Let $x, u \in \mathbb{R}_0$ be such that $x < u$. Let $v, w \in \mathbb{R}_0$ be such that $v + w = u$. Then there are $y, z \in \mathbb{R}_0$ such that $y \le v$, $z \le w$ and $y + z = x$. Since f and g are nondecreasing, we have $f(v) + g(w) \ge f(y) + g(z) \ge h(x)$. Since $v, w \in \mathbb{R}_0$ were arbitrary, we obtain $h(u) \ge h(x)$. $\square$

Lemma 3. *Let $f, g \in \Phi$ and let f and g be both increasing. Then $h = f_\square g$ is increasing.*

Proof. (By contradiction.) By the Lemma 2 the function h is nondecreasing and hence $h \in \mathscr{M}^*$. Let $x, u \in \mathbb{R}_0$ be such that $x < u$ and $h(x) = h(u)$. Then there are the convergence sequences $\{v_n\}_{n=1}^\infty$ and $\{w_n\}_{n=1}^\infty$ such that for each $n \in \mathbb{N}$ there holds $v_n, w_n \in [0,u]$, $v_n + w_n = u$, $f(v_n) + g(w_n) < h(u) + \frac{1}{n}$. Denote their limits by v, w, respectively. Hence $v_n \to v$, $w_n \to w$, where $v, w \in [0,u]$. Evidently $v + w = u$. We suppose that $u > 0$ and $w > 0$. (The cases $v = 0$, resp. $w = 0$ are similar.) Let $r, s \in [0,u]$ be such that $r < v$, $s < w$, $x < r + s$. Let $n \in \mathbb{N}$. Then there is $k \in \mathbb{N}$, $k \ge n$, such that $r < v_k$, $s < w_k$. Hence $h(u) = h(r+s) \le f(r) + g(s) < f(v_k) + g(w_k) < h(u) + \frac{1}{k} \le h(u) + \frac{1}{n}$. Because $n \in \mathbb{N}$ was arbitrary we have $h(u) = f(r) + g(s)$. Put $y = \min(x, r)$, $z = x - y$. Hence $h(u) = h(x) \le f(y) + g(z) < f(r) + g(s) = h(u)$. It is a contradiction. $\square$

By [5, Theorem 2.4] and [4, Theorem 1] it is not difficult to verify the following theorem:

Theorem 2. *The classes $\mathscr{M}_{a,b}$, $\mathscr{M}^*$, $\mathscr{M}^*_<$, and $\{\min(a,x) : a \in \mathbb{R}^+\}$, are closed subclasses of the class $\mathscr{M}$ in the sense of infimal convolution.*

Remark. The classes $\{\frac{ax}{b+cx} : a,b,c \subset \mathbb{R}^+\}$ and $\{x^a : a \in (0,1)\}$ are not closed classes of the class $\mathscr{M}$ in the sense of infimal convolution.

It is true because $f_\square g(x) = \min\{f(x), g(x)\}$ for these classes.

Problem. *Is the class $\mathscr{M}$ closed in the sense of infimal convolution?*

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