

ALGEBRAIC PROPERTIES
OF WEAK PERRON NUMBERS

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ABSTRACT. We study algebraic properties of real positive algebraic numbers which are not less than the moduli of their conjugates. In particular, we are interested in the relation of these numbers to Perron numbers.

1. Introduction

Some important classes of algebraic integers can be specified by the sizes of their conjugates, e.g., Pisot, Salem or Perron numbers. Sometimes certain refinements of these classes of numbers are necessary. For instance, strong Pisot numbers play a prominent role in the study of limit points of sequences of fractional parts of algebraic integers [2].

A *Perron number* is a real positive algebraic integer which is strictly larger in modulus than its other conjugates (see, e.g., [13, Definition 11.1.2]). Perron numbers play a role in various theories, for instance primitive integral matrices (e.g., [13]), Mahler measures (e.g., [3]), self-similar tilings (e.g., [8]) or numeration systems (e.g., [15]). Here we are concerned with a slight modification of Perron numbers, namely weak Perron numbers. A *weak Perron number* is a real positive algebraic integer α such that α^n is a Perron number for some positive integer n . D. Lind [12, p. 590] introduced weak Perron numbers in his studies of spectral radii of matrices with nonnegative integer entries.

On the other hand, another notion was coined by D. Lind and B. Marcus [13, Exercise 11.1.12]: The real positive algebraic integer α is called an *almost Perron number* if it satisfies

$$\alpha \geq |\beta| \tag{1}$$

for each conjugate β of α . Clearly, every Perron number is almost Perron because then we have strict inequality for $\beta \neq \alpha$ in (1).

Almost Perron numbers were characterized by A. Schinzel [14, Theorem]; observe that in [14] these numbers are called K-numbers. According to this work almost Perron numbers have been introduced and investigated by I. Korec [9] in connection with generalized Pascal triangles. A. Schinzel [14] settled several problems mentioned in [9], [10].

Based on results of [5], [14] we show that the classes of almost Perron and weak Perron numbers coincide (Theorem 2). Further we collect some topological and algebraic properties of weak Perron numbers (Theorem 5 and 6). The author is indebted to an anonymous referee for the recommendation to extend these considerations to the wider concept of algebraic numbers.

2. Algebraic properties of weak Perron numbers

Following [13] we denote by \mathbb{P} (\mathbb{P}_w , respectively) the set of Perron numbers (weak Perron numbers, respectively). Clearly, $\mathbb{P} \subset \mathbb{P}_w$, and this inclusion is strict; e.g., we have $\sqrt{2} \in \mathbb{P}_w \setminus \mathbb{P}$.

Our first result is directly deduced from [5].

LEMMA 1. *Let α be an almost Perron number of degree $d \geq 2$ and put*

$$n = \#\{i \in \{1, \dots, d\} : \alpha = |\alpha_i|\},$$

where $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ are the conjugates of α and $\#S$ denotes the cardinality of the set S . Then $\alpha^n \in \mathbb{P}$.

Proof. The statement is trivial for $n = 1$. We now let $n > 1$ and $\lambda = \alpha^n$. We tacitly use ideas of the proof of [5, Theorem].

If n is even then the minimal polynomial of α can be written in the form $g(X^2)$ with an irreducible $g \in \mathbb{Z}[X]$, and g has $n/2$ roots of modulus α^2 one of which is real and positive. By induction hypothesis we have $\lambda = (\alpha^2)^{n/2} \in \mathbb{P}$.

Let now n be odd. Then renaming the conjugates of α if necessary we can write

$$\alpha_i = \zeta^{i-1} \alpha_1 \quad (i = 1, \dots, n)$$

with some primitive n th root of unity ζ . Thus $n = d$ or the moduli of $\alpha_{n+1}^n, \dots, \alpha_d^n$ are different from λ , hence $\lambda \in \mathbb{P}$. \square

The essence of the following characterization result is completely proved in [14]; here we further mention the computability of the real numbers occurring in Theorem 2 (iv). We denote by \mathbb{N} the set of nonnegative rational integers.

THEOREM 2. *Let $\alpha \in \mathbb{R}_{>0}$. The following statements are equivalent.*

- (i) α is an almost Perron number.
- (ii) α is a weak Perron number.

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- (iii) *There exist effectively computable $\gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{Z}[\alpha] \cap \mathbb{R}_{>0}$ and $c_{ijk} \in \mathbb{N}$ ($i, j, k = 1, \dots, n$) with the property: For all $i, j \in \{1, \dots, n\}$ we have*

$$\gamma_i \gamma_j = \sum_{k=1}^n c_{ijk} \gamma_k. \quad (2)$$

In particular, one can take $\gamma_1 = 1$ and $\gamma_2 = \alpha$.

- (iv) *There exist $\gamma_1, \gamma_2, \dots, \gamma_n \in \mathbb{R}_{>0}$ with the following properties:*
 (a) *We have*

$$\alpha = \sum_{k=1}^n c_k \gamma_k$$

for some $c_1, \dots, c_n \in \mathbb{N}$.

- (b) *For all $i, j \in \{1, \dots, n\}$ there exist $c_{ijk} \in \mathbb{N}$ such that (2) holds.*

P r o o f.

(i) \implies (ii) Clear by Lemma 1. We mention that this result can alternatively be deduced from [14, Lemma 4].

(ii) \implies (i) This can easily be checked by the definitions.

(i) \implies (iii) The statement is trivial for $d = 1$ where we denote by d the degree of α . Let $d > 1$. By Lemma 1 we have $\lambda = \alpha^m \in \mathbb{P}$ for some computable $m \in \mathbb{N}_{>0}$. Set

$$M = \max \{ |\mu| : \mu \text{ conjugate to } \lambda, \mu \neq \lambda \},$$

and pick $\ell \in \mathbb{N}$ with

$$\ell > \max \left\{ \frac{3 \log 2}{\log \lambda}, \frac{2 \log 2}{\log \lambda - \log M} \right\}.$$

Observe

$$\lambda^\ell - 3M^\ell > 2,$$

hence we can choose some $L \in \mathbb{N}$ with

$$M^\ell < L < \frac{1}{2}(\lambda^\ell - M^\ell).$$

Let $\beta_1 = \beta, \beta_2, \dots, \beta_k$ be the conjugates of $\beta := \lambda^\ell - L = \alpha^{\ell m} - L \in \mathbb{Z}[\alpha] \cap \mathbb{R}_{>0}$. As shown in the proof of [14, Lemma 5] we have

$$\beta > |\beta_j|, \quad \Re \beta_j < 0 \quad (j = 2, \dots, k).$$

For $j = 2, \dots, k$ we write

$$\beta_j = |\beta_j| \exp(2\pi i \varphi_j), \quad \varphi_j \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right),$$

thus

$$|\varphi_j - z| > \frac{1}{4}$$

for all $z \in \mathbb{Z}$.

Choose

$$N \in \mathbb{N}_{\geq 4} \quad \text{with} \quad \frac{1}{N} \leq \frac{1}{2} \min \{|\varphi_j - z| : z \in \mathbb{Z}, j = 2, \dots, k\} - \frac{1}{8}.$$

We can compute $\nu \in \mathbb{N}_{>0}$ and $z_j \in \mathbb{Z}$ with

$$|(\nu + 1)\varphi_j - z_j| \leq \frac{1}{N} \quad (j = 2, \dots, k)$$

by an algorithm as described in [6], [7]. The proof of [14, Lemma 3] then shows

$$\Re\beta_j^\nu \leq \frac{1}{2} |\beta_j| \Re\beta_j^{\nu-1} \quad (j = 2, \dots, k).$$

Finally, following the proof of [14, Theorem] the vector

$$\begin{aligned} & (\gamma_1, \dots, \gamma_n) \\ & := (1, \alpha, \dots, \alpha^{\ell m-1}, \beta, \beta\alpha, \dots, \beta\alpha^{\ell m-1}, \dots, \beta^{k\nu-1}, \beta^{k\nu-1}\alpha, \dots, \beta^{k\nu-1}\alpha^{\ell m-1}) \end{aligned}$$

satisfies our requirements where we set $n := k\ell m\nu$.

(iii) \implies (iv) Trivial.

(iv) \implies (i) [14, Theorem]. □

Remark 3. Further characterizations of weak Perron numbers are given in [13, Theorem 11.1.5] and [13, Exercise 11.1.12].

The well-known Lind-Boyd conjecture [1] lists the minimal polynomials of smallest Perron numbers of given degree. In [16] the smallest Perron numbers of degree at most 24 is computed and it is verified that all these numbers satisfy the Lind-Boyd conjecture. On the other hand, using Lemma 4 below it is easy to check that the smallest weak Perron number of degree $d \geq 2$ is $2^{1/d}$.

LEMMA 4. *Let $c \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_{>0}$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be given by $f(x_1, \dots, x_n) = x_1$. Then we have*

$$\begin{aligned} & f(c^{1/n}, \dots, c^{1/n}) \\ & = \min \left\{ \left\{ f(x_1, \dots, x_n) : x_1 \dots x_n = c, \quad x_1 \geq x_i > 0 \quad (i = 1, \dots, n) \right\} \right\}. \end{aligned}$$

Proof. Obvious. □

Now we carry over some topological and algebraic properties of Perron numbers to weak Perron numbers (cf. [11, Section 5]).

THEOREM 5.

(i) *For every $d \in \mathbb{N}_{>0}$ the set*

$$\{\alpha \in \mathbb{P}_w : \deg(\alpha) \leq d\}$$

is a discrete subset of $[1, \infty)$.

(ii) *If F is an algebraic number field, then $F \cap \mathbb{P}_w$ is a discrete subset of $[1, \infty)$.*

Proof.

- (i) Similarly as the proof of [11, Proposition 3].
- (ii) The proof of [11, Proposition 4] applies equally well here.

□

Following the line proposed by A. Dubickas and C. J. Smyth ([4], see also [3]) we now drop the restriction to algebraic integers. We say that the real positive algebraic number α is called a *generalized weak Perron number* if there exists a positive integer n such that α^n is strictly larger in modulus than its other conjugates. The following results are straightforward extensions of well-known facts: Theorem 6 (ii) goes back to D. Lind [11], Theorem 6 (iii) to A. Korec [9], and Theorem 6 (iv), (v), (vi) to A. Schinzel [14].

THEOREM 6.

- (i) *The following statements are equivalent for an algebraic number α .*
 - (a) *α is a generalized weak Perron number.*
 - (b) *There exists a positive rational integer q such that $m q \alpha$ is a weak Perron number for all $m \in \mathbb{N}_{>0}$.*
 - (c) *There exists a positive rational integer q such that $q \alpha$ is a weak Perron number.*
 - (d) *For each conjugate β of α inequality (1) holds.*
- (ii) *The set of generalized weak Perron numbers is closed under addition and multiplication.*
- (iii) *Let α be a generalized weak Perron number and $n \in \mathbb{N}_{\geq 2}$. Then $\alpha^{1/n}$ is a generalized weak Perron number.*
- (iv) *Let α, β be generalized weak Perron numbers. Then we have $\mathbb{Q}(\alpha + \beta) = \mathbb{Q}(\alpha, \beta)$.*
- (v) *If α, β are weak Perron numbers with $\alpha\beta \in \mathbb{N}$, then there exists some $n \in \mathbb{N}_{>0}$ such that $\alpha^n \in \mathbb{N}$ and $\beta^n \in \mathbb{N}$.*
- (vi) *If α, β are generalized weak Perron numbers with $\alpha\beta \in \mathbb{Q}$, then there exists some $n \in \mathbb{N}_{>0}$ such that $\alpha^n \in \mathbb{Q}$ and $\beta^n \in \mathbb{Q}$.*

Proof. We denote by $\hat{\mathbb{P}}_w$ the set of generalized weak Perron numbers.

- (i) (a) \implies (b) Let $n \in \mathbb{N}_{>0}$ such that α^n is strictly larger than all its other algebraic conjugates, and pick $q \in \mathbb{N}_{>0}$ such that $q\alpha$ is an algebraic integer. For each $m \in \mathbb{N}_{>0}$ the positive algebraic integer $(mq\alpha)^n$ dominates all its other conjugates, thus we have $m q \alpha \in \mathbb{P}_w$.
- (b) \implies (c) Trivial.

(c) \implies (d) By Theorem 2 we know that $q\alpha$ is an almost Perron number, hence (1) holds for each conjugate β of α .

(d) \implies (a) Let $q \in \mathbb{N}_{>0}$ such that $q\alpha$ is an algebraic integer, hence an almost Perron number by definition and therefore $q\alpha \in \mathbb{P}_w$ by Theorem 2. Clearly, we then have $\alpha \in \hat{\mathbb{P}}_w$.

- (ii) Copying the proof of [11, Proposition 1] we see that \mathbb{P}_w is closed under addition and multiplication. Let $\alpha, \beta \in \hat{\mathbb{P}}_w$. By (i) there is some $t \in \mathbb{N}_{>0}$ such that $t\alpha, t\beta \in \mathbb{P}_w$, hence

$$t(\alpha + \beta) = t\alpha + t\beta \in \mathbb{P}_w$$

which implies $\alpha + \beta \in \hat{\mathbb{P}}_w$ by (i). Similarly we show $\alpha\beta \in \hat{\mathbb{P}}_w$.

- (iii) This is clear by (i) and [9, Theorem 4.5 (b)].

- (iv) Again let $t \in \mathbb{N}_{>0}$ with $t\alpha, t\beta \in \mathbb{P}_w$. Then we find by Theorem 2 and [14, Corollary 2]

$$\mathbb{Q}(\alpha + \beta) = \mathbb{Q}(t(\alpha + \beta)) = \mathbb{Q}(t\alpha, t\beta) = \mathbb{Q}(\alpha, \beta).$$

- (v) Clear by Theorem 2 and [14, Corollary 3].

- (vi) By (i) there is some $t \in \mathbb{N}_{>0}$ such that $t\alpha, t\beta \in \mathbb{P}_w$. Let $s \in \mathbb{N}_{>0}$ such that $st\alpha, st\beta \in \mathbb{N}$. Using (v) we find $n \in \mathbb{N}_{>0}$ with $(st\alpha)^n, (st\beta)^n \in \mathbb{N}$, hence $\alpha^n, \beta^n \in \mathbb{Q}$.

□

We observe that Theorem 6 (v) cannot be extended to generalized weak Perron numbers. For instance, $\sqrt{2}/2$ and $2\sqrt{2}$ are generalized weak Perron numbers with integer product, but no power of $\sqrt{2}/2$ is an integer.

Finally, we point out that surely not all algebraic properties of Perron numbers can be expected to hold for weak Perron numbers.

Remark 7.

- (i) Observe that [11, Proposition 5] cannot be extended to weak Perron numbers, i.e., $\alpha, \beta \in \mathbb{P}_w$ does not imply $\alpha \in \mathbb{Q}(\alpha\beta)$ or $\beta \in \mathbb{Q}(\alpha\beta)$. For instance, take $\alpha = \beta = \sqrt{2}$.
- (ii) It does not seem useful to carry over the notion of irreducibility of Perron numbers (see [11, p. 293]) to weak Perron numbers: If $\alpha \in \mathbb{P}_w$ and $n \in \mathbb{N}_{>0}$, then we have the trivial factorization

$$\alpha = \left(\alpha^{\frac{1}{n}}\right)^n$$

with $\alpha^{1/n} \in \mathbb{P}_w$ by Theorem 5 (iii).

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REFERENCES

- [1] BOYD, D. W.: *The maximal modulus of an algebraic integer*, Math. Comp. **45** (1985), 243–249.
- [2] DUBICKAS, A.: *A note on powers of Pisot numbers*, Publ. Math. Debrecen **56** (2000), 141–144.
- [3] DUBICKAS, A.: *Mahler measures generate the largest possible groups*, Math. Res. Lett. **11** (2004), 279–283.
- [4] DUBICKAS, A.—SMYTH, C. J.: *On the Remak height, the Mahler measure and conjugate sets of algebraic numbers lying on two circles*, Proc. Edinburgh Math. Soc. (2) **44** (2001), 1–17.
- [5] FERGUSON, R.: *Irreducible polynomials with many roots of equal modulus.*, Acta Arith. **78** (1997), 221–225.
- [6] JURKAT, W.—KRATZ, W.—PEYERIMHOFF, A.: *On best two-dimensional Dirichlet-approximations and their algorithmic calculation*, Math. Ann. **244** (1979), 1–32.
- [7] JUST, B.: *Generalizing the continued fraction algorithm to arbitrary dimensions*, SIAM J. Comput. **21** (1992), 909–926.
- [8] KENYON, R.: *The construction of self-similar tilings*, Geom. Funct. Anal. **6** (1996), 471–488.
- [9] KOREC, I.: *Irrational speeds of configurations growth in generalized Pascal triangles*, Theoret. Comput. Sci. **112** (1993), 399–412.
- [10] ———, *Open problems more or less related to generalized Pascal triangles*, Preprint 22/1996, Mathematical Institute SAS, Bratislava.
- [11] LIND, D. A.: *The entropies of topological Markov shifts and a related class of algebraic integers*, Ergodic Theory Dynam. Systems **4** (1984), 283–300.
- [12] ———, *Entropies of automorphisms of a topological Markov shift*, Proc. Amer. Math. Soc. **99** (1987), 589–595.
- [13] LIND, D.—MARCUS, B.: *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, Cambridge, 1995.
- [14] SCHINZEL, A.: *A class of algebraic numbers*, Tatra Mt. Math. Publ. **11** (1997), 35–42.
- [15] VERGER-GAUGRY, J.-L.: *Uniform distribution of Galois conjugates and beta-conjugates of a Parry number near the unit circle and dichotomy of Perron numbers*, Unif. Distrib. Theory **3** (2008), 157–190.
- [16] WU, Q.: *The smallest Perron numbers*, Math. Comp. **79** (2010), 2387–2394.

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