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ON SOME PROPERTIES OF ONE SPECIAL TYPE OF POLYNOMIALS AND NUMBERS

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ABSTRACT. We concentrate on polynomials given by the relation $\mathcal{J}_n(x) = \sum_{i=0}^{n-2} \binom{n}{i} x^{n-i}$, $n \in \mathbb{N}$, $n \geqq 2$, with $\mathcal{J}_0(x) = \mathcal{J}_1(x) = 0$. We have found some their properties and some recurrences which they satisfy. Further the numbers given by $\mathfrak{J}_n = \mathcal{J}_n(1)$ are investigated. Some results about their divisibility and primality are shown.

1. Introduction

Bernoulli's inequality, which is named after Jacob Bernoulli (1654–1705) (see [1], p. 380), is usually presented in the form

$$(\forall n \in \mathbb{N})\big(\forall x \in [-1, \infty)\big)\ (1+x)^n \geqq 1 + nx\,. \tag{1}$$

For $x = -1$ the proposition (1) is obvious and for $x \in (-1, \infty)$ it can be proved by the mathematical induction on n.

As $\mathcal{J}_n(x)$ polynomials we will denote polynomials

$$\mathcal{J}_0(x) = \mathcal{J}_1(x) = 0\,, \qquad \mathcal{J}_n(x) = \sum_{i=0}^{n-2} \binom{n}{i} x^{n-i}\,, \quad n \geqq 2\,,$$

therefore

$$\mathcal{J}_0(x) = 0\,, \qquad \mathcal{J}_n(x) = (1+x)^n - 1 - nx\,, \quad n \in \mathbb{N}\,. \tag{2}$$

For every positive integer n we will denote numbers $\mathfrak{J}_n$

$$\mathfrak{J}_n = \mathcal{J}_n(1) = 2^n - 1 - n\,.$$

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The first goal in this paper is to find some properties of polynomials $\mathcal{J}_n(x)$, especially their generating function and the ones for multiple convolutions of them. The second goal is to investigate the divisibility and primality of the numbers $\mathfrak{J}_n$. Our main results are Theorem 2 and its Corollary, Theorem 5 and the list of all primes given in the form $2^n - 1 - n$ for positive integers $n < 15000$.

2. Connection of the numbers $\mathfrak{J}_n$ with some other types of numbers

For *Eulerian numbers* $\left\langle {n \atop k} \right\rangle$, which represent the number of permutations $(p_1, p_2, \cdots, p_n)$ of $\{1, 2, \ldots, n\}$ that have k *ascents*, namely k places where $p_j < p_{j+1}$, the following can be proved:

$$\mathfrak{J}_n = \left\langle {n \atop 1} \right\rangle .$$

Mersenne numbers $\mathfrak{M}_n$ can be defined as numbers of the form $2^n - 1$, where $n \in \mathbb{N}$. It is easy to see that

$$\mathfrak{J}_n = \mathfrak{M}_n - n ,$$

and consequently

$$\mathfrak{J}_n \equiv \mathfrak{M}_n \pmod{n} .$$

THEOREM 1. *Let p be any prime, l any nonnegative integer and k any natural number. Then*

$$\mathfrak{M}_{kp^l} \equiv \mathfrak{M}_k \pmod{p} . \tag{3}$$

P r o o f . We use the mathematical induction on l. For $l = 0$ the statement (3) obviously holds. If we assume that (3) holds for $l - 1 \geqq 0$ we obtain

$$\mathfrak{M}_{kp^l} = 2^{kp^l} - 1 = \left(2^{kp^{l-1}}\right)^p - 2^{kp^{l-1}} + 2^{kp^{l-1}} - 1 \equiv \mathfrak{M}_{kp^{l-1}} \equiv \mathfrak{M}_k \pmod{p} .$$

Hence (3) holds for l, too. □

COROLLARY 1. *Let p be any prime, l any nonnegative integer number and k any natural number. Then*

$$\mathfrak{J}_{kp^l} \equiv \mathfrak{M}_k \pmod{p} .$$

P r o o f .

$$\mathfrak{J}_{kp^l} = 2^{kp^l} - 1 - kp^l \equiv \mathfrak{M}_{kp^l} \equiv \mathfrak{M}_k \pmod{p} .$$

□

Corollary 2. *Let p be any prime and l any natural number. Then $p^l \nmid \mathfrak{J}_{p^l}$. Especially, $p \nmid \mathfrak{J}_p$.*

P r o o f. For $k = 1$, (in Corollary 1)

$$\mathfrak{J}_{p^l} \equiv \mathfrak{M}_1 \equiv 1 \pmod{p},$$

therefore $p^l \nmid \mathfrak{J}_{p^l}$. □

Theorem 2. *Let n be any natural number, $n > 1$. Then $n \nmid \mathfrak{M}_n$.*

P r o o f. As $\mathfrak{M}_n$ is odd for any n, the assertion holds for every even n.

Let us consider that the following holds for any odd n:

$$n \mid \mathfrak{M}_n .$$

Every odd natural number n can be written in the standard form

$$n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} ,$$

with integers $a_i > 0$, $i = 1, 2, \dots s$, and primes $2 < p_1 < p_2 < \cdots < p_s$. Thus $p_1 \mid \mathfrak{M}_n$, too. It means $2^n \equiv 1 \pmod{p_1}$. But we can show that this congruence does not hold.

The group of numbers relatively prime to p_1 under multiplication mod p_1 has the order $p_1 - 1$. By Lagrange Theorem $m \mid p_1 - 1$, where m is the order of the cyclic subgroup generated by number 2 under multiplication mod p_1.

Thus the last congruence can be true if and only if

$$m \mid n . \tag{4}$$

For $m = 1$, the congruence $2^1 \equiv 1 \pmod{p_1}$ does not hold. It means that $1 < m < p_1 < p_2 < \cdots < p_s$. Hence (4) cannot be true. This is a contradiction and n does not divide $\mathfrak{M}_n$. □

Corollary. *Let n be any natural number, $n > 1$. Then $n \nmid \mathfrak{J}_n$.*

P r o o f. It is clear with respect to $\mathfrak{J}_n \equiv \mathfrak{M}_n \pmod{n}$. □

3. Primality of $\mathfrak{J}_n$ numbers

Now we will investigate which of $\mathfrak{J}_n$ numbers are primes. The computer testing of their primality is easier by using the following theorems.

Theorem 3. *Let n be any natural number. Then*

$$\mathfrak{J}_n \equiv 0 \pmod{2} \iff n \equiv 1 \pmod{2}, \quad \mathfrak{J}_n \equiv 1 \pmod{2} \iff n \equiv 0 \pmod{2}.$$

P r o o f. Let k be any natural number. Then

$$\mathfrak{J}_{2k} = 2^{2k} - 1 - 2k \equiv -1 \equiv 1 \pmod{2}, \quad \mathfrak{J}_{2k+1} = 2^{2k+1} - 1 - (2k+1) \equiv 0 \pmod{2}.$$

□

Theorem 4. *Let n be any natural number. Then*

$$\begin{aligned}\mathfrak{J}_n \equiv 0 \pmod 3 &\iff n \equiv 0,1 \pmod 6\,,\\ \mathfrak{J}_n \equiv 1 \pmod 3 &\iff n \equiv 2,3 \pmod 6\,,\\ \mathfrak{J}_n \equiv 2 \pmod 3 &\iff n \equiv 4,5 \pmod 6\,.\end{aligned}$$

Proof. Let k be any natural number. By the mathematical induction it is easy to prove the following:

$$\begin{aligned}\mathfrak{J}_{6k} &= 64^k - 6k - 1 \equiv 0 \pmod 3\,,\\ \mathfrak{J}_{6k+1} &= 2(64^k - 3k - 1) \equiv 0 \pmod 3\,,\\ \mathfrak{J}_{6k+2} &= 4\cdot 64^k - 6k - 3 \equiv 1 \pmod 3\,,\\ \mathfrak{J}_{6k+3} &= 8\cdot 64^k - 6k - 4 \equiv 1 \pmod 3\,,\\ \mathfrak{J}_{6k+4} &= 16\cdot 64^k - 6k - 5 \equiv 2 \pmod 3\,,\\ \mathfrak{J}_{6k+5} &= 32\cdot 64^k - 6k - 6 \equiv 2 \pmod 3\,.\end{aligned}$$

□

Theorem 5. *Let n be any natural number. Then*

$$\begin{aligned}2 \mid \mathfrak{J}_n &\iff n \equiv 1 \pmod 2\,,\\ 3 \mid \mathfrak{J}_n &\iff n \equiv 0,1 \pmod{2\cdot 3}\,,\\ 5 \mid \mathfrak{J}_n &\iff n \equiv 0,1,7,18 \pmod{4\cdot 5}\,,\\ 7 \mid \mathfrak{J}_n &\iff n \equiv 0,1,17 \pmod{3\cdot 7}\,,\\ 11 \mid \mathfrak{J}_n &\iff n \equiv 0,1,4,17,49,68,73,75,96,102 \pmod{10\cdot 11}\,,\\ 13 \mid \mathfrak{J}_n &\iff n \equiv 0,1,5,8,22,28,69,71,102,111,127,146 \pmod{12\cdot 13}\,,\\ 17 \mid \mathfrak{J}_n &\iff n \equiv 0,1,46,75,100,122,127,133 \pmod{8\cdot 17}\,,\\ 19 \mid \mathfrak{J}_n &\iff n \equiv 0,1,6,8,48,87,148,161,176,207,211,\\ &\qquad 221,232,241,244,273,299,326 \pmod{18\cdot 19}\,,\\ 23 \mid \mathfrak{J}_n &\iff n \equiv 0,1,15,17,97,140,150,168,192,233,241 \pmod{11\cdot 23}\,.\end{aligned}$$

Proof. The proof is similar to the proof of the previous theorem. □

We used some of the previous statements for the computer testing of the primality of $\mathfrak{J}_n$ numbers in the following way.

The conditions of the divisibility by numbers 2 and 3 lead to the simple fact that every prime in the form $\mathfrak{J}_n$ must be in the form $\mathfrak{J}_{6k+2}$ or $\mathfrak{J}_{6k+4}$. Similarly, with respect to the divisibility by numbers 2, 3 and 5, every prime in the form $\mathfrak{J}_n$ must be in the form $\mathfrak{J}_{60k\pm2}$, $\mathfrak{J}_{60k\pm4}$, $\mathfrak{J}_{60k\pm10}$, $\mathfrak{J}_{60k\pm14}$, $\mathfrak{J}_{60k\pm16}$, $\mathfrak{J}_{60k\pm22}$,

$\mathfrak{J}_{60k\pm 26}$, $\mathfrak{J}_{60k+32}$ or $\mathfrak{J}_{60k-8}$. Using computer we have found that $\mathfrak{J}_4$, $\mathfrak{J}_{10}$, $\mathfrak{J}_{14}$, $\mathfrak{J}_{16}$, $\mathfrak{J}_{26}$, $\mathfrak{J}_{50}$, $\mathfrak{J}_{56}$, $\mathfrak{J}_{70}$, $\mathfrak{J}_{116}$, $\mathfrak{J}_{2072}$, $\mathfrak{J}_{6250}$, $\mathfrak{J}_{13670}$, $\mathfrak{J}_{14216}$ and $\mathfrak{J}_{14626}$ are the only primes with the index less than 15000.

The following theorem shows the relation between the mentioned divisors and the modulus of the congruences in Theorem 5.

Theorem 6. *For any odd prime p there exists a natural number $k > 1$ such that*

$$p \mid \mathfrak{J}_{kp}\,.$$

The minimal number k is equal to the order m of the cyclic subgroup generated by number 2 under multiplication mod p.

Proof.

$$\mathfrak{J}_{kp} = 2^{kp} - 1 - kp = \left(2^k\right)^p - 2^k + 2^k - 1 - kp \equiv 2^k - 1 \pmod{p}\,.$$

The congruence $2^k \equiv 1 \pmod{p}$ is valid for every multiple of the order m. □

4. Some relations for polynomials $\mathcal{J}_n(x)$

(i) Recurrence relations.

With respect to (2), these polynomials satisfy a nonhomogeneous linear difference equation of the second order, and the roots of the characteristic equation of the corresponding homogeneous difference equation must be $\lambda_1 = x + 1$ and $\lambda_2 = 1$. Hence this homogeneous difference equation has the form

$$y_{n+2}(x) - (x+2)\,y_{n+1}(x) + (x+1)\,y_n(x) = 0\,. \tag{5}$$

Thus we obtain the sought relation after recurrence of polynomials $\mathcal{J}_n(x)$ by putting (2) into the left side of (5)

$$\begin{aligned}\mathcal{J}_{n+2}(x) &- (x+2)\mathcal{J}_{n+1}(x) + (x+1)\mathcal{J}_n(x)\\ &= \left[(1+x)^{n+2} - 1 - (n+2)\,x\right] - (x+2)\left[(1+x)^{n+1} - 1 - (n+1)\,x\right]\\ &\quad + (x+1)\left[(1+x)^n - 1 - n\,x\right] = x^2.\end{aligned}$$

Therefore polynomials $\mathcal{J}_n(x)$ satisfy the recurrence

$$\mathcal{J}_{n+2}(x) - (x+2)\,\mathcal{J}_{n+1}(x) + (x+1)\,\mathcal{J}_n(x) = x^2,$$

with the initial conditions $\mathcal{J}_0(x) = \mathcal{J}_1(x) = 0$.

By an analogous procedure we can find other recurrences, for example

$$\mathcal{J}_{n+1}(x) - (x+1)\,\mathcal{J}_n(x) = nx^2,$$

where $\mathcal{J}_0(x) = 0$ for every $x \in \mathbb{R}$.

(ii) The sum of polynomials $\mathcal{J}_n(x)$.

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{J}_k(x) &= \sum_{k=0}^{n} (1+x)^k - (n+1) - x \sum_{k=0}^{n} k \\
&= \frac{(1+x)^{n+1} - 1}{x} - (n+1) - x\frac{n(n+1)}{2} \\
&= \frac{1}{x}\mathcal{J}_{n+1}(x) - \binom{n+1}{2} x, \qquad x \not0 .
\end{aligned} \tag{6}$$

For $x = 0$ clearly $\sum\limits_{k=0}^{n} \mathcal{J}_k(0) = 0$.

(iii) The generating function for polynomials $\mathcal{J}_n(x)$.

In the following we will use a well-known formula for $b, c \in \mathbb{R}$,

$$\frac{1}{(1-bz)^c} = \sum_{n=0}^{\infty} \binom{n+c-1}{c-1} b^n z^n .$$

For $|t| < 1$, $|(1+x)t| < 1$, the following holds

$$\begin{aligned}
\mathcal{J}(t) = \sum_{n=0}^{\infty} \mathcal{J}_n(x) t^n &= \sum_{n=0}^{\infty} \big((1+x)^n - nx - 1\big) t^n \\
&= \sum_{n=0}^{\infty} \big((1+x)t\big)^n - xt \sum_{n=0}^{\infty} n t^{n-1} - \sum_{n=0}^{\infty} t^n \\
&= \frac{1}{1-(1+x)t} - \frac{xt}{(1-t)^2} - \frac{1}{1-t} = \frac{x^2t^2}{(1-t)^2\big(1-t(x+1)\big)} .
\end{aligned} \tag{7}$$

(iv) The use of the generating function for polynomials $\mathcal{J}_n(x)$.

First, we will again derive (6) by the generating function method. After multiplying of (7) by the generating function $1/(1-t)$ of the sequence $\{1\}_{n=0}^{\infty}$, we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \mathcal{J}_k(x) \right) t^n = \frac{1}{1-t}\mathcal{J}(t) = \frac{x^2t^2}{(1-t)^3\big(1-t(x+1)\big)} .$$

The right side can be expressed as a sum of partial fractions ($x \neq 0$)

$$\frac{x^2t^2}{(1-t)^3\big(1-t(x+1)\big)} = \frac{(x+1)/x}{1-t(x+1)} - \frac{1/x}{1-t} + \frac{x-1}{(1-t)^2} - \frac{x}{(1-t)^3} ,$$

and after an expansion of these fractions into power series

$$\begin{aligned}\frac{x+1}{x}\sum_{n=0}^{\infty}\big(t(x+1)\big)^n-\frac{1}{x}\sum_{n=0}^{\infty}t^n+(x-1)\sum_{n=0}^{\infty}(n+1)t^n-x\sum_{n=0}^{\infty}\tbinom{n+2}{2}t^n\\ =\sum_{n=0}^{\infty}\Big(\frac{(1+x)^{n+1}}{x}-\frac{1}{x}+(x-1)(n+1)-x\tbinom{n+2}{2}\Big)t^n\\ =\sum_{n=0}^{\infty}\Big(\frac{1}{x}\mathcal{J}_{n+1}(x)-\tbinom{n+2}{2}x\Big)t^n,\end{aligned}$$

we get

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\mathcal{J}_k(x)\Big)t^n=\sum_{n=0}^{\infty}\Big(\frac{1}{x}\mathcal{J}_{n+1}(x)-\tbinom{n+2}{2}x\Big)t^n,$$

therefore we have again (6).

Now we will find another, more complicated, relation with the help of the generating function. We will consider the convolution of two polynomials $\mathcal{J}_n(x)$. Hence

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\mathcal{J}_{n-k}(x)\mathcal{J}_k(x)\Big)t^n=\big(\mathcal{J}(t)\big)^2=\frac{x^4t^4}{(1-t)^4\big(1-t(x+1)\big)^2}.$$

The term on the right side can be expressed as a sum of partial fractions

$$\begin{aligned}\frac{x^4t^4}{(1-t)^4\big(1-t(x+1)\big)^2}=\frac{4/x}{1-t}+\frac{x^2-2x+3}{(1-t)^2}+\frac{2x(x-1)}{(1-t)^3}+\frac{x^2}{(1-t)^4}\\ -\frac{4(x+1)/x}{\big(1-t(x+1)\big)}+\frac{1}{\big(1-t(x+1)\big)^2},\end{aligned}$$

and after an expansion of these fractions into power series ($x\neq 0$)

$$\begin{aligned}\frac{4}{x}\sum_{n=0}^{\infty}t^n+(x^2-2x+3)\sum_{n=0}^{\infty}(n+1)t^n-2x(x-1)\sum_{n=0}^{\infty}\tbinom{n+2}{2}t^n\\ +x^2\sum_{n=0}^{\infty}\tbinom{n+3}{3}t^n-\frac{4(x+1)}{x}\sum_{n=0}^{\infty}\big(t(x+1)\big)^n+\sum_{n=0}^{\infty}(n+1)\big(t(x+1)\big)^n\\ =\sum_{n=0}^{\infty}\Big(\frac{4}{x}+(x^2-2x+3)(n+1)-2x(x-1)\tbinom{n+2}{2}\\ +x^2\tbinom{n+3}{3}-\frac{4(x+1)}{x}(1+x)^n+(n+1)(1+x)^n\Big)t^n\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\mathcal{J}_{n-k}(x)\mathcal{J}_k(x)\right)t^n=\sum_{n=0}^{\infty}\Big(\frac{4}{x}(1-(1+x)^{n+1})+(x^2-2x+3)(n+1)$$
$$-2x(x-1)\tbinom{n+2}{2}+x^2\tbinom{n+3}{3}+(n+1)(1+x)^n\Big)t^n,$$

hence

$$\sum_{k=0}^{n}\mathcal{J}_{n-k}(x)\mathcal{J}_k(x)=\frac{4(1-(1+x)^{n+1})}{x}+(x^2-2x+3)(n+1)$$
$$-2x(x-1)\binom{n+2}{2}+x^2\binom{n+3}{3}+(n+1)(1+x)^n.$$

We can obtain similar relations for multiple convolutions, for example

$$\sum_{\substack{i,j,k\\ i+j+k=n}}\mathcal{J}_i(x)\mathcal{J}_j(x)\mathcal{J}_k(x)=\frac{-15(n+1)}{x}-\frac{6(n+1)(1+x)^{n+1}}{x}-\frac{3(5x+7)}{x^2}$$
$$+\frac{3(1+x)^n\,(5x^2+12x+7)}{x^2}+(1+x)^n\binom{n+2}{2}$$
$$+(x^3-3x^2+6x-10)\binom{n+2}{2}$$
$$-3x(x^2-2x+2)\binom{n+3}{3}+3x^2(x-1)\binom{n+4}{4}$$
$$-x^3\binom{n+5}{5}.$$

5. Further properties of polynomials $\mathcal{J}_n(x)$

Theorem 7. *Let n be a positive integer. Then the following holds:*

(a) $\mathcal{J}_0(x)\geqq 0$ *and* $\mathcal{J}_1(x)\geqq 0$ *on* $\mathbb{R}$,

(b) *if n is even, then $\mathcal{J}_n$ are decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$,*

(c) *if n is odd ($n\neq 1$), then $\mathcal{J}_n$ are decreasing on $(-2,0)$ and increasing on $(-\infty,-2)$ and on $(0,\infty)$.*

Proof. As $\mathcal{J}_0(x)=\mathcal{J}_1(x)=0$ for all $x\in\mathbb{R}$, (a) is clear.

For all $x\in\mathbb{R}$ we have $\mathcal{J}_n'(x)=n(1+x)^{n-1}-n$, hence $\mathcal{J}_n$ is increasing if $(1+x)^{n-1}>1$, $\mathcal{J}_n$ is decreasing if $(1+x)^{n-1}<1$. From the known monotonousness of the function $f(y) = y^m$ for even and odd exponent m we get immediately the intervals where the polynomials $\mathcal{J}_n(x)$ are increasing or decreasing, respectively. □

Corollary of Theorem 7. *If n is odd, then $\mathcal{J}_n(x)$ has the global minimum on $[-1,\infty)$ at $x = 0$, thus $\mathcal{J}_n(x) \geqq 0$ on $[-1,\infty)$. If n is even, then $\mathcal{J}_n(x)$ has the global minimum at $x = 0$, thus $\mathcal{J}_n(x) \geqq 0$ on $\mathbb{R}$.*

Theorem 8. *For every integer $n \geq 2$ and a real number $x > 0$ the inequality $\mathcal{J}_n(x) < \mathcal{J}_{n+1}(x)$ holds.*

P r o o f. The inequality is a direct consequence of the definition of polynomials $\mathcal{J}_n(x)$. □

6. Some applications of polynomials $\mathcal{J}_n(x)$

(i) Polynomials $\mathcal{J}_n(x)$ and Bernoulli's inequality.

We will show that the following holds:

$$(\forall n \in \mathbb{N})\big(\forall x \in [-2,\infty)\big)\ (1+x)^n \geqq 1 + nx\,. \tag{8}$$

For $x \in [-2,-1)$, we put $x = -1-\varepsilon$, with $0 < \varepsilon \leqq 1$. According to (1) and

$$\mathcal{J}_n(-1-\varepsilon) = (-\varepsilon)^n - 1 + n(1+\varepsilon) \geqq \varepsilon + (-\varepsilon)^n \geqq \varepsilon - \varepsilon^n = \varepsilon(1-\varepsilon^{n-1}) \geqq 0$$

(8) holds.

Theorem 9. *No further extension of* (1) *is possible.*

P r o o f. Let us consider only odd n $(n = 2m-1,\ m \in \mathbb{N})$ and $x \in (-\infty,-2)$. Thus we take $x = -2-\delta$, with $\delta > 0$. For every $\delta > 0$, there exists a natural k such that $\mathcal{J}_k(x) = (1+x)^k - 1 - k\,x < 0$ as

$$\begin{aligned}\lim_{m\to\infty} \mathcal{J}_{2m-1}(-2-\delta) &= \lim_{m\to\infty} \big((-1)^{2m-1}(1+\delta)^{2m-1} - 1 + (2m-1)(2+\delta)\big)\\ &= -\lim_{m\to\infty}(1+\delta)^{2m-1}\left(1 + \frac{1}{(1+\delta)^{2m-1}} - (2+\delta)\frac{2m-1}{(1+\delta)^{2m-1}}\right) = -\infty\,.\end{aligned}$$

□

(ii) Other examples.

Now let us turn to the two well-known relations (e.g., see [2]), but let us use a little different way of their derivations.

Let c, d be positive real numbers. Then for every $n \in \mathbb{N}$, $\left(\frac{c}{d}\right)^{\frac{1}{n}} - 1 > -1$, and therefore $\mathcal{J}_{n+1}\left(\left(\frac{c}{d}\right)^{\frac{1}{n}} - 1\right) \geq 0$. It means that $\left(\left(\frac{c}{d}\right)^{\frac{1}{n}}\right)^{n+1} - (n+1)\left(\frac{c}{d}\right)^{\frac{1}{n}} + n + 1 - 1 \geq 0$ and after a simple calculation

$$\left(n + 1 - \frac{c}{d}\right)\left(\frac{c}{d}\right)^{\frac{1}{n}} \leq n\,. \tag{9}$$

a) Let $c = n + 1$, $d = n + 2$. Then the inequality

$$\left(n + 1 - \frac{n+1}{n+2}\right)\left(\frac{n+1}{n+2}\right)^{\frac{1}{n}} \leq n$$

can be subsequently modified to the form

$$\left(1 + \frac{1}{n+1}\right)^{n+1} \geq \left(1 + \frac{1}{n}\right)^{n}.$$

This relation shows the well-known fact that the sequence $\left\{(1 + \frac{1}{n})^n\right\}$ is monotone. As $\mathcal{J}_{n+1}\left(\left(\frac{c}{d}\right)^{\frac{1}{n}} - 1\right) = 0$ only for $c = d$, the sequence is increasing.

b) The arithmetic and the geometric means of positive real numbers $a_1, a_2, \dots$ $\dots, a_n$ are defined by relations

$$A_n = \frac{a_1 + a_2 + \cdots + a_n}{n}\,, \quad G_n = \sqrt[n]{a_1 a_2 \cdots a_n} = (a_1 a_2 \cdots a_n)^{\frac{1}{n}}\,,$$

respectively.

It is easy to see that $a_{n+1} = \frac{G_{n+1}^{n+1}}{G_n^n}$ for every $n \in \mathbb{N}$. If we put $c = a_{n+1}$, $d = A_{n+1}$ in (9), the following inequality is obtained:

$$\left(n + 1 - \frac{a_{n+1}}{A_{n+1}}\right)\left(\frac{a_{n+1}}{A_{n+1}}\right)^{\frac{1}{n}} \leq n\,.$$

And after its simplification we get

$$\left(\frac{A_n}{G_n}\right)^{n} \leq \left(\frac{A_{n+1}}{G_{n+1}}\right)^{n+1}.$$

As $\left(\frac{A_1}{G_1}\right)^1 = \frac{a_1}{a_1} = 1$ and n is an arbitrary natural number, the inequality $\frac{A_n}{G_n} \geq 1$, resp. $A_n \geq G_n$, holds for every $n \in \mathbb{N}$. It is a well-known arithmetic-geometric inequality. We can suppose that $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$. Then $\frac{c}{d} = \frac{a_{n+1}}{A_{n+1}} = 1$ holds only for $a_1 = a_2 = \cdots = a_{n+1}$. Therefore the equality $A_n = G_n$ holds only in this case, otherwise $A_n > G_n$.

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