Tatra Mt. Math. Publ. **44** (2009), 105–113
DOI: 10.2478/v10127-009-0051-4

Tatra Mountains
Mathematical Publications

AN INTEGRAL FOR A BANACH VALUED FUNCTION

Giuseppa Riccobono

ABSTRACT. Using partitions of the unity ((PU)-partition), a new definition of an integral is given for a function $f\colon [a,b] \to X$, where X is a Banach space, and it is proved that this integral is equivalent to the Bochner integral.

Introduction

Using the idea of an integral given by Feauveau in [5], an integral ($(PU)^*$-integral) for a Banach valued function is defined on an interval $[a,b]$, as a limit of Riemann-type sums, using partitions of the unity ((PU)-partition) rather than the usual partitions of the interval. Using some properties of the (PU)-integral of real valued functions defined on a compact Hausdorff space (see [10]), the measurability of a $(PU)^*$-integrable function and the equivalence of this integral with the Bochner integral are proved.

We observe that, while the (PU)-integral is equivalent to the Lebesgue integral and hence to the McShane integral, the $(PU)^*$-integral is equivalent to the Bochner integral but not to the McShane one.

Preliminaries

In this paper, X denotes a Banach space with a norm $|| \ ||_X$ and X^* its dual space, $[a,b]$ is a real interval with $a < b$, $\mathcal{M}$ is a σ-algebra of subsets of $[a,b]$ to which the open sets belong, μ is a non-atomic, finite, complete Radon measure on $\mathcal{M}$.

2000 Mathematics Subject Classification: 26A39, 28B05.
Keywords: Banach valued function, (PU)-partition, $(PU)^*$-integral, Bochner-integral.

Definition 1 ([10]). A partition of the unity ((PU)-partition) of [a,b] is a finite collection $P = \{(\theta_i, x_i)\}_{i=1}^{p}$, where $x_i \in [a,b]$ and θ_i are non negative, μ-measurable and μ-integrable real functions on $[a,b]$ such that $\sum_{i=1}^{p} \theta_i(x) = 1$ a.e. in [a,b].

Definition 2 ([9]). A gauge δ on $[a,b]$ is a map from [a,b] to $(0,+\infty)$.

Definition 3 ([10]). If δ is a gauge on $[a,b]$, a (PU)-partition $P = \{(\theta_i, x_i)\}_{i=1}^{p}$ is δ-fine if $S_{\theta_i} \subset \big(x_i - \delta(x_i), x_i + \delta(x_i)\big)$ $(i = 1, 2, \ldots, p)$, where $S_{\theta_i} = \big\{x \in [a,b]$ such that $\theta_i(x) \neq 0\big\}$.

Now, we will give the following definition:

Definition 4. A function $f\colon [a,b] \to X$ is $(PU)^*$-integrable on $[a,b]$ if for every given $\epsilon > 0$ there is a gauge δ_ϵ such that $\sum_{i=1}^{p} \big\| \big(f(x_i) - f(x_i')\big) \int_a^b \theta_i, d\mu \big\| < \epsilon$ for each couple $P = \{(\theta_i, x_i)\}_i$ and $P' = \{(\theta_i, x_i')\}_i$ of δ_ϵ-fine (PU)-partitions.

We say that δ_ϵ is ϵ-adapted to f.

Main results

1. The $(PU)^*$-integral

Proposition 1.1. *Let $f\colon [a,b] \to X$ be a $(PU)^*$-integrable function and, for every $\epsilon > 0$, δ_ϵ be a gauge ϵ-adapted to f. For every (PU)-partition $P_\epsilon = \{(\theta_{i,\epsilon}, x_{i,\epsilon})\}$ δ_ϵ-fine, put $\sigma(f, P_\epsilon) = \sum_i f(x_{i,\epsilon}) \int_a^b \theta_{i,\epsilon}\, d\mu$. If ϵ converges to 0, then the function $\epsilon \to \sigma(f, P_\epsilon)$ converges; this limit does not depend on the chosen family $\{P_\epsilon, \epsilon > 0\}$ and it is called the $(PU)^*$-integral of f. Denote this limit by $(PU)^* \int_a^b f$.*

Proof. The proof is similar to that in [5, Theorem 2.1]. First, consider an increasing family of gauges (δ_ϵ) with $\epsilon > 0$, so $\delta_\alpha \leq \delta_\gamma$ for $0 < \alpha < \gamma$.

For $0 < \alpha < \gamma$, consider two (PU)-partitions $P_\alpha = (\theta_{i,\alpha}, x_{i,\alpha})_{i=1}^{n}$ and $P_\gamma = (\theta'_{j,\gamma}, x'_{j,\gamma})_{j=1}^{p}$ δ_α-fine and δ_γ-fine respectively, and, since $\delta_\alpha \leq \delta_\gamma$, P_α is also δ_γ-fine.

Using the same construction as in [10, Proposition 1.5], it is possible to construct two partitions $P'_\gamma = (h_{ij,\gamma}, x_i)_{i,j}$ and $P''_\gamma = (h_{ij,\gamma}, x'_j)_{i,j}$ δ_γ-fine with $h_{ij} = (\theta_i \cdot \theta'_j)$ $i = 1, 2, \ldots n$ $j = 1, \ldots p$ and such that

$$\|\sigma(f, P_\alpha) - \sigma(f, P_\gamma)\| = \|\sigma(f, P'_\gamma) - \sigma(f, P''_\gamma)\| \leq \gamma.$$

Since the family $\{\sigma(f, P_\epsilon)\}_\epsilon$ satisfies the Cauchy-property, it converges.

Now, let $(\delta_\epsilon)_{\epsilon>0}$ be a general family of gauges adapted to f and $(\epsilon_n)_{n\in N}$ be a decreasing sequence of positive numbers converging to 0. The sequence $\big(\sigma(f, P_{\epsilon_n})\big)_{n\in N}$ is convergent.

In fact, define a new family of gauges $(\bar{\delta}_\epsilon)_{\epsilon>0}$ as follows:

$\bar{\delta}_\epsilon = \delta_{\epsilon_0}$ for every $\epsilon \geq \epsilon_0$,

$\bar{\delta}_\epsilon = \min\{\delta_{\epsilon_0}, \delta_{\epsilon_1}, \dots, \delta_{\epsilon_n}\}$ for $\epsilon \in [\epsilon_n, \epsilon_{n-1}[$ for every $n > 0$.

The family $(\bar{\delta}_\epsilon)_{\epsilon>0}$ is increasing and adapted to f. So, the family $\big\{\sigma(f, P_{\epsilon_n})\big\}_n$ converges, and the function $\epsilon \to \sigma(f, P_\epsilon)$ is convergent for $\epsilon \to 0$.

Now, we will prove that the limit is independent from the choice of the family $\{P_\epsilon\}$.

Let $(\delta_\epsilon)_{\epsilon>0}$ and $(\delta'_\epsilon)_{\epsilon>0}$ be two families of gauges adapted to f, and $\mathcal{P} = \big\{P_\epsilon = (\theta_{i,\epsilon}, x_{i,\epsilon})_i\big\}_\epsilon$, $\mathcal{P}' = \big\{P'_\epsilon = (\theta'_{i,\epsilon}, x'_{i,\epsilon})\big\}_\epsilon$ be two families of δ_ϵ and $\delta'_\epsilon-$fine (PU)-partitions, respectively, for varying $\epsilon > 0$. For every $\epsilon > 0$, define a family of gauges $\{\delta''_\epsilon\}$ and a family of δ''_ϵ-fine (PU)-partitions $\mathcal{P}'' = \big\{P''_\epsilon = (x''_{i,\epsilon}, \theta''_{i,\epsilon})_i\big\}_\epsilon$ as follows:

$\delta''_\epsilon = \delta_\epsilon$ and $P''_\epsilon = P_\epsilon$ if ϵ is a positive rational,

$\delta''_\epsilon = \delta'_\epsilon$ and $P''_\epsilon = P'_\epsilon$ if ϵ is a positive irrational.

The function $\epsilon \to \sigma(f, P''_\epsilon)$ is convergent for $\epsilon \to 0$, and its limit is equal to the limit of $\{\sigma(f, P'_\epsilon)\}$ and to the limit of $\{\sigma(f, P_\epsilon)\}$. □

Proposition 1.2. *If f, g are two $(PU)^*$-integrable functions on $[a, b]$, the following classical results for an integral are verified:*

α) $f + g$ and kf, for k real number, are $(PU)^*$-integrable,

β) the operator $f \to (PU)^* \int_a^b f$ is linear,

γ) f is $(PU)^*$-integrable in every subinterval $[c, d]$,

δ) if $c \in (a, b)$, then f is $(PU)^*$-integrable in $[a, c]$ and $[c, b]$ and satisfies the Chasles relation

$$(PU)^*\int_a^b f = (PU)^*\int_a^c f + (PU)^*\int_c^b f.$$

Proposition 1.3. *If f is $(PU)^*$-integrable, then for $\epsilon > 0$ and for every δ_ϵ-fine partition $\big\{(\chi_{I_i}, c_i)\big\}$, where χ_{I_i} is the characteristic function of the interval I_i, we have*

$$\sum_i \left\| f(c_i) \int_a^b \chi_{I_i}\, d\mu - (PU)^* \int_{x_{i-1}}^{x_i} f \right\| \leq \epsilon,$$

where $(I_i = [x_{i-1}, x_i], c_i)_i$ is a δ_ϵ-fine partition of $[a, b]$.

Proof. The proof follows the idea of the classical Saks-Henstock lemma, (see [5, p. 922]). □

Proposition 1.4. *If f is $(PU)^*$-integrable on $[a,b]$, then the function $F\colon t \to (PU)^* \int_a^t f$ with $t \in [a,b]$ is absolutely continuous.*

Proof. The proof is analogous to that in [5, Theorem 3.3], but (PU)-partitions $\{(\chi_{I_i})\}$ are used instead of the usual partitions. □

Proposition 1.5. *Let $f\colon [a,b] \to X$ be a continuous and differentiable function with the derivative f' $(PU)^*$-integrable on $[a,b]$. Then*

$$(PU)^* \int_a^b f' = f(b) - f(a).$$

Proof. The proof is analogous to that in [5, Theorem 3.4]. □

Proposition 1.6. *If f is $(PU)^*$-integrable, then the function $F(t) = (PU)^* \int_a^t f$ is differentiable a.e. on $[a,b]$ and $F'(t) = f(t)$ a.e. $t \in [a,b]$.*

Proof. The proof is the same as for the real valued functions (see [7, p. 145]). □

2. Integrability of the norm

Proposition 2.1. *If f is $(PU)^*$-integrable, then the function $\|f\|\colon [a,b] \to [0,+\infty)$ is Lebesgue-integrable and we have the relation*

$$\left\| (PU)^* \int_a^b f \right\| \le \int_a^b \|f\| \, d\mu.$$

Proof. By [10], a real function is (PU)-integrable on $[a,b]$ if there is a real number λ such that for every $\epsilon > 0$, there exists a gauge δ on $[a,b]$ with the property that for every δ-fine (PU)-partition $\{(\theta_i, c_i)\}_i$ of the interval $[a,b]$, the following inequality

$$\left| \sum_i f(c_i) \int_a^b \theta_i \, d\mu - \lambda \right| \le \epsilon$$

holds.

The $(PU)^*$-integrability of $\|f\|$ is an immediate consequence of the definition of $(PU)^*$-integrability of f and the (PU)-integrability of $\|f\|$ (see[10, Prop. 1.5 and 3.3]). Moreover, observe that a gauge δ_ϵ, ϵ – adapted to f, is also ϵ – adapted to $\|f\|$, so the last relation is a consequence of the inequality

$$\left\| \sum_i f(c_i) \int_a^b \theta_i \, d\mu \right\| \le \sum_i \|f(c_i)\| \int_a^b \theta_i \, d\mu$$

for every δ-fine (PU)-partition. □

3. Measurability of a $(PU)^*$-integrable function

,

Proposition 3.1. *If f is a.e. a null function then it is $(PU)^*$-integrable.*

Proof. If f is a.e. a null function, the real function $\|f\|$ is null a.e. and, by the completeness of the measure, it is μ-measurable and μ-integrable and, by [10, Proposition 2.2], it is (PU)-integrable on $[a,b]$. So, for $\epsilon > 0$, there exists a gauge δ_ϵ such that

$$\sum_i \|f(c_i)\| \int_a^b \theta_i \, d\mu \le \epsilon$$

for each δ-fine (PU)-partition $\{(\theta_i, c_i)\}_i$.

Then, for each couple of δ-fine (PU)-partitions $P = \{(\theta_i, c_i)\}_i$ and $P' = \{(\theta_i, c_i')\}_i$, we have

$$\sum_i \|f(c_i) - f(c_i')\| \int_a^b \theta_i \, d\mu \le \sum_i \|f(c_i)\| \int_a^b \theta_i \, d\mu + \sum_i \|f(c_i')\| \int_a^b \theta_i \, d\mu \le 2\epsilon.$$

So, f is $(PU)^*$-integrable and, by Proposition 2.1, $(PU)^* \int_a^b f = 0$. □

Proposition 3.2. *If f is $(PU)^*$-integrable and if $g = f$ a.e. in $[a,b]$, then g is $(PU)^*$-integrable and $(PU)^* \int_a^b f = (PU)^* \int_a^b g$.*

Proof. It is an immediate consequence of the previous proposition and of the linearity of the integral. □

Let us recall the classical definition of the measurability in the strong sense.

Definition 3.1. A function $f\colon [a,b] \to X$ is measurable if it is the limit of a sequence of simple measurable functions a.e. in $[a,b]$.

Proposition 3.3. *If f is a $(PU)^*$-integrable function, then f is measurable.*

P r o o f. We will use the "Petti's measurability theorem" (see [4, p. 42]).

For $x^* \in X^*$, if f is $(PU)^*$-integrable, the real function $(x^* \circ f)$ is (PU)--integrable; in fact, for $\epsilon > 0$ there is a gauge $\delta > 0$ such that

$$\sum_i \left\| \big(f(c_i) - f(c_i')\big) \int_a^b \theta_i \, d\mu \right\| \le \epsilon$$

for each couple of δ-fine partitions $P = \big\{(\theta_i, c_i)\big\}_i$, $P' = \big\{(\theta_i, c_i')\big\}_i$.

Consider the relation:

$$\begin{aligned}
&\left| \sum_i \Big[x^*\big(f(c_i)\big) - x^*\big(f(c_i')\big) \Big] \int_a^b \theta_i \, d\mu \right| \\
&= \left| x^* \left[\sum_i \big(f(c_i) - f(c_i')\big) \int_a^b \theta_i \, d\mu \right] \right| \\
&\le \|x^*\|_{X^*} \left\| \sum_i \big(f(c_i) - f(c_i')\big) \int_a^b \theta_i \, d\mu \right\| \\
&\le \|x^*\|_{X^*} \sum_i \left\| \big(f(c_i) - f(c_i')\big) \int_a^b \theta_i \, d\mu \right\| \\
&\le \|x^*\|_{X^*} \epsilon.
\end{aligned}$$

So, by [10, Proposition 1.5], $(x^* \circ f)$ is (PU)-integrable and, hence, μ-measurable.

To prove that f is μ-essentially separably valued, observe that the continuity of the function F on $[a, b]$ implies the compactness of $F\big([a, b]\big)$. Thus $F\big([a, b]\big)$ is separable.

If $\mathcal{V}\big(F([a, b])\big)$ is a closed linear space spanned by $F\big([a, b]\big)$, then $\mathcal{V}\big(F([a, b])\big)$ is separable and contains the set $\mathcal{H} = \big\{f(t) : F'(t) = f(t),\ t \in [a, b]\big\}$.

So, $\mathcal{H}$ is separable and f is measurable. □

4. Convergence theorems

Definition 4.1. A sequence f_n of $(PU)^*$-integrable functions is uniformly $(PU)^*$-integrable on $[a, b]$ if for each $\epsilon > 0$ there is δ_ϵ such that

$$\sum_i \left\| \big(f_n(c_i) - f_n(c_i')\big) \int_a^b \theta_i \, d\mu \right\| \le \epsilon$$

for each n and for each couple of δ-fine (PU)-partitions $P = \{(\theta_i, c_i)\}_i$ and $P' = \{(\theta_i, c'_i)\}_i$.

Proposition 4.1. *Let f_n be a sequence of uniformly $(PU)^*$-integrable functions defined on $[a,b]$, pointwise convergent to a function f. Then*

1) *f is $(PU)^*$-integrable on $[a,b]$,*
2) *$(PU)^* \int_a^b f = \lim_n (PU)^* \int_a^b f_n$,*
3) *$\lim_n \int_a^b \|f_n - f\| \, d\mu = 0$.*

Proof.

1) Fix $\epsilon > 0$. Let δ_ϵ be a gauge ϵ-adapted to every f_n. So, for every couple $P = \{(\theta_i, c_i)\}_i$, $P' = \{(\theta_i, c'_i)\}_i$ of δ-fine (PU)-partitions and for every $n \in N$, we have

$$\sum_i \left\| \big(f_n(c_i) - f_n(c'_i)\big) \int_a^b \theta_i \, d\mu \right\| \leq \epsilon$$

and the limit for $n \to +\infty$ gives the condition of integrability for f.

2) If $P = \{(\theta_i, c_i)\}_i$ is a δ-fine partition, then $\sum_i f_n(c_i) \int_a^b \theta_i \, d\mu$ converges to $\sum_i f(c_i) \int_a^b \theta_i \, d\mu$. So, for $\epsilon > 0$ fixed, the following relations are verified for all $n > n_\epsilon$, with n_ϵ suitable:

$$\left\| \sum_i f_n(c_i) \int_a^b \theta_i \, d\mu - (PU)^* \int_a^b f_n \right\| \leq \epsilon,$$

$$\left\| \sum_i f(c_i) \int_a^b \theta_i \, d\mu - (PU)^* \int_a^b f \right\| \leq \epsilon,$$

$$\left\| \sum_i f_n(c_i) \int_a^b \theta_i \, d\mu - \sum_i f(c_i) \int_a^b \theta_i \, d\mu \right\| \leq \epsilon.$$

By these three relations, it follows

$$\left\| (PU)^* \int_a^b f_n - (PU)^* \int_a^b f \right\| \leq 3\epsilon.$$

3) The sequence $(f_n - f)_n$ is uniformly $(PU)^*$-integrable, hence, the sequence $\|f_n - f\|$ is uniformly (PU)-integrable and converges to 0 (see [11, Proposition 1]). □

Dominated convergence theorem. Let $f_n\colon [a,b] \to X$ be a sequence of $(PU)^*$-integrable functions on $[a,b]$ converging to f, and let $g \geq 0$ be a μ-integrable real function defined on $[a,b]$ such that $\|f_n\| \leq g$ a.e. in $[a,b]$ for every n. Then f is $(PU)^*$integrable on $[a,b]$ and we have:

$$\lim_n \int_a^b \|f_n - f\| \, d\mu = 0, \qquad \text{and} \qquad \lim_n (PU)^* \int_a^b f_n = (PU)^* \int_a^b f.$$

P r o o f. The proof is similar to the proof for real valued functions. □

5. Equivalence with the Bochner integral

By [4], a measurable function $f\colon [a,b] \to X$ is Bochner integrable if there exists such a sequence of simple measurable functions $f_n\colon [a,b] \to X$ that

$$\lim_n \int_a^b \|f_n - f\| \, d\mu = 0,$$

and the Bochner integral of f is

$$B \int_a^b f = \lim_n \int_a^b f_n.$$

We prove the following result.

Proposition 5.1. *A function $f\colon [a,b] \to X$ is $(PU)^*$-integrable if and only if it is Bochner-integrable. Moreover, the integrals coincide.*

P r o o f. Suppose that f is Bochner-integrable.

By the measurability of f, there exists a sequence $(s_n)_n$ of simple measurable functions such that $\lim_n \|s_n - f\| = 0$ a.e. in $[a,b]$ and it implies that there exists n_ϵ such that for all $n > n_\epsilon$ we have $\|s_n\| \leq \|f\| + 1$ a.e. in $[a,b]$, and according to the dominated convergence theorem, f is $(PU)^*$-integrable. □

The converse is a consequence of the Bochner's Characterization Theorem and the Propositions 2.1 and 3.3.

Finally, since f is the limit of simple functions and the integral of a simple function is the same for any type of integral, by the dominated convergent theorem, we have the equality of the integrals.

REFERENCES

[1] AUBIN, J. P.: *Applied Abstract Analysis*. John Wiley & Sons, New York, 1977.
[2] CAO, S.: *The Henstock integral for Banach-valued functions*, Southeast Asian Bull. Math. **16** (1992), 35–40.
[3] CONGXIN, W.—XIAOBO, Y.: *A Riemann-type definition of the Bochner integral*, J. Math. Study **27** (1994), 32–36.
[4] DIESTEL, J.—UHL, J. J., JR.: *Vector Measures*. Math. Surveys Monogr., Vol. 15, Amer. Math. Soc., Providence, RI, 1977.
[5] FEAUVEAU, J.C.: *A generalized Riemann integral for Banach-valued functions*, Real Anal. Exchange **25** (2000), 919–930.
[6] FREMLIN, D. H.: *The generalized McShane integral*, Illinois J. Math. **39** (1995), 39–67.
[7] GORDON, R. A.: *The Integrals of Lebesgue, Denjoy, Perron and Henstock*. Grad. Stud. Math., Vol. 4, Amer. Math. Soc., Providence, RI, 1994.
[8] GORDON, R. A.: *The McShane integral of Banach-valued functions*, Illinois J. Math. **34** (1990), 557–564.
[9] PFEFFER, W. F.: *The Riemann Approach to Integration: Local Geometric Theory*, Cambridge University Press, Cambridge, 1993.
[10] RICCOBONO, G.: *A PU-integral on a compact Hausdorff space*, Atti Accad. Sci. Lett. Arti Palermo, V. Ser. **22** (2002), 53–69.
[11] RICCOBONO, G.: *Convergent theorems for the PU-integral*, Math. Bohemica **125** (2000), 77–86.
[12] SKVORTSOV, V. A.—SOLODOV, A. P.: *Variational integral for Banach-valued functions*, Real Anal. Exchange **24** (1998), 799–805.

Received December 1, 2008

Dipartimento di Matematica
Universitá di Palermo
Via Archirafi 34
90123 Palermo
ITALY
E-mail: ricco@math.unipa.it