

BAYESIAN ESTIMATE OF PARAMETERS FOR ARMA MODEL FORECASTING

ZUL AMRY

Department of Mathematics, Faculty of Mathematics and Natural Sciences, State University of Medan, Medan, INDONESIA

ABSTRACT. This paper presents a Bayesian approach to finding the Bayes estimator of parameters for ARMA model forecasting under normal-gamma prior assumption with a quadratic loss function in mathematical expression. Obtaining the conditional posterior predictive density is based on the normal-gamma prior and the conditional predictive density, whereas its marginal conditional posterior predictive density is obtained using the conditional posterior predictive density. Furthermore, the Bayes estimator of parameters is derived from the marginal conditional posterior predictive density.

1. Introduction

Bayesian inference is a method of analysis that combines information collected from experimental data with the knowledge that one has prior to performing the experiment. The Bayesian approach in general requires an explicit formulation of a model and conditioning on known quantities, in order to draw inferences about the unknown. The forecasting model is the most often recognized as Bayesian forecasting when a probability distribution is used to describe uncertainty regarding the unknown parameters and when Bayes theorem is applied. For forecasting problems, Bayesian approach combining all the information and sources of uncertainty into a predictive distribution of the future values to estimate the parameters in model. The main idea of Bayesian forecasting is the predictive distribution of the future given past data following directly from the joint probabilistic model. The predictive distribution is derived from the sampling predictive density, weighted by the posterior distribution.

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This paper is done referring to Liu discussion on Bayesian analysis for one-step ahead forecast in ARMA model. The other papers related to this research are Fan and Yao, Kleibergen and Hoek, and Turbey also discussed the Bayesian analysis for ARMA model. This paper focuses on finding the mathematical expression of the Bayes estimator of parameters for the ARMA model forecasting under normal-gamma prior assumption with a quadratic loss function.

2. Materials and methods

The materials in this paper are some theories in mathematics and statistics such as the ARMA model, Bayes theorem, repeated integration, gamma distribution and the multivariate t-distribution. The method is a study of literature by applying the Bayesian analysis under normal-gamma prior assumption.

DEFINITION 2.1. The ARMA (p, q) model is defined by

$$y_t = \sum_{i=1}^p \phi_i y_{t-i} + \sum_{j=1}^q \theta_j e_{t-j} + e_t \quad (1)$$

where $\{e_t\}$ is a sequence of *iid* normal random variables with

$$e_t \sim N(0, \tau^{-1}), \quad \tau > 0$$

and unknowns ϕ_i and θ_j are parameters.

DEFINITION 2.2. The Bayes theorem can be expressed as

$$(posterior) \propto (prior) \times (likelihood). \quad (2)$$

DEFINITION 2.3. A positive random quantity ϕ is said to have a *gamma distribution* with parameter $n > 0$ and $d > 0$ if it has the probability density function

$$p(\phi) = \frac{d^n}{\Gamma(n)} d^{n-1} \exp(-\phi d) \quad (3)$$

the mean is $E(\phi) = \frac{n}{d}$ and the variance is $\text{Var}(\phi) = \frac{n}{d^2}$.

DEFINITION 2.4. A random p -vector \underline{X} is said to have a joint *student-t distribution* on n degrees of freedom with mode μ and scale matrix Ω if it has the probability density function

$$p(\underline{X}) = \frac{\Gamma(\frac{n+p}{2}) n^{\frac{n+p-1}{2}}}{\Gamma(\frac{n}{2}) \pi^{\frac{p}{2}} |\Omega|^{\frac{p}{2}}} (n + (x - \mu)^T \Omega^{-1} (x - \mu))^{-\frac{n+p}{2}}. \quad (4)$$

3. Construction of the estimator

The k -step-ahead point forecast of y_{n+k} is defined by

$$\hat{y}(k) = E(y_{n+k} | S_n^*), \quad (5)$$

where

$$S_n^* = (y_1, y_2, \dots, y_{n+k-1}).$$

Using the equation (1) one obtains residuals as

$$e_t = y_t - \sum_{i=1}^p \phi_i y_{t-i} - \sum_{j=1}^q \theta_j e_{t-j} \quad (6)$$

and by conditioning the first p observations and letting $e_p = e_{p-1} = \dots = e_r = 0$, where $r = \min(0, p+1-q)$, one may approximate by Box and Jenkins in [5], the likelihood function for parameters $\Psi = (\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q)$ and τ based on S_n^* is

$$L(\Psi, \tau | S_n^*) \propto \tau^{\frac{(n+k-1)-p}{2}} \exp \left(-\frac{\tau}{2} \left(\sum_{t=p+1}^{n+k-1} \left(y_t - \sum_{i=1}^p \phi_i y_{t-i} - \sum_{j=1}^q \theta_j e_{t-j} \right)^2 \right) \right). \quad (7)$$

The equation (7) can be expressed as

$$L(\Psi, \tau | S_n^*) \propto \tau^{\frac{(n+k-1)-p}{2}} \exp \left(-\frac{\tau}{2} \left(\sum_{t=p+1}^{n+k-1} y_t^2 - 2\Psi^T \sum_{t=p+1}^{n+k-1} y_t B_{t-1} + \sum_{t=p+1}^{n+k-1} (\Psi^T B_{t-1})^2 \right) \right), \quad (8)$$

where $B_t = (y_t, y_{t-1}, \dots, y_{t+1-p}, e_t, e_{t-1}, \dots, e_{t+1-q})$. By letting

$$U = \begin{pmatrix} y_p & y_{p+1} & \cdots & y_{n+k-2} \\ y_{p-1} & y_p & \cdots & y_{n+k-3} \\ \vdots & \vdots & \vdots & \vdots \\ y_1 & y_2 & \cdots & y_{n+k-1-p} \\ e_p & e_{p+1} & \cdots & e_{n+k-2} \\ e_{p-1} & e_p & \cdots & e_{n+k-3} \\ \vdots & \vdots & \vdots & \vdots \\ e_1 & e_2 & \cdots & e_{n+k-1-p} \end{pmatrix}, \quad X_0 = \begin{pmatrix} y_{p+1} \\ y_{p+2} \\ \vdots \\ y_{n+k-1} \end{pmatrix},$$

where $e_t = y_t - \sum_{i=1}^p \tilde{\phi}_i y_{t-i} - \sum_{j=1}^q \tilde{\theta}_j e_{t-j}$, $t = p+1, p+2, \dots, n$, $\tilde{\phi}_i$ and $\tilde{\theta}_j$ are maximum likelihood estimator of ϕ_i and θ_j . $e_t, e_{t-1}, \dots, e_{t-q}$ is obtained via

$$e_t = y_t - \tilde{\Psi}^T B_{t-1}, \quad (9)$$

where $\tilde{\Psi} = (\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_p, \tilde{\theta}_1, \tilde{\theta}_2, \dots, \tilde{\theta}_q)$.

From the likelihood function in equation (8) one can obtain:

$$\begin{aligned}
 \sum_{t=p+1}^{n+k-1} y_t B_{t-1} &= y_{p+1} B_p + y_{p+2} B_{p+1} + \cdots + y_{n+k-1} B_{n+k-2} \\
 &= y_{p+1} (y_p, y_{p-1}, \dots, y_1, e_p, e_{p-1}, \dots, e_{p+1-q}) \\
 &\quad + y_{p+2} (y_{p+1}, y_p, \dots, y_2, e_{p+1}, e_p, \dots, e_{p+2-q}) + \cdots \\
 &\quad + y_{n+k-1} (y_{n+k-2}, y_{n+k-3}, \dots, y_{n-k-1-p}, \\
 &\quad e_{n+k-2}, e_{n+k-3}, \dots, e_{n+k-1-q}) \\
 &= U^T X_0 \\
 \sum_{t=p+1}^{n+k-1} (\Psi^T B_{t-1})^2 &= (\Psi^T B_p)^2 + (\Psi^T B_{p+1})^2 + \cdots + (\Psi^T B_{n+k-2})^2 \\
 &= ((\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q))^T \\
 &\quad (y_p, y_{p-1}, \dots, y_1, e_p, e_{p-1}, \dots, e_{p+1-q}))^2 \\
 &\quad + ((\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q))^T \\
 &\quad (y_{p+2}, y_{p+1}, \dots, y_2, e_{p+2}, e_{p+1}, \dots, e_{p+2-q}))^2 + \cdots \\
 &\quad + ((\phi_1, \phi_2, \dots, \phi_p, \theta_1, \theta_2, \dots, \theta_q))^T \\
 &\quad (y_{n+k-2}, y_{n+k-3}, \dots, y_{n-k-1-p}, e_{n+k-2}, e_{n+k-3}, \dots, \\
 &\quad e_{n+k-1-q}))^2 \\
 &= \Psi^T (U U^T) \Psi
 \end{aligned}$$

such that the likelihood function in equation (8) can be expressed as

$$L(\Psi, \tau | S_n^*) \propto \tau^{\frac{(n+k-1)-p}{2}} \exp \left(-\frac{\tau}{2} \left(\sum_{t=p+1}^{n+k-1} y_t^2 - 2\Psi^T V + \Psi^T W \Psi \right) \right), \quad (10)$$

where $V = U^T X_0$ and $W = U U^T$.

3.1. Posterior distribution

According to suggestion by Broemeling and Shaarawy in [2], the normal-gamma prior of parameters Ψ and τ is

$$\begin{aligned}
 \xi(\Psi, \tau) &= \xi_1(\Psi | \tau) \cdot \xi_2(\tau) \propto \tau^{\frac{p+2\alpha-2}{2}} \\
 &\quad \exp \left(-\frac{\tau}{2} (\Psi^T Q \Psi - \Psi^T Q \mu - \mu^T Q \Psi + \mu^T Q \mu + 2\beta) \right),
 \end{aligned} \quad (11)$$

where $\xi_1 \sim N(\mu, (\tau Q)^{-1})$, $\xi_2 \sim GAM(\alpha, \beta)$, Q is a positive definite matrix of the order $(p+q)$, and α and β are parameters.

THEOREM 3.1. *The posterior of Ψ and τ parameters is*

$$\pi(\Psi, \tau | S_n^*) \propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} (\Psi^T P \Psi - \Psi^T (V + Q\mu) - (V + Q\mu)^T \Psi + K) \right), \quad (12)$$

where $P = W + Q$ and $K = \sum_{t=p+1}^{n+k-1} y_t^2 + \mu^T Q \mu + 2\beta$.

P r o o f. By applying the Bayes theorem, the posterior of Ψ and τ is

$$\begin{aligned} \pi(\Psi, \tau | S_n^*) &\propto \tau^{\frac{(n+k-1)-p}{2}} \exp \left(-\frac{\tau}{2} \left(\sum_{t=p+1}^{n+k-1} y_t^2 - 2\Psi^T V + \Psi^T W \Psi \right) \right) \\ &\quad \times \tau^{\frac{p+2\alpha-2}{2}} \exp \left(-\frac{\tau}{2} (\Psi^T Q \Psi - \Psi^T Q \mu - \mu^T Q \Psi + \mu^T Q \mu + 2\beta) \right) \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left(\sum_{t=p+1}^{n+k-1} y_t^2 - 2\Psi^T V + \Psi^T W \Psi \right. \right. \\ &\quad \left. \left. + \Psi^T Q \Psi - \Psi^T Q \mu - \mu^T Q \Psi + \mu^T Q \mu + 2\beta \right) \right) \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left(\Psi^T (W + Q) \Psi - \Psi^T V \right. \right. \\ &\quad \left. \left. - \Psi^T Q \mu - \Psi^T V - \mu^T Q \Psi + \sum_{t=p+1}^{n+k-1} y_t^2 + \mu^T Q \mu + 2\beta \right) \right) \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left(\Psi^T (W + Q) \Psi - \Psi^T (V + Q\mu) \right. \right. \\ &\quad \left. \left. - V^T \Psi - (Q\mu)^T \Psi + \sum_{t=p+1}^{n+k-1} y_t^2 + \mu^T Q \mu + 2\beta \right) \right) \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left(\Psi^T (W + Q) \Psi - \Psi^T (V + Q\mu) \right. \right. \\ &\quad \left. \left. - (V + Q\mu)^T \Psi + \sum_{t=p+1}^{n+k-1} y_t^2 + \mu^T Q \mu + 2\beta \right) \right) \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left(\Psi^T P \Psi - \Psi^T (V + Q\mu) \right. \right. \\ &\quad \left. \left. - (V + Q\mu)^T \Psi + K \right) \right). \end{aligned}$$

□

3.2. Marginal posterior distribution

The marginal posterior distribution is obtained by integrating the posterior distribution in (12).

THEOREM 3.2. *The marginal posterior distribution of Ψ is a multivariate t-distribution on $(n+k+2\alpha-1-p)$ degrees of freedom with mode $P^{-1}(V+Q\mu)$ and scale matrix $(\frac{n+k+2\alpha-1-p}{K-(V+Q\mu)^TP^{-1}(V+Q\mu)}P)^{-1}$ and can be expressed as*

$$\pi(\Psi|S_n^*) \propto \left((n+k+2\alpha-1-p) + (\Psi - P^{-1}(V+Q\mu))^T \right. \\ \left. \frac{n+k+2\alpha-1-p}{K-(V+Q\mu)^TP^{-1}(V+Q\mu)} P (\Psi - P^{-1}(V+Q\mu)) \right)^{-\frac{n+k+2\alpha-1}{2}}. \quad (13)$$

P r o o f. The marginal posterior distribution of Ψ is obtained by integrating the posterior distribution in equation (12) to τ

$$\pi(\Psi|S_n^*) = \int_0^\infty \pi(\Psi, \tau|S_n^*) d\tau = \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(\Psi^T P \Psi - \Psi^T(V+Q\mu) \right. \\ \left. - (V+Q\mu)^T \Psi + K)\right) d\tau \\ = \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(\Psi^T P \Psi - \Psi^T(V+Q\mu) - (V+Q\mu)^T \Psi + K)\right) d\tau \\ = \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(\Psi^T P \Psi - \Psi^T(V+Q\mu) \right. \\ \left. - (V+Q\mu)^T \Psi + V^T P^{-1}(V+Q\mu) - (V+Q\mu)^T P^{-1}(V+Q\mu) + K)\right) d\tau \\ = \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(\Psi^T P \Psi - \Psi^T P P^{-1}(V+Q\mu) \right. \\ \left. - (V+Q\mu)^T P^{-1} P \Psi + (V+Q\mu)^T P^{-1}(V+Q\mu) - (V+Q\mu)^T P^{-1} \right. \\ \left. \times (V+Q\mu) + K)\right) d\tau \\ = \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(\Psi^T P \Psi - \Psi^T P P^{-1}(V+Q\mu) \right. \\ \left. - (V+Q\mu)^T (P^{-1})^T P \Psi + (V+Q\mu)^T (P^{-1})^T (V+Q\mu) - (V+Q\mu)^T \right. \\ \left. \times P^{-1}(V+Q\mu) + K)\right) d\tau$$

$$\begin{aligned}
 &= \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left(\Psi^T P \Psi - \Psi^T P P^{-1} (V + Q\mu) \right. \right. \\
 &\quad \left. \left. - (P^{-1}(V + Q\mu))^T P \Psi + (P^{-1}(V + Q\mu))(V + Q\mu) - (V + Q\mu)^T P^{-1} \right. \right. \\
 &\quad \times (V + Q\mu) + K \right) d\tau \\
 &= \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\frac{\tau}{2} \left((\Psi - P^{-1}(V + Q\mu))^T \right. \right. \\
 &\quad \left. \left. P(\Psi - P^{-1}(V + Q\mu)) + K - (V + Q\mu)^T P^{-1}(V + Q\mu) \right) \right) d\tau \\
 &= \int_0^\infty \tau^{\frac{n+k+2\alpha-3}{2}} \exp \left(-\tau \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T}{2} \right. \right. \\
 &\quad \left. \left. P(\Psi - P^{-1}(V + Q\mu)) + K - (V + Q\mu)^T P^{-1}(V + Q\mu) \right) \right) d\tau \\
 &= \int_0^\infty \tau^{\frac{n+k+2\alpha-1}{2}-1} \exp \left(-\tau \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T}{2} \right. \right. \\
 &\quad \left. \left. P(\Psi - P^{-1}(V + Q\mu)) + K - (V + Q\mu)^T P^{-1}(V + Q\mu) \right) \right) d\tau.
 \end{aligned}$$

By applying the formula of gamma distribution to the last equation one obtains

$$\begin{aligned}
 \pi(\Psi | S_n^*) &\propto \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T P(\Psi - P^{-1}(V + Q\mu)) + K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{-\frac{n+k+2\alpha-1}{2}} \\
 &\propto \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T P(\Psi - P^{-1}(V + Q\mu)) + K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
 &\quad \times \left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
 &\quad \times \left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{\frac{(n+k+2\alpha-1-p)+p}{2}} \\
 &\quad \times (n+k+2\alpha-1-p)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \times (n+k+2\alpha-1-p)^{\frac{(n+k+2\alpha-1-p)+p}{2}}
 \end{aligned}$$

$$\begin{aligned}
& \propto \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T P (\Psi - P^{-1}(V + Q\mu)) + K - }{2} \right. \\
& \quad \left. - \frac{(V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \quad \times \left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \quad \times (n + k + 2\alpha - 1 - p)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \propto \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T P (\Psi - P^{-1}(V + Q\mu)) + K - }{2} \right. \\
& \quad \left. - \frac{(V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \quad \times \left(\left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2} \right)^{-1} \right)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \quad \times (n + k + 2\alpha - 1 - p)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \propto \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T P (\Psi - P^{-1}(V + Q\mu)) + K - }{K - (V + Q\mu)^T P^{-1}(V + Q\mu) + 1} \right)^{-\frac{(n+k+2\alpha-1-p)+p}{2}} \\
& \quad \times (n + k - 1 - p + 2\alpha)^{-\frac{(n+k-1-p+2\alpha)+p}{2}} \\
& \propto \left(\frac{(\Psi - P^{-1}(V + Q\mu))^T (n + k + 2\alpha - 1 - p) P (\Psi - P^{-1}(V + Q\mu))}{K - (V + Q\mu)^T P^{-1}(V + Q\mu)} \right. \\
& \quad \left. + (n + k + 2\alpha - 1 - p) \right)^{-\frac{(n+k-1-p+2\alpha)+p}{2}} \\
& \propto \left((n + k + 2\alpha - 1 - p) + l (\Psi - P^{-1}(V + Q\mu))^T \right. \\
& \quad \left. \times \frac{n + k + 2\alpha - 1 - p}{K - (V + Q\mu)^T P^{-1}(V + Q\mu)} P (\Psi - P^{-1}(V + Q\mu)) \right)^{-\frac{n+k+2\alpha-1}{2}} \\
& \propto \left((n + k - 1 - p + 2\alpha) + (\Psi - P^{-1}(V + Q\mu))^T \right. \\
& \quad \left. \frac{n + k - 1 - p + 2\alpha}{K - (V + Q\mu)^T P^{-1}(V + Q\mu)} P (\Psi - P^{-1}(V + Q\mu)) \right)^{-\frac{n+k+2\alpha-1}{2}}.
\end{aligned}$$

□

THEOREM 3.3. *The marginal posterior distribution of τ is a gamma distribution with parameters $(\frac{n+k+2\alpha-1}{2}, \frac{K-(V+Q\mu)^T P^{-1}(V+Q\mu)}{2})$ and is expressed as*

$$\pi(\tau|S_n^*) \propto \tau^{\frac{n+k+2\alpha-1}{2}} \exp\left(-\tau\left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2}\right)\right). \quad (14)$$

Proof. The marginal posterior distribution of τ is obtained by integrating the posterior distribution to Ψ

$$\begin{aligned} \pi(\tau|S_n^*) &= \int_{-\infty}^{\infty} \pi(\Psi, \tau|S_n^*) d\Psi \\ &\propto \int_{-\infty}^{\infty} \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(\Psi^T P \Psi - 2\Psi^T(V + Q\mu) + K)\right) d\Psi \\ &\propto \int_{-\infty}^{\infty} \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}\left((\Psi - P^{-1}(V + Q\mu))^T\right.\right. \\ &\quad \left.\left.P(\Psi - P^{-1}(V + Q\mu)) + K - (V + Q\mu)^T P^{-1}(V + Q\mu)\right)\right) d\Psi \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\frac{\tau}{2}(K - (V + Q\mu)^T P^{-1}(V + Q\mu))\right) \\ &\quad \int_{-\infty}^{\infty} \exp\left(\left((\Psi - P^{-1}(V + Q\mu))^T P(\Psi - P^{-1}(V + Q\mu))\right)\right) d\Psi \\ &\propto \tau^{\frac{n+k+2\alpha-3}{2}} \exp\left(-\tau\left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2}\right)\right) \\ &\quad \int_{-\infty}^{\infty} \exp\left(\frac{1}{2}\left((\Psi - P^{-1}(V + Q\mu))^T (-\tau^{-1}P^{-1})^{-1}\right.\right. \\ &\quad \left.\left.(\Psi - P^{-1}(V + Q\mu))\right)\right) d\Psi \\ &\propto \tau^{\frac{n+k+2\alpha-1}{2}-1} \exp\left(-\tau\left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2}\right)\right) \times 1 \\ &\propto \tau^{\frac{n+k+2\alpha-1}{2}-1} \exp\left(-\tau\left(\frac{K - (V + Q\mu)^T P^{-1}(V + Q\mu)}{2}\right)\right). \end{aligned}$$

□

The Bayes estimator of Ψ is mean posterior of the marginal posterior distribution $\pi(\Psi|S_n^*)$ that is $\hat{\Psi} = P^{-1}(V + Q\mu)$ and the Bayes estimator of τ is mean posterior of the marginal posterior distribution $\pi(\tau|S_n^*)$, that is

$$\hat{\tau} = (K - (V + Q\mu)^T P^{-1}(V + Q\mu))^{-1} (n + k + 2\alpha - 1).$$

4. Conclusions

This paper mainly studies the Bayesian approach to finding the mathematical expression of the Bayes estimator of Ψ and τ parameters for ARMA Model Forecasting under normal-gamma prior. Mathematical analysis used to obtain the Bayes estimator is simple such as the repeated integral in calculus and manipulation with matrices in algebra. The Bayes estimator of Ψ is derived using the marginal posterior distribution $\pi(\Psi|S_n^*)$ which has a multivariate t-distribution, where the Bayes estimator of τ is derived using the marginal posterior distribution $\pi(\tau|S_n^*)$ which has a gamma distribution.

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*Faculty of Mathematics and Natural Sciences
State University of Medan
Jl. Willem Iskandar Pasar V
Medan Estate, 20222
INDONESIA
E-mail: zul.amry@gmail.com*