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TATRA MOUNTAINS
Mathematical Publications
DOI: 10.2478/tmmp-2020-0009
Tatra Mt. Math. Publl. **75** (2020), 135–146

NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF SECOND-ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this work, we obtain necessary and sufficient conditions for the oscillation of all solutions of second-order half-linear delay differential equation of the form

$$\big(r(x')^{\gamma}\big)'(t) + q(t)x^{\alpha}(\tau(t)) = 0\,.$$

Under the assumption $\int^{\infty}\big(r(\eta)\big)^{-1/\gamma}\mathrm{d}\eta = \infty$, we consider the two cases when $\gamma > \alpha$ and $\gamma < \alpha$. Further, some illustrative examples showing applicability of the new results are included, and state an open problem.

1. Introduction

The main feature of this article is having an oscillation condition that is necessary and sufficient at the same time. We mainly consider the following second-order half-linear delay differential equation

$$\big(r(x')^{\gamma}\big)'(t) + q(t)x^{\alpha}(\tau(t)) = 0\,, \tag{1.1}$$

by considering two cases: $\gamma > \alpha$ and $\gamma < \alpha$. We suppose that the following assumptions hold:

(A1) γ and α are the quotient of two odd positive integers, $r, q \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $r(t) > 0$ and q is not identically zero eventually, $\tau \in C([t_0,\infty), R_+)$ such that $\tau(t) \le t$ for $t \ge t_0$, $\tau(t) \to \infty$ as $t \to \infty$.

(A2) $r(t) > 0$ and $\int_0^{\infty}(r(\eta))^{-1/\gamma}\,\mathrm{d}\eta = \infty$. Letting $R(t) = \int_0^{t}(r(\eta))^{-1/\gamma}\,\mathrm{d}\eta$, we have $\lim_{t\to\infty} R(t) = \infty$.

2010 Mathematics Subject Classification: 34C10, 35K40, 34K11.

Keywords: oscillation, non-oscillation, delay, linear, Lebesgue's dominated convergence theorem.

Initially single delay has taken and in the later section one can see the effect for several delays. As example of function satisfying (A2), we have

$$r(t) = e^{-t} \quad \text{or} \quad r(t) = 1.$$

The interest in the study of functional differential equations comes from their applications to engineering and natural sciences. Equations involving arguments that are delayed, advanced or a combination of both arise in models such as the lossless transmission lines in high speed computers that interconnect switching circuits. Moreover, delay differential equations play an important role in modelling virtually every physical, technical, and biological process, from celestial motion, to bridge design, to interactions between neurons.

In what follows, we provide some background details regarding the study of oscillation of second-order differential equations which motivated our study. Brands [5] showed that for bounded delays, the solutions to

$$x''(t) + q(t)x(t - \tau(t)) = 0 \tag{1.2}$$

are oscillatory if and only if solutions to $x''(t) + q(t)x(t) = 0$ are oscillatory. Recently, Chatzarakis et al. [6] have established sufficient conditions for the oscillation and asymptotic behaviour of all solutions of second-order half-linear differential equations of the form

$$\big(r(x')^{\alpha}\big)'(t) + q(t)y^{\alpha}(\tau(t)) = 0. \tag{1.3}$$

In another paper, Chatzarakis et al. [7] have considered (1.3) and established new oscillation criteria for (1.3). Fisnarova and Marik [10] considered the half-linear differential equation

$$\big(r(t)\Phi(z'(t))\big)' + c(t)\Phi\big(x(\tau(t))\big) = 0, \quad z(t) = x(t) + b(t)x(\tau(t)),$$

where $\Phi(t) = |t|^{p-2}t$, $p \geq 2$. Karpuz and Santra [13] have established sufficient conditions for oscillation and asymptotic behaviour of solutions to the equation

$$\Big[r(t)\big(x(t) + p(t)x(\tau(t))\big)'\Big]' + \sum_{i=1}^{m} q_i(t)G_i\big(x(\tau_i(t))\big) = 0. \tag{1.4}$$

Wong [23] studied necessary and sufficient conditions for the oscillation of solutions to

$$\big(x(t) + px(t - \tau)\big)'' + q(t)H\big(x(t - \sigma)\big) = 0,$$

where the constant p satisfies $-1 < p < 0$.

Oscillation criteria for second-order delay differential equations have been reported in [1, 2, 3, 4, 8, 9, 12, 15, 16, 19, 20, 22]. Note that most publications consider only sufficient conditions, and just a few of them consider necessary and sufficient conditions.

By a solution to the equation (1.1), we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, where $T_x \ge t_0$, such that $rx' \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$, and satisfies (1.1) on the interval $[T_x,\infty)$. A solution x of (1.1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| : \ t \ge T\} > 0$ for all $T \ge T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be non-oscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

2. Main results

Lemma 2.1. *Assume that* (A1) *and* (A2) *hold. If x is an eventually positive solution of* (1.1), *then x satisfies*

$$x'(t) > 0 \quad \text{and} \quad \big(r(x')^\gamma\big)'(t) < 0 \quad \text{for all large } t. \tag{2.1}$$

Proof. Since $x(t)$ is an eventually positive solution of (1.1). Then there exists $t_0 \ge 0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \ge t_0$. From (1.1), it follows that

$$\big(r(x')^\gamma\big)'(t) = -q(t)x^\alpha(\tau(t)) \le 0 \quad \text{for } t \ge t_0. \tag{2.2}$$

Therefore, $\big(r(x')^\gamma\big)(t)$ is non-increasing. We claim that $\big(r(x')^\gamma\big)(t) > 0$ for $t \ge t_0$. On the contrary, assume that $\big(r(x')^\gamma\big)(t) \le 0$ for some $t \ge t_0$, then we can find $t^* \ge t_0$ and $\kappa_1 > 0$ such that $\big(r(x')^\gamma\big)(t) \le -\kappa_1$ for all $t \ge t^*$. Integrating the inequality $x'(t) \le (-\kappa_1/r(t))^{1/\gamma}$, from t^* to t $(t > t^*)$, by (A2) we obtain

$$x(t) \le x(t^*) - \kappa_1^{1/\gamma}\int_{t^*}^{t}(r(\eta))^{-1/\gamma}\,\mathrm{d}\eta \to -\infty, \quad \text{as } t \to \infty.$$

This contradicts $x(t)$ being a positive solution. So, $\big(r(x')^\gamma\big)(t) > 0$ for $t \ge t_0$. Since $r(t) \ge 0$, then $x'(t) \ge 0$ for $t \ge t_0$. □

2.1. The Case $\gamma > \alpha$.

In this subsection, we assume that there exists a constant β such that $0 < \alpha < \beta < \gamma$ and

$$u^{\alpha-\beta} \ge v^{\alpha-\beta}, \quad \text{for } 0 < u \le v\,. \tag{2.3}$$

Lemma 2.2. *Assume that all conditions of Lemma 2.1 hold. Then there exists $t_1 \ge t_0$ and $\kappa > 0$ such that for $t \ge t_1$, the following holds*

$$x(t) \le \kappa^{1/\gamma}R(t) \tag{2.4}$$

$$\big(R(t) - R(t_1)\big)\left[\int_t^\infty q(\zeta)\big(\kappa^{1/\gamma}R(\tau(\zeta))\big)^{\alpha-\beta}x^\beta(\tau(\zeta))\,\mathrm{d}\zeta\right]^{1/\gamma} \le x(t). \tag{2.5}$$

P r o o f. By Lemma 2.1, $r(t)(x'(t))^\gamma$ is positive and non-increasing. Then there exists
$$\kappa > 0 \quad \text{and} \quad t_1 \geq t_0$$
such that
$$r(t)(x'(t))^\gamma \leq \kappa.$$

Integrating the inequality $x'(t) \leq (\kappa/r(t))^{1/\gamma}$, we have
$$x(t) \leq x(t_1) + \kappa^{1/\gamma}\big(R(t) - R(t_1)\big).$$
Since $\lim_{t\to\infty} R(t) = \infty$, then the last inequality becomes that
$$x(t) \leq \kappa^{1/\gamma} R(t) \quad \text{for } t \geq t_1\,,$$
which is (2.4). Note that κ depends on the solution x evaluated at a time t_0. Thus, the condition (2.7) must include all possible κ's.

By (2.4) and the assumption (2.3), we have
$$x^\alpha(\tau(t)) = x^{\alpha-\beta}(\tau(t))x^\beta(\tau(t)) \geq \big(\kappa^{1/\gamma}R(\tau(t))\big)^{\alpha-\beta} x^\beta(\tau(t))\,.$$
Integrating (1.1) from t to ∞, we have
$$\lim_{A\to\infty}\big[\big(r(x')^\gamma\big)(\eta)\big]_t^A + \int_t^\infty q(\eta)\big(\kappa^{1/\gamma}R(\tau(\eta))\big)^{\alpha-\beta} x^\beta(\tau(\eta))\,\mathrm{d}\eta \leq 0.$$
Using that $\big(r(x')^\gamma\big)(t)$ is positive and non-increasing, we have
$$\int_t^\infty q(\eta)\big(\kappa^{1/\gamma}R(\tau(\eta))\big)^{\alpha-\beta} x^\beta(\tau(\eta))\,\mathrm{d}\eta \leq \big(r(x')^\gamma\big)(t) \quad \text{for } t \geq t_1\,.$$
Therefore,
$$x'(t) \geq \left[\frac{1}{r(t)}\int_t^\infty q(\eta)\big(\kappa^{1/\gamma}R(\tau(\eta))\big)^{\alpha-\beta} x^\beta(\tau(\eta))\,\mathrm{d}\eta\right]^{1/\gamma}. \tag{2.6}$$
Since $x(t) \geq 0$. Integrating (2.6) from t_1 to t, we obtain
$$\begin{aligned} x(t) &\geq \int_{t_1}^t \left[\frac{1}{r(\eta)}\int_\eta^\infty q(\zeta)\big(\kappa^{1/\gamma}R(\tau(\zeta))\big)^{\alpha-\beta} x^\beta(\tau(\zeta))\mathrm{d}\zeta\right]^{1/\gamma} \mathrm{d}\eta \\ &\geq \big(R(t) - R(t_1)\big)\left[\int_t^\infty q(\zeta)\big(\kappa^{1/\gamma}R(\tau(\zeta))\big)^{\alpha-\beta} x^\beta(\tau(\zeta))\,\mathrm{d}\zeta\right]^{1/\gamma}, \end{aligned}$$
which is (2.5). □

Theorem 2.1. *Under assumptions* (A1) *and* (A2)*, every solution of* (1.1) *is oscillatory if and only if*

$$\int_0^\infty q(\eta)R^\alpha(\tau(\eta))\mathrm{d}\eta = +\infty. \tag{2.7}$$

P r o o f. To prove sufficiency by contradiction, assume that x is a non-oscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is eventually positive. Then Lemmas 2.1 and 2.2 hold for $t \ge t_1$. So,

$$x(t) > \bigl(R(t) - R(t_1)\bigr)w^{1/\gamma}(t) \quad \text{for all} \quad t \ge t_1,$$

where

$$w(t) = \int_t^\infty q(\zeta)\bigl(\kappa^{1/\gamma}R(\tau(\zeta))\bigr)^{\alpha-\beta}x^\beta(\tau(\zeta))\,\mathrm{d}\zeta \ge 0\,.$$

Since $\lim_{t\to\infty} R(t) = \infty$, there exists $t_2 \ge t_1$, such that $R(t) - R(t_1) \ge \frac{1}{2}R(t)$ for $t \ge t_2$. Then

$$x(t) > \frac{1}{2}(t)w^{1/\gamma}(t) \quad \text{for } t \ge t_2\,, \quad \text{and} \quad x^\beta/(\kappa^{1/\gamma}R)^\beta \ge w^{\beta/\gamma}/(2\kappa^{1/\gamma})^\beta.$$

Taking the derivative of w we have

$$\begin{aligned} w'(t) &= -q(t)\bigl(\kappa^{1/\gamma}R(\tau(t))\bigr)^{\alpha-\beta}x^\beta(\tau(t)) \\ &\le -q(t)\bigl(\kappa^{1/\gamma}R(\tau(t))\bigr)^{\alpha}w^{\beta/\gamma}(\tau(t))(2\kappa^{1/\gamma})^{-\beta} \le 0\,. \end{aligned}$$

Therefore, $w(t)$ is non-increasing so $w^{\beta/\gamma}(\tau(t))/w^{\beta/\gamma}(t) \ge 1$, and

$$\bigl(w^{1-\beta/\gamma}(t)\bigr)' = (1-\beta/\gamma)w^{-\beta/\gamma}(t)w'(t) \le -(1-\beta/\gamma)2^{-\beta}\kappa^{(\alpha-\beta)/\gamma}q(t)R^\alpha(\tau(t))\,.$$

Integrating this inequality form t_2 to t, we have

$$\bigl[w^{1-\beta/\gamma}(\eta)\bigr]_{t_2}^{t} \le -(1-\beta/\gamma)2^{-\beta}\kappa^{(\alpha-\beta)/\gamma}\int_{t_2}^{t} q(\eta)R^\alpha(\tau(\eta))\,\mathrm{d}\eta\,.$$

Since $\beta/\gamma < 1$ and $w(t)$ is positive and non-increasing, we have

$$\int_{t_2}^{t} q(\eta)R^\alpha(\tau(\eta))\,\mathrm{d}\eta \le \frac{2^\beta\kappa^{(\beta-\alpha)/\gamma}}{(1-\beta/\gamma)}w^{1-\beta/\gamma}(t_2)\,.$$

This contradicts (2.7) and proves the oscillation of all solutions.

Next, we show that (2.7) is necessary. Suppose that (2.7) does not hold; so for some $\lambda > 0$ the integral in (2.7) is finite. Then there exists $T \ge t_0$ such that

$$\int_T^\infty q(\eta)R^\alpha(\tau(\eta))\mathrm{d}\eta \le \frac{\lambda^{1-\alpha/\gamma}}{2}. \tag{2.8}$$

Let us consider the closed subset of continuous functions

$$M = \Big\{ x \in C([t_0, +\infty), \mathbb{R}) : x(t) = 0 \text{ for } t_0 \le t < T \quad \text{and} \\ \Big(\frac{\lambda}{2}\Big)^{1/\gamma} [R(t) - R(T)] \le x(t) \le \lambda^{1/\gamma} [R(t) - R(T)] \text{ for } t \ge T \Big\}.$$

Then we define the operator $\Phi : M \to C([t_0, +\infty), \mathbb{R})$ by

$$(\Phi x)(t) = \begin{cases} 0, & t_0 \le t < T, \\ \int_T^t \Big[\frac{1}{r(\eta)} \Big[\frac{\lambda}{2} + \int_\eta^\infty q(\zeta) x^\alpha(\tau(\zeta)) \mathrm{d}\zeta \Big] \Big]^{1/\gamma} \mathrm{d}\eta, & t \ge T. \end{cases}$$

For $x \in M$ and $t \ge T$, we have

$$(\Phi x)(t) \ge \int_T^t \Big[\frac{1}{r(\eta)} \frac{\lambda}{2} \Big]^{1/\gamma} \mathrm{d}\eta = \Big(\frac{\lambda}{2}\Big)^{1/\gamma} [R(t) - R(T)] .$$

For $x \in M$ and $t \ge T$, we have $x(t) \le \lambda^{1/\gamma} R(t)$ and $x^\alpha(\tau(t)) \le \big(\lambda^{1/\gamma} R(\tau(t))\big)^\alpha$. Then using (2.8) we have

$$(\Phi x)(t) \le \int_T^t \Big[\frac{1}{r(\eta)} \Big(\frac{\lambda}{2} + \frac{\lambda}{2} \Big) \Big]^{1/\gamma} \mathrm{d}\eta = \lambda^{1/\gamma} [R(t) - R(T)].$$

Thus, $\Phi x \in M$. Let us define now a sequence of continuous function $v_n : [t_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$v_1(t) = \begin{cases} 0, & t \in [t_0, T) \\ \big(\frac{\lambda}{2}\big)^{1/\gamma} [R(t) - R(T)], & t \ge T. \end{cases}$$

$$v_n(t) = (\Phi v_{n-1})(t) \quad \text{for } n > 1.$$

By induction, it is easy to verify that for $n > 1$,

$$\Big(\frac{\lambda}{2}\Big)^{1/\gamma} [R(t) - R(T)] \le v_{n-1}(t) \le v_n(t) \le \lambda^{1/\gamma} [R(t) - R(T)].$$

Therefore, the point-wise limit of the sequence exists. Let $\lim_{n\to\infty} v_n(t) = v(t)$ for $t \ge t_0$. By Lebesgue's dominated convergence theorem $v \in M$ and $(\Phi v)(t) = v(t)$, where $v(t)$ is a solution of equation (1.1) on $[T, \infty)$. Hence, (2.7) is a necessary condition. This completes the proof. □

Example 2.1. Consider the delay differential equation

$$(e^{-t}(x'(t))^{3/5})' + (t+1)(x(t-2))^{1/3} = 0, \quad t \ge 0 . \tag{2.9}$$

Here

$$\gamma = 3/5, \quad \alpha = 1/3, \quad r(t) = e^{-t}, \quad \tau(t) = t-2, \quad R(t) = \int_0^t e^{5s/3}\,\mathrm{d}s = \frac{3}{5}\big(e^{5t/3} - 1\big).$$

For $\beta = 1/2$, we have

$$0 < \alpha < \beta < \gamma \quad \text{and} \quad u^{\alpha-\beta} = u^{-1/6}$$

which is a decreasing function. To check (2.7) we have

$$\int_0^\infty q(\eta) R^\alpha(\tau(\eta)) \mathrm{d}\eta = \int_0^\infty (\eta+1)\Big(\frac{3}{5}\big(e^{5(\eta-2)/3}-1\big)\Big)^{1/3} \mathrm{d}\eta = \infty,$$

because the integrand approaches $+\infty$ as $\eta \to +\infty$. So that all the assumptions in Theorem 2.1 hold. Thus, every solution of (2.9) oscillates.

2.2. For the Case $\gamma < \alpha$.

In this subsection, we assume that there exists $\alpha > \beta > \gamma > 0$ such that

$$u^{\alpha-\beta} \le v^{\alpha-\beta}, \quad \text{for } 0 < u \le v\,. \tag{2.10}$$

Lemma 2.3. *Assume that all conditions of Lemma 2.1 hold. Then there exists $t_1 \ge t_0$ and $\kappa > 0$ such that for $t \ge t_1$, the following holds:*

$$x^\alpha(\tau(t)) \ge \kappa^{\alpha-\beta} x^\beta(\tau(t))\,. \tag{2.11}$$

Proof. By Lemma 2.1, it follows that $x'(t) > 0$, so x is increasing and $x(t) \ge x(t_0)$ for $t \ge t_0$. Thus

$$x(\tau(t)) \ge x(\tau(t_0)) := \kappa > 0 \quad \text{for } t \ge t_1 := t_0\,.$$

From (2.10), we have

$$x^\alpha(\tau(t)) = x^{\alpha-\beta}(\tau(t)) x^\beta(\tau(t)) \ge \kappa^{\alpha-\beta} x^\beta(\tau(t)) \ \text{ for } t \ge t_1,$$

which is (2.11). □

Theorem 2.2. *Under assumptions* (A1), (A2) *and* $r'(t) \ge 0$, *every solution of* (1.1) *is oscillatory if and only if*

$$\int_T^\infty \left[\frac{1}{r(\eta)} \int_\eta^\infty q(\zeta)\,\mathrm{d}\zeta\right]^{1/\gamma} \mathrm{d}\eta = +\infty \quad \textit{for all} \quad T > 0. \tag{2.12}$$

Proof. To prove sufficiency by contradiction, assume that x is a non-oscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is eventually positive. Then Lemmas 2.1 and 2.3 hold for $t \ge t_1$. Using (2.11) in (1.1) and then integrating the final inequality from t to ∞, we have

$$\lim_{A\to\infty} \Big[\big(r(x')^\gamma\big)'(\eta)\Big]_t^A + \kappa^{\alpha-\beta} \int_t^\infty q(\eta) x^\beta(\tau(\eta))\,\mathrm{d}\eta \le 0.$$

Using that $\big(r(x')^\gamma\big)(t)$ is positive and non-increasing, and $r'(t) \ge 0$, we have

$$\kappa^{\alpha-\beta}\int_t^\infty q(\eta)x^\beta(\tau(\eta))\,\mathrm{d}\eta \le \big(r(x')^\gamma\big)(t) \le \big(r(x')^\gamma\big)(\tau(t)) \le r(t)\big(x'(\tau(t))\big)^\gamma$$

for all $t \ge t_1$. Therefore,

$$\kappa^{(\alpha-\beta)/\gamma}\left[\frac{1}{r(t)}\int_t^\infty q(\eta)x^\beta(\tau(\eta))\,\mathrm{d}\eta\right]^{1/\gamma} \le x'(\tau(t))$$

implies that

$$\kappa^{(\alpha-\beta)/\gamma}\left[\frac{1}{r(t)}\int_t^\infty q(\eta)\,\mathrm{d}\eta\right]^{1/\gamma} \le \frac{x'(\tau(t))}{x^{\beta/\gamma}(\tau(t))}\,. \tag{2.13}$$

Integrating (2.13) from t_1 to ∞, we have

$$\kappa^{(\alpha-\beta)/\gamma}\int_{t_1}^\infty\left[\frac{1}{r(\eta}\int_\eta^\infty q(\zeta)\,\mathrm{d}\zeta\right]^{1/\gamma}\mathrm{d}\eta \le \frac{x^{1-\beta/\gamma}(\tau(t_1))}{\beta/\gamma-1} < \infty\,.$$

This contradicts (2.12) and proves the oscillation of all solutions.

Next, we show that (2.12) is necessary. Suppose that (2.12) does not hold; so for each $\lambda > 0$, there exists $T \ge t_0$ such that

$$\int_T^\infty\left[\frac{1}{r(\eta)}\int_\eta^\infty q(\zeta)\mathrm{d}\zeta\right]^{1/\gamma}\mathrm{d}\eta \le \frac{\lambda^{1-\alpha/\gamma}}{2}\,. \tag{2.14}$$

Let us consider the closed subset of continuous functions

$$M=\left\{x\in C([t_0,+\infty),\mathbb{R}) : x(t)=\frac{\lambda}{2} \text{ for } t\in[t_0,T) \text{ and } \frac{\lambda}{2}\le x(t)\le\lambda \text{ for } t\ge T\right\}.$$

Then we define the operator $\Phi : M \to C([t_0,+\infty),\mathbb{R})$ by

$$(\Phi x)(t) = \begin{cases} \lambda/2, & t_0 \le t < T \\ \lambda/2 + \int_T^t\left[\frac{1}{r(\eta)}\int_\eta^\infty q(\zeta)x^\alpha(\tau(\zeta))\mathrm{d}\zeta\right]^{1/\gamma}\mathrm{d}\eta & t \ge T\,. \end{cases}$$

Note that for $x \in M$, we have $(\Phi x)(t) \ge \lambda/2$. Also for $x \in M$ and $t \ge T$, we have $x(t) \le \lambda$ and by (2.14), $(\Phi x)(t) \le \lambda$. Therefore, $\Phi x \in M$. Analogously to the proof of Theorem 2.1, the mapping Φ has a fixed point $v \in M$; that is, $(\Phi v)(t) = v(t)$ for $t \ge t_0$. It can be easily verified that $u(t)$ is a solution of (1.1), such that $\lambda/2 \le v(t) \le \lambda$ for $t \ge T$. Thus, we have a non-oscillatory solution to (1.1). This completes the proof. □

Example 2.2. Consider the delay differential equation

$$\big((x'(t))^{1/5}\big)' + (t+1)(x(t-2))^{5/3} = 0, \quad t \geq 0\,. \tag{2.15}$$

Here

$$\gamma = 1/5, \quad \alpha = 5/3, \quad \tau(t) = t-2.$$

For $\beta = 4/3$, we have

$$\alpha > \beta > \gamma > 0 \quad \text{and} \quad u^{\alpha-\beta} = u^{1/3}$$

which is an increasing function. The integral in (2.12) is equal to

$$\int_2^\infty \left[\int_\eta^\infty (\zeta+1)\mathrm{d}\zeta\right]^5 \mathrm{d}\eta = \infty.$$

So, all the assumptions in Theorem 2.2 hold. Thus, every solution of (2.15) oscillates.

3. Final Comment

In this section, we will be giving one remark and two examples to conclude the paper.

Remark 3.1. The results of this paper also hold for equations of the form

$$\big(r(x')^\gamma\big)'(t) + \sum_{i=1}^m q_i(t)x^{\alpha_i}(\tau_i(t)) = 0\,, \tag{3.1}$$

where r, q_i, α_i, τ_i $(i = 1, 2, \dots, m)$ satisfy the assumptions in (A1) and (A2), (2.3) and (2.10). In order to extend Theorem 2.1 and Theorem 2.2, there exists an index i such that q_i, α_i, τ_i fulfils (2.7) and (2.12).

We finalize the paper by presenting two examples, which show how Remark 3.1 can be applied.

Example 3.1. Consider the delay differential equation

$$\big(e^{-t}(x'(t))^{3/5}\big)' + \frac{1}{t+1}(x(t-2))^{1/3} + \frac{1}{t+2}(x(t-1))^{1/5} = 0, \quad t \geq 0\,. \tag{3.2}$$

Here

$$\gamma = 3/5, \quad r(t) = e^{-t}, \quad \tau_1(t) = t-2, \quad \tau_2(t) = t-1,$$

$$R(t) = \int_0^t e^{5s/3}\,\mathrm{d}s = \frac{3}{5}\big(e^{5t/3} - 1\big) \quad \text{and} \quad i = 1, 2.$$

$$\alpha_1 = 1/3 \quad \text{and} \quad \alpha_2 = 1/5.$$

For $\beta = 1/2$, we have

$$0 < \alpha_1, \quad \alpha_2 < \beta < \gamma \quad \text{and} \quad u^{\alpha_1-\beta} = u^{-1/6} \quad \text{and} \quad u^{\alpha_2-\beta} = u^{-3/10}$$

which both are decreasing functions.

To check (2.7) we have

$$\begin{aligned}\int_0^\infty \sum_{i=1}^m q_i(\eta)R^{\alpha_i}(\tau_i(\eta))\mathrm{d}\eta &\ge \int_0^\infty q_1(\eta)R^{\alpha_1}(\tau_1(\eta))\mathrm{d}\eta\\ &= \int_0^\infty \frac{1}{\eta+1}\Big(\frac{3}{5}\big(e^{5(\eta-2)/3}-1\big)\Big)^{1/3}\mathrm{d}\eta\\ &= \infty,\end{aligned}$$

because the integrand approaches $+\infty$ as $\eta \to +\infty$. So that all the assumptions in Theorem 2.1 hold. Thus, every solution of (3.2) oscillates.

EXAMPLE 3.2. Consider the delay differential equation

$$\big((x'(t))^{3/5}\big)' + t(x(t-2))^{5/3} + (t+1)(x(t-1))^{7/3} = 0, \quad t \ge 0\,. \tag{3.3}$$

Here

$$\gamma = 3/5, \quad \tau_1(t) = t-2, \quad \tau_2(t) = t-1, \quad i = 1,2.$$

$$\alpha_1 = 5/3 \quad \text{and} \quad \alpha_2 = 7/3.$$

For $\beta = 4/3$, we have

$$\alpha_1, \alpha_2 > \beta > \gamma > 0, \quad u^{\alpha_1-\beta} = u^{1/3} \quad \text{and} \quad u^{\alpha_2-\beta} = u$$

which both are increasing functions. Clearly, all the assumptions in Theorem 2.2 hold. Thus, every solution of (3.3) oscillates.

OPEN PROBLEM. From this article and from [3, 4, 7, 8, 9, 13, 19] we have a common question: Can we find necessary and sufficient conditions for the oscillation of solutions to second-order differential equation

$$\Big[r(t)\big((x(t)+p(t)x(\tau(t)))'\big)^\gamma\Big]' + \sum_{i=1}^m q_i(t)x^{\alpha_i}(\tau_i(t)) = 0 \quad \text{for} \quad p \in C(\mathbb{R}_+, \mathbb{R})?$$

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Received March 30, 2019

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