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STRICT DENSITY TOPOLOGY OF THE PLANE. CATEGORY CASE

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ABSTRACT. We study the properties of category density topology of the plane generated by a restricted convergence in the category of double sequences of characteristic functions

$$\left\{\chi_{((n,m)\cdot(A-(x_0,y_0)))\cap([-1,1]\times[-1,1])}\right\}_{n,m\in\mathbb{N}}$$

and more interesting topology generated by a strict convergence in the category of the same sequences, which is a natural modification of a previous one. Similar problems for measure density were considered in [M. Filipczak, W. Wilczyński: *Strict density topology on the plane. Measure case* (in preparation)].

It is well-known that if (X, S, μ) is a finite measure space, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ of S-measurable real functions defined on X converges in measure to a function $f\colon X \to \mathbb{R}$ if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ converges to f μ-almost everywhere. From the above it follows that the convergence in measure can be described in terms of σ-algebra S and σ-ideal of μ-null sets (without the measure μ itself). Suppose now that (X, τ) is a topological space, $\mathcal{B} \subset 2^X$ is a σ-algebra of the sets having the Baire property, and $\mathcal{I} \subset \mathcal{B}$ - a σ-ideal of the sets of first category. We will say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{B}$-measurable (having the Baire property) real functions defined on X converges to a function $f\colon X \to \mathbb{R}$ in category if and only if for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ such that $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ converges to f except on a set of first category (in abbr. $\mathcal{I}$-a.e.). This kind of convergence has been studied in [W]. In the sequel, we will use the convergence in category when $X = \mathbb{R}$ or $\mathbb{R}^2$, in both cases equipped with the natural topology.

The classical density topology has been described in [GNN]. For the convenience of the reader, we will recall basic definition and properties of this topology (compare [O, Chapter 22]).

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Let $\mathcal{L}$ be a σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$ and λ—a linear Lebesgue measure. A point $x_0 \in \mathbb{R}$ is called a density point of a set $A \in \mathcal{L}$ if and only if

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x_0 - h, x_0 + h])}{2h} = 1.$$

If $\Phi(A) = \{x \in \mathbb{R} : x \text{ is a density point of } A\}$ for $A \in \mathcal{L}$, then the operator $\Phi\colon \mathcal{L} \to 2^{\mathbb{R}}$ has the following properties:

1) for each $A \in \mathcal{L}$, $\lambda\big(A \triangle \Phi(A)\big) = 0$ (the Lebesgue Density Theorem),
2) for each $A, B \in \mathcal{L}$, if $\lambda(A \triangle B) = 0$, then $\Phi(A) = \Phi(B)$,
3) $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$,
4) for each $A, B \in \mathcal{L}$, $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

Moreover, the family $\mathcal{T}_d = \{A \in \mathcal{L} : A \subset \Phi(A)\}$ is a topology of the real line stronger than the natural topology ($\mathcal{T}_d$ is called the density topology). Observe that $\mathcal{T}_d \subset \mathcal{L}$ is closed under arbitrary unions, while $\mathcal{L}$ is only a σ-algebra. Observe also that in fact $\Phi\colon \mathcal{L} \to \mathcal{L}$, which follows immediately from LDT.

In [PWW], it was observed that the definition of a density point can be formulated in terms of the convergence in measure. Indeed, if we denote $n \cdot A = \{nt : t \in A\}$ and $A - x = \{t - x : t \in A\}$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $A \subset \mathbb{R}$, then the following conditions are equivalent for $A \in \mathcal{L}$ and $x_0 \in \mathbb{R}$:

a) $\lim_{h\to 0^+} \frac{\lambda(A\cap[x_0-h,x_0+h])}{2h} = 1$,
b) $\lim_{n\to\infty} \frac{\lambda(A\cap[x_0-\frac{1}{n},x_0+\frac{1}{n}])}{2\cdot\frac{1}{n}} = 1$,
c) $\lim_{n\to\infty} \lambda\big(n \cdot (A - x_0) \cap [-1, 1]\big) = 2$,
d) a sequence $\big\{\chi_{(n\cdot(A-x_0)\cap[-1,1])}\big\}_{n\in\mathbb{N}}$ of characteristic functions converges in measure to $\chi_{[-1,1]}$.

Since the convergence in measure (as we observed earlier) can be formulated without using the measure, it follows that the definition of a density point requires only to take subsequences from subsequences of the sequence of characteristic functions of expanded and translated sets. This observation was the starting point in [PWW] for the construction of a category analogue of the density topology of the real line. Namely, a point x_0 is an $\mathcal{I}$-density point of a set $A \subset \mathbb{R}$ having the Baire property if and only if the sequence of characteristic functions described in d) converges in category to $\chi_{[-1,1]}$. The $\mathcal{I}$-density topology is defined as $\mathcal{T}_{\mathcal{I}} = \{A \in \mathcal{B} : A \subset \Phi_{\mathcal{I}}(A)\}$, where $\Phi_{\mathcal{I}}(A)$ denotes the set of all $\mathcal{I}$-density points of A.

In this paper, we present a category density-type topology of the plane related to the strong $\mathcal{I}$-density topology considered in [CW]. To make the text more self-explaining, we start with the definition of the strong density topology in $\mathbb{R}^2$ and its category analogue.

Let $\mathcal{L}_2$ be a σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^2$ and λ_2—a two-dimensional Lebesgue measure. We say that $(x_0, y_0) \in \mathbb{R}^2$ is a strong density point of $A \in \mathcal{L}_2$ if and only if

$$\lim_{\substack{h \to 0^+ \\ k \to 0^+}} \frac{\lambda_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]))}{4hk} = 1.$$

It is known that if $\Phi_s(A) = \big\{(x, y) : (x, y) \text{ is a strong density point of } A\big\}$ for $A \in \mathcal{L}_2$, then the operator $\Phi_s \colon \mathcal{L}_2 \to 2^{\mathbb{R}^2}$ has the following properties:

1) for each $A \in \mathcal{L}_2$, $\lambda_2\big(A \triangle \Phi_s(A)\big) = 0$ (the Lebesgue Density Theorem, see [S, p. 129]),
2) for each $A, B \in \mathcal{L}_2$, if $\lambda_2(A \triangle B) = 0$, then $\Phi_s(A) = \Phi_s(B)$,
3) $\Phi_s(\emptyset) = \emptyset$, $\Phi_s(\mathbb{R}^2) = \mathbb{R}^2$,
4) for each $A, B \in \mathcal{L}_2$, $\Phi_s(A \cap B) = \Phi_s(A) \cap \Phi_s(B)$.

Moreover, the family $\mathcal{T}_s = \big\{A \in \mathcal{L}_2 : A \subset \Phi_s(A)\big\}$ is a topology stronger than the natural topology of the plane called the strong density topology (compare [GNN]).

For our purposes, it will be more convenient to use the following (equivalent) definition of a strong density point (compare [FW]): $(x_0, y_0) \in \mathbb{R}^2$ is a strong density point of $A \in \mathcal{L}_2$ if and only if

$$\lim_{\substack{n \to \infty \\ m \to \infty}} \frac{\lambda_2(A \cap ([x_0 - \frac{1}{n}, x_0 + \frac{1}{n}] \times [y_0 - \frac{1}{m}, y_0 + \frac{1}{m}]))}{4 \cdot \frac{1}{n} \cdot \frac{1}{m}} = 1.$$

The following observation will also be useful: a double sequence of real numbers $\{s_{n,m}\}_{n,m \in \mathbb{N}}$ converges to g if and only if for all increasing sequences of positive integers $\{n_k\}_{k \in \mathbb{N}}$, $\{m_k\}_{k \in \mathbb{N}}$, we have $\lim_{k \to \infty} s_{n_k, m_k} = g$.

Let now $\mathcal{B}_2$ be a σ-algebra of subsets of $\mathbb{R}^2$ having the Baire property and $\mathcal{I}_2$—a σ-ideal of first category subsets of $\mathbb{R}^2$. We shall use the following denotation: $(n, m) \cdot A = \big\{(nx, my) : (x, y) \in A\big\}$ and $A - (x_0, y_0) = \big\{(x - x_0, y - y_0) : (x, y) \in A\big\}$ for $A \subset \mathbb{R}^2$, $n, m \in \mathbb{N}$ and $(x_0, y_0) \in \mathbb{R}^2$. We say (compare [CW]) that $(x_0, y_0) \in \mathbb{R}^2$ is a strong $\mathcal{I}_2$-density point of $A \in \mathcal{B}_2$ if and only if a sequence $\{f_{n,m}\}_{n,m \in \mathbb{N}}$, where $f_{n,m} = \chi_{((n,m) \cdot (A - (x_0, y_0)) \cap ([-1,1] \times [-1,1]))}$, converges in category to $\chi_{[-1,1] \times [-1,1]}$. We will use the following denotation:

$$\chi_{((n,m) \cdot (A - (x_0, y_0)) \cap ([-1,1] \times [-1,1]))} \xrightarrow[n,m \to \infty]{\mathcal{I}_2} \chi_{[-1,1] \times [-1,1]}.$$

Observe that it means that for all increasing sequences $\{n_k\}_{k \in \mathbb{N}}$, $\{m_k\}_{k \in \mathbb{N}}$ there exists an increasing sequence $\{k_p\}_{p \in \mathbb{N}}$ such that $\{f_{n_{k_p}, m_{k_p}}\}_{p \in \mathbb{N}}$ converges to $\chi_{[-1,1] \times [-1,1]}$ $\mathcal{I}_2$-a.e. If $\Phi_{\mathcal{I}_s}(A) = \{(x, y) : (x, y) \text{ is a strong } \mathcal{I}_2\text{-density point of } A\}$ for $A \in \mathcal{B}_2$, then the operator $\Phi_{\mathcal{I}_s} : \mathcal{B}_2 \to 2^{\mathbb{R}^2}$ has the following properties (see [CW]):

1) for each $A \in \mathcal{B}_2$, $A \triangle \Phi_{\mathcal{I}_s}(A) \in \mathcal{I}_2$,
2) for each $A, B \in \mathcal{B}_2$, if $A \triangle B \in \mathcal{I}_2$, then $\Phi_{\mathcal{I}_s}(A) = \Phi_{\mathcal{I}_s}(B)$,
3) $\Phi_{\mathcal{I}_s}(\emptyset) = \emptyset$, $\Phi_{\mathcal{I}_s}(\mathbb{R}^2) = \mathbb{R}^2$,
4) for each $A, B \in \mathcal{B}_2$, $\Phi_{\mathcal{I}_s}(A \cap B) = \Phi_{\mathcal{I}_s}(A) \cap \Phi_{\mathcal{I}_s}(B)$.

Also, the family $\mathcal{T}_{\mathcal{I}_s} = \{A \in \mathcal{B}_2 : A \subset \Phi_{\mathcal{I}_s}(A)\}$ is a topology stronger than the natural topology in the plane ($\mathcal{T}_{\mathcal{I}_s}$ is called the strong $\mathcal{I}_2$-density topology).

In [Ch, p. 18] one can find the following definition: a double sequence

$$\{s_{n,m}\}_{n,m \in \mathbb{N}}$$

of real numbers converges in the restricted sense to s if and only if for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|s_{n,m} - s| < \epsilon$ whenever $n + m \geq n_0$. It is not difficult to observe that the above condition is equivalent to the conjunction of the following three conditions:

a) $\lim_{n\to\infty} s_{n,m} = s$ for each $m \in \mathbb{N}$,
b) $\lim_{m\to\infty} s_{n,m} = s$ for each $n \in \mathbb{N}$,
c) $\lim_{\substack{n\to\infty \\ m\to\infty}} s_{n,m} = s$.

The convergence in the restricted sense of the mean density of A on rectangles has been used for the construction of topologies in [FW].

It is natural also to consider the convergence in the restricted sense of a double sequence of real functions. Using the above observations, we will introduce the following definition:

Definition 1. We say that a point $(x_0, y_0) \in \mathbb{R}^2$ is a point of restricted $\mathcal{I}_2$-density of a set $A \in \mathcal{B}_2$ if and only if:

1^o for each $m \in \mathbb{N}$, $\chi_{((n,m)\cdot(A-(x_0,y_0)))\cap([-1,1]\times[-1,1]))} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}$,

2^o for each $n \in \mathbb{N}$, $\chi_{((n,m)\cdot(A-(x_0,y_0)))\cap([-1,1]\times[-1,1]))} \xrightarrow[m\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}$,

3^o $\chi_{((n,m)\cdot(A-(x_0,y_0)))\cap([-1,1]\times[-1,1]))} \xrightarrow[n,m\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}$.

We say that a point (x_0, y_0) is a point of restricted $\mathcal{I}_2$-dispersion of a set $A \in \mathcal{B}_2$ if and only if it is a point of restricted $\mathcal{I}_2$-density of $\mathbb{R}^2 \setminus A$.

Example 1. There exists a set $A \subset \mathbb{R}^2$ having the Baire property such that $(0,0)$ is a point of restricted $\mathcal{I}_2$-dispersion of A and for any $h > 0$, the set $A \cap \big([-h,h] \times [-h,h]\big)$ is of second category.

Let

$$A = \big\{(x,y) : 0 \leq x,\ 0 \leq y,\ x \leq y \leq x + x^2\big\}.$$

For any fixed m and $n > m$,

$$(n,m)\cdot A \cap \big([-1,1]\times[-1,1]\big) = \left\{(x,y)\in\big([0,1]\times[0,1]\big) : \frac{m}{n}x \le y \le \frac{m}{n}\left(x+\frac{x^2}{n}\right)\right\}.$$

Hence, $\lim_{n\to\infty}\big((n\cdot m)\cdot A\cap([-1,1]\times[-1,1])\big) = \big\{(0,0)\big\}\in\mathcal{I}_2$, and

$$\chi_{((n,m)\cdot A\cap([-1,1]\times[-1,1]))} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{\phi}.$$

It means that the set $\mathbb{R}^2\setminus A$ fulfills condition 1^o. In an analogous way, we check that $\mathbb{R}^2\setminus A$ fulfills 2^o.

We will prove that $(0,0)$ is a strong $\mathcal{I}_2$-dispersion point of A.

Take two increasing sequences $\{n_k\}_{k\in\mathbb{N}}, \{m_k\}_{k\in\mathbb{N}}$. We want to show that there exists an increasing sequence $\{k_p\}_{p\in\mathbb{N}}$ such that

$$\limsup_{p\to\infty}\big((n_{k_p},m_{k_p})\cdot A\big)\cap\big([-1,1]\times[-1,1]\big)\in\mathcal{I}_2.$$

Observe that there exists a sequence $\{k_p\}_{p\in\mathbb{N}}$ such that either

$$\lim_{p\to\infty}\frac{m_{k_p}}{n_{k_p}} = a\in(0,\infty),\quad\text{or}\quad \lim_{p\to\infty}\frac{m_{k_p}}{n_{k_p}} = 0,\quad\text{or}\quad \lim_{p\to\infty}\frac{m_{k_p}}{n_{k_p}} = +\infty.$$

In the first case, we have

$$\lim_{p\to\infty}\big((n_{k_p},m_{k_p})\cdot A\big)\cap\big([-1,1]\times[-1,1]\big)\subset\left\{(x,ax) : x\in\left[0,\frac{1}{a}\right]\right\}\in\mathcal{I}_2.$$

Indeed, the Hausdorff distance between the set

$$\big((n_{k_p},m_{k_p})\cdot A\big)\cap\big([-1,1]\times[-1,1]\big)\ \text{and}\ \left\{(x,ax) : x\in\left[0,\frac{1}{a}\right]\right\}$$

tends to zero when p tends to infinity. Similarly, one can prove that in the second and the third case, we have

$$\limsup_{p\to\infty}\big((n_{k_p},m_{k_p})\cdot A\big)\cap\big([-1,1]\times[-1,1]\big)\subset\big\{(0,0)\big\}.$$

Thus, the set $\mathbb{R}^2\setminus A$ fulfils 3^o.

Example 2. Let $B = \big\{(x,y) : x^2\le|y|\le\sqrt{|x|}\big\}$ and $A=\mathbb{R}^2\setminus B$. It is easy to verify that conditions 1^o and 2^o are fulfilled for $(0,0)$ while $(0,0)$ is not a strong $\mathcal{I}_2$-density point of A.

Let $\Phi_{\mathcal{I}_r}(A) = \big\{(x,y)\in\mathbb{R}^2 : (x,y)\text{ is a restricted }\mathcal{I}_2\text{-density point of }A\big\}$ for $A\in\mathcal{B}_2$. We observe at once that $\Phi_{\mathcal{I}_r}(A)\subset\Phi_{\mathcal{I}_s}(A)$ for each $A\in\mathcal{B}_2$.

Remark 1. Observe also that condition 1^o holds if and only if

$$\chi_{((n,1)\cdot(A-(x_0,y_0))\cap([-1,1]\times[-1,1]))} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}.$$

The necessity is obvious. To prove the sufficiency, take $m \in \mathbb{N}$. To simplify the denotations, put

$$f_{n,m} = \chi_{((n,m)\cdot(A-(x_0,y_0))\cap([-1,1]\times[-1,1]))}.$$

Then, we have

$$f_{n,1} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]},$$

so

$$f_{n,1|[-1,1]\times[-\frac{1}{m},\frac{1}{m}]} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-\frac{1}{m},\frac{1}{m}]}.$$

However,

$$f_{n,m}(x,y) = f_{n,1}\left(x, \frac{y}{m}\right) \quad \text{for } (x,y) \in [-1,1]\times[-1,1],$$

so

$$f_{n,m} \xrightarrow[n\to\infty]{} \overset{\mathcal{I}_2}{\to} \longrightarrow \chi_{[-1,1]\times[-1,1]}.$$

The same remark concerns 2^o.

Theorem 1. *The operator $\Phi_{\mathcal{I}_r} : \mathcal{B}_2 \to 2^{\mathbb{R}^2}$ has the following properties:*

2) *for each $A, B \in \mathcal{B}_2$, if $A \triangle B \in \mathcal{I}_2$, then $\Phi_{\mathcal{I}_r}(A) = \Phi_{\mathcal{I}_r}(B)$,*
3) *$\Phi_{\mathcal{I}_r}(\emptyset) = \emptyset$, $\Phi_{\mathcal{I}_r}(\mathbb{R}^2) = \mathbb{R}^2$,*
4) *for each $A, B \in \mathcal{B}_2$, $\Phi_{\mathcal{I}_r}(A \cap B) = \Phi_{\mathcal{I}_r}(A) \cap \Phi_{\mathcal{I}_r}(B)$.*

The proof is straightforward. □

Remark 2. $\Phi_{\mathcal{I}_r}\big((-\infty,1]\times\mathbb{R}\big) = (-\infty,0]\times\mathbb{R}$, so the analogon of LDT does not hold.

Theorem 2. *The family $\mathcal{T}_{\mathcal{I}_r} = \big\{A \in \mathcal{B}_2 : A \subset \Phi_{\mathcal{I}_r}(A)\big\}$ is a topology.*

Proof. $\emptyset$ and $\mathbb{R}^2$ belong to $\mathcal{T}_{\mathcal{I}_r}$ by virtue of 3. $\mathcal{T}_{\mathcal{I}_r}$ is closed under finite intersections by virtue of 4. If $\mathcal{A} \subset \mathcal{T}_{\mathcal{I}_r}$, then also $\mathcal{A} \subset \mathcal{T}_{\mathcal{I}_s}$, because $\Phi_{\mathcal{I}_r}(A) \subset \Phi_{\mathcal{I}_s}(A)$ for $A \in \mathcal{B}_2$. Hence, $\cup\mathcal{A} \in \mathcal{T}_{\mathcal{I}_s} \subset \mathcal{B}_2$. Since $A \subset \Phi_{\mathcal{I}_r}(A)$ for $A \in \mathcal{A}$, we have also $A \subset \Phi_{\mathcal{I}_r}(\cup\mathcal{A})$ for $A \in \mathcal{A}$, because $\Phi_{\mathcal{I}_r}$ is monotone (which follows immediately from 4)). Finally, $\cup\mathcal{A} \subset \Phi_{\mathcal{I}_r}(\cup\mathcal{A})$, which finishes the proof. □

Theorem 3. *If $A \in \mathcal{B}_2$, then $\Phi_{\mathcal{I}_r}(A) \in \mathcal{B}_2$.*

Proof. Suppose that $A \in \mathcal{B}_2$ and put

$$\Psi_1(A) = \left\{(x,y) \in \mathbb{R}^2 : \chi_{((n,1)\cdot(A-(x,y))\cap([-1,1]\times[-1,1]))} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}\right\},$$

$$\Psi_2(A)$$
$$= \left\{(x,y) \in \mathbb{R}^2 : \chi_{((1,m)\cdot(A-(x,y))\cap([-1,1]\times[-1,1]))} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}\right\}.$$

Since $\Phi_{\mathcal{I}_r}(A) = \Psi_1(A)\cap\Psi_2(A)\cap\Phi_{\mathcal{I}_s}(A)$, it is sufficient to prove that $\Psi_1(A) \in \mathcal{B}_2$ and $\Psi_2(A) \in \mathcal{B}_2$. We shall prove this for Ψ_1, the proof for Ψ_2 remains being similar.

If $A \in \mathcal{B}_2$, then $A = (G\setminus P_1)\cup P_2$, where G is a regular open set in the natural topology in $\mathbb{R}^2$, $P_1, P_2 \in \mathcal{I}_2$. It is easy to check that $\Psi_1(A) = \Psi_1(G)$, so it is sufficient to prove that $\Psi_1(G) \in \mathcal{B}_2$.

Let $G_1 = \operatorname{Int}(\mathbb{R}^2 \setminus G)$. We have $\mathbb{R}^2 = G \cup \operatorname{Fr} G \cup G_1$ (a disjoint union).

For any square $Q = (a,b) \times (c,d)$ we will denote by Q^* the rectangle

$$(a,b) \times (c-1, d+1).$$

Let $G_2 = \cup Q^*$, where the union is taken over all squares $Q \subset G_1$. Obviously, G_2 is an open set. We will show that $\Psi_1(G) \cap G_2 = \emptyset$.

Fix a point $(x,y) \in G_2$. There exists a square $Q = (a,b) \times (c,d) \subset G_1$ such that $(x,y) \in Q^*$. There are three cases: $d \le y < d+1$, $c < y < d$ or $c-1 < y \le c$. If $d < y < d+1$, then $d - y > -1$, and

$$(n,1)\cdot\big(G-(x,y)\big)\cap\big([-1,1]\times(-1,d-y)\big) = \emptyset$$

for sufficiently big n (such that $\frac{1}{n} < \min\{b-x, x-a\}$). Hence (in the denotation of Remark 1), the sequence $\{f_{n,1}\}$ does not converge in category to $\chi_{[-1,1]\times[-1,1]}$ and $(x,y) \notin \Psi_1(G)$.

If $c < y < d$ then, for sufficiently big n,

$$(n,1)\cdot\big(G-(x,y)\big)\cap\big([-1,1]\times(c-y,d-y)\big) = \emptyset$$

and again, $(x,y) \notin \Psi_1(G)$. The third case is analogous to the first one.

Let $G_3 = \operatorname{Int}(\mathbb{R}^2 \setminus G_2)$. Since $\mathbb{R}^2 \setminus G_2 \subset \mathbb{R}^2 \setminus G_1 = G \cup \operatorname{Fr} G = \overline{G}$ and G is a regular open set, we have $G_3 \subset G$. Obviously, G_3 is an open set, so for any $(x,y) \in G_3$ there exists a square

$$Q = (x-h, x+h) \times (y-h, y+h) \subset G_3.$$

It is not difficult to observe that $Q^* \subset G$. From this it immediately follows that $(x,y) \in \Psi_1(G)$.

Finally, we have $G_3 \subset \Psi_1(G) \subset \mathbb{R}^2 \setminus G_2$ and $\mathbb{R}^2 \setminus G_2 = G_3 \cup \operatorname{Fr} G_2$, so $\mathbb{R}^2 \setminus G_2$ differs from G_3 on nowhere dense set. Hence, $\Psi_1(G) \in \mathcal{B}_2$. □

Theorem 4. $\mathcal{T}_{\mathcal{I}_r} = \{\emptyset\} \cup \{\mathbb{R}^2 \setminus P : P \in \mathcal{I}_2\}$.

Proof. Suppose that $A \in \mathcal{T}_{\mathcal{I}_r}$ and $\mathbb{R}^2 \setminus A \notin \mathcal{I}_2$. Then there exists a square $Q = (a,b) \times (c,d)$ such that $Q \cap A \in \mathcal{I}_2$.

From the proof of the previous theorem, it follows that $Q^* \cap \Phi_{\mathcal{I}_r}(A) = \emptyset$, so also $Q^* \cap A = \emptyset$. If $Q_1 = (a,b) \times (e,f)$ is an arbitrary square included in Q^*, then again we have $Q_1^* \cap A = \emptyset$. Repeating this argument, we obtain that the vertical strip $(a,b) \times \mathbb{R}$ is disjoint with A. Starting from the square $(a,b) \times (c,d)$ for arbitrary $(c,d) \subset \mathbb{R}$, we similarly obtain that the horizontal strip $\mathbb{R} \times (c,d)$ is disjoint with A. Hence, $A = \emptyset$. □

A similar result for measure has been obtained in [FW].

From the latter theorem, it follows that the topology $\mathcal{T}_{\mathcal{I}_r}$ is not very interesting. We will consider some modification of definition which leads to the topology between the natural topology and the strong $\mathcal{I}_2$-density topology in $\mathbb{R}^2$.

Definition 2. We say that a point (x_0, y_0) is a strict $\mathcal{I}_2$-density point of a set $A \in \mathcal{B}_2$ if and only if:

1^o there exists $m_0 \in \mathbb{N}$ such that
$$\chi_{((n,m_0)\cdot(A-(x_0,y_0))\cap([-1,1]\times[-1,1]))} \xrightarrow[n\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]},$$

2^o there exists $n_0 \in \mathbb{N}$ such that
$$\chi_{((n_0,m)\cdot(A-(x_0,y_0))\cap([-1,1]\times[-1,1]))} \xrightarrow[m\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]},$$

3^o $$\chi_{((n,m)\cdot(A-(x_0,y_0))\cap([-1,1]\times[-1,1]))} \xrightarrow[n,m\to\infty]{\mathcal{I}_2} \chi_{[-1,1]\times[-1,1]}.$$

Observe that (similarly like after the previous definition) the convergence for m_0 in 1^o implies the convergence for each $m \geq m_0$, and similarly in 2^o.

Let $\Phi_{\mathcal{I}_{st}}(A) = \big\{(x,y) : (x,y) \text{ is a strict } \mathcal{I}_2\text{-density point of } A \in \mathcal{B}_2\big\}$.

Theorem 5. *The operator $\Phi_{\mathcal{I}_{st}} : \mathcal{B}_2 \to 2^{\mathbb{R}^2}$ has the following properties:*

1) *for each $A \in \mathcal{B}_2$, $A \triangle \Phi_{\mathcal{I}_{st}}(A) \in \mathcal{I}_2$,*
2) *for each $A, B \in \mathcal{B}_2$, if $A \triangle B \in \mathcal{I}_2$, then $\Phi_{\mathcal{I}_{st}}(A) = \Phi_{\mathcal{I}_{st}}(B)$,*
3) *$\Phi_{\mathcal{I}_{st}}(\emptyset) = \emptyset$, $\Phi_{\mathcal{I}_{st}}(\mathbb{R}^2) = \mathbb{R}^2$,*
4) *for each $A, B \in \mathcal{B}_2$, $\Phi_{\mathcal{I}_{st}}(A \cap B) = \Phi_{\mathcal{I}_{st}}(A) \cap \Phi_{\mathcal{I}_{st}}(B)$.*

P r o o f. The proofs of 2), 3) and 4) are straightforward.

1) Let $A = (G \setminus P_1) \cup P_2$, where G is open in the natural topology of $\mathbb{R}^2$, P_1 and P_2 are of first category. Then, it is easy to see that $G \subset \Phi_{\mathcal{I}_{st}}(A) \subset \overline{G}$, so $G \triangle \Phi_{\mathcal{I}_{st}}(A) \subset \overline{G} \setminus G \in \mathcal{I}_2$, and finally, $A \triangle \Phi_{\mathcal{I}_{st}}(A) \in \mathcal{I}_2$ since $A \triangle G \in \mathcal{I}_2$. □

Corollary. *If $A \in \mathcal{B}_2$, then $\Phi_{\mathcal{I}_{st}}(A) \in \mathcal{B}_2$.*

P r o o f. It follows from 1). □

Theorem 6. *The family $\mathcal{T}_{\mathcal{I}_{st}} = \big\{A \in \mathcal{B}_2 : A \subset \Phi_{\mathcal{I}_{st}}(A)\big\}$ is a topology stronger than the natural topology $\mathcal{T}$ in $\mathbb{R}^2$ and weaker than the strong $\mathcal{I}_2$-density topology $\mathcal{T}_{\mathcal{I}_s}$.*

Proof. $\emptyset$ and $\mathbb{R}^2$ belong to $\mathcal{T}_{\mathcal{I}_{st}}$ by virtue of 3). The family $\mathcal{T}_{\mathcal{I}_{st}}$ is closed with respect to finite intersections by virtue of 4). If $\mathcal{A} \subset \mathcal{T}_{\mathcal{I}_{st}}$, then $\mathcal{A} \subset \mathcal{T}_{\mathcal{I}_s}$ and so $\cup\mathcal{A} \in \mathcal{B}_2$. From 4) it also follows that if $A \subset B$, then $\Phi_{\mathcal{I}_{st}}(A) \subset \Phi_{\mathcal{I}_{st}}(B)$, so $\Phi_{\mathcal{I}_{st}}(A) \subset \Phi_{\mathcal{I}_{st}}(\cup\mathcal{A})$ for each $A \in \mathcal{A}$. Since also $A \subset \Phi_{\mathcal{I}_{st}}(A)$ from the definition of $\mathcal{T}_{\mathcal{I}_{st}}$, we immediately have $\cup\mathcal{A} \subset \Phi_{\mathcal{I}_{st}}(\cup\mathcal{A})$ and $\cup\mathcal{A} \in \mathcal{T}_{\mathcal{I}_{st}}$, which means that $\mathcal{T}_{\mathcal{I}_{st}}$ is closed under arbitrary unions.

The set $(\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q})$, where $\mathbb{Q}$ stands for the set of all rational numbers, belongs to $\mathcal{T}_{\mathcal{I}_{st}}$ but not to the natural topology in $\mathbb{R}^2$.

Let $E = \cup_{k=1}^{\infty}(a_k, b_k)$, where $b_{k+1} < a_k < b_k$ for $k \in \mathbb{N}$, be a set for which 0 is the right-hand $\mathcal{I}$-density point, $F = E \cup (-E)$ and $H = (F \times F) \cup \{(0,0)\}$. Then, $(0,0)$ is a strong $\mathcal{I}_2$-density point of H and, since $H \setminus \{(0,0)\}$ is open in the natural topology on the plane, $H \in \mathcal{T}_{\mathcal{I}_{st}}$.

On the other hand, for any fixed $m_0 \in \mathbb{N}$, the second category set

$$[-1,1] \times \left(m_0 \cdot \left(\bigcup_{k=1}^{\infty}(b_{k+1}, a_k)\right)\right)$$

is disjoint from the set

$$(n, m_0) \cdot H \cap \big([-1,1] \times [-1,1]\big)$$

for all $n \in \mathbb{N}$. Therefore, condition 1^o from Definition 2 is not fulfilled and, consequently, H does not belong to $\mathcal{T}_{\mathcal{I}_{st}}$. $\square$

Theorem 7. *$(\mathbb{R}^2, \mathcal{T}_{\mathcal{I}_{st}})$ is a Hausdorff but not regular space.*

Proof. Since $\mathcal{T}_{\mathcal{I}_{st}}$ is stronger than the natural topology in $\mathbb{R}^2$, it is obviously Hausdorff.

The set $A = (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0,0)\}$ is $\mathcal{T}_{\mathcal{I}_{st}}$-closed and cannot be separated from $(0,0)$. Indeed, if $U \in \mathcal{T}_{\mathcal{I}_{st}}$, $U \supset A$ and $U = (G_1 \setminus P_1) \cup P_2$, where G_1 is open in the natural topology in $\mathbb{R}^2$, $P_1, P_2 \in \mathcal{I}_2$, then G_1 is dense in $\mathbb{R}^2$. Also, if $V \in \mathcal{T}_{\mathcal{I}_{st}}$, $(0,0) \in V$, and $V = (G_2 \setminus P_3) \cup P_4$, where G_2 is open, $P_3, P_4 \in \mathcal{I}_2$, then $G_2 \neq \emptyset$. Hence, $G_1 \cap G_2 \neq \emptyset$, and finally $U \cap V \neq \emptyset$. $\square$

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