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# NONMEASURABLE SETS WITH REGULAR SECTIONS

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ABSTRACT. We improve and generalize the result of Kirchheim and Natkaniec [Kirchheim, B., Natkaniec, T.: *Exceptional directions for Sierpiński's nonmeasurable sets*, Fund. Math. **140** (1992), 237–245] by proving that there exists  $D \subset [0,\pi)$  of full outer measure such that the class  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap \bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$  contains a nonmeasurable set. The present result is obtained within ZFC.

Let  $\theta \in [0, \pi)$  and  $n \in \mathbb{N}$ . By  $\mathcal{G}_n(\theta)$  we denote the family of all  $E \subset \mathbb{R}^2$  such that for any line  $\ell$  in the direction  $\theta$ , the set  $E \cap \ell$  is open in  $\ell$  and has at most n connected components. For any two different points  $x, y \in \mathbb{R}^2$ ,  $\ell(x, y)$  denotes the line through x and y, and for each  $x \in \mathbb{R}^2$  and for each  $\theta \in [0, \pi)$ ,  $\ell_{\theta}(x)$  denotes the line through x in the direction  $\theta$ . We let dir $(\ell)$  denote the direction of the line  $\ell \subset \mathbb{R}^2$  and for each  $\theta \in [0, \pi)$ , we denote by  $L(\theta)$  the family of all lines  $\ell \subset \mathbb{R}^2$  with a direction  $\theta$ . For  $D \subset [0, \pi)$ , let  $C_D$  be the set  $\bigcup \{\ell_{\theta}(0) \colon \theta \in D\}$ .

The  $\sigma$ -algebra of all Borel subsets of the space X will be written as  $\mathcal{B}(X)$ . We denote by  $\mathcal{N}$  and  $\mathcal{M}$  the  $\sigma$ -ideal of Lebesgue null sets and of meager sets in  $\mathbb{R}$ , respectively.

Sierpiński [5] constructed an example of Lebesgue nonmeasurable and without the Baire property subset of the plane whose complement belongs to the class  $\bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$ . Kirchheim, Natkaniec [4] and Frantz [3] were interested in the smaller class  $\bigcap_{\theta \in [0,\pi)} \mathcal{G}_2(\theta)$ . In [3], Frantz proved that if  $[0,\pi) \setminus D$  is a Lebesgue null set, then every set in  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta)$  is measurable (with respect to Lebesgue measure) subset of the plane. In [4], the authors asked if the full outer Lebesgue measure of D is sufficient for measurability of sets in  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap \bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$  and they showed that if A(m) holds, then the answer is negative, where A(m) stands for the proposition that the union of less than  $\mathfrak{c}$  measure zero sets has measure zero. They remarked that the category analogue of this result is true. Our aim is to improve and generalize these results and give an answer which is obtained within ZFC.

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**DEFINITION 1** ([2]). Let  $\mathcal{I}$  and  $\mathcal{J}$  be proper  $\sigma$ -ideals of subsets of  $\mathbb{R}$  and let

$$\mathcal{I} \otimes \mathcal{J} = \left\{ A \subset \mathbb{R}^2 \colon \{ x \in \mathbb{R} \colon A_x \notin \mathcal{J} \} \in \mathcal{I} \right\},\$$

where  $A_x = \{y \in \mathbb{R} \colon (x, y) \in A\}$ . We define

$$\mathcal{I}\tilde{\otimes}\mathcal{J} = \left\{ A \subset \mathbb{R}^2 \colon \left( \exists B \in \mathcal{B}(\mathbb{R}^2) \cap (\mathcal{I} \otimes \mathcal{J}) \right) A \subset B \right\}.$$

It is easy to verify that  $\mathcal{N} \otimes \mathcal{N}$  and  $\mathcal{M} \otimes \mathcal{M}$  produce exactly  $\sigma$ -ideals of Lebesgue null sets and of meager sets in  $\mathbb{R}^2$ . On the other hand, if  $\mathcal{I} = \mathcal{N} \cap \mathcal{M}$ , then it is not true that  $\mathcal{I} \otimes \mathcal{I} = (\mathcal{N} \otimes \mathcal{N}) \cap (\mathcal{M} \otimes \mathcal{M})$ . Indeed, suppose that  $A \subset \mathbb{R}$  is a Lebesgue null set of second category and suppose that  $B \subset \mathbb{R}$  is a meager set but not a Lebesgue null set. According to the Fubini theorem and to the Kuratowski-Ulam theorem, respectively,  $E = (A \times B) \cup (B \times A)$  is a meager set and a Lebesgue null set. However, E does not belong to  $\sigma$ -ideal  $\mathcal{I} \otimes \mathcal{I}$ .

Throughout the paper,  $\mathcal{I}$  will stand for a proper  $\sigma$ -ideal of subsets of  $\mathbb{R}$  which contains all countable subsets of  $\mathbb{R}$ ,  $\mathcal{K}$  will stand for the smallest  $\sigma$ -algebra containing  $\mathcal{I} \cup \mathcal{B}(\mathbb{R})$  and  $\mathcal{K}_2$  will stand for the smallest  $\sigma$ -algebra containing  $(\mathcal{I} \otimes \mathcal{I}) \cup \mathcal{B}(\mathbb{R}^2)$ . If it is not explicitly said, we assume that  $\sigma$ -ideal  $\mathcal{I}$  satisfies the following conditions:

- (1)  $\mathcal{I}$  does not contain nondegenerate subintervals of  $\mathbb{R}$ .
- (2)  $\mathcal{I}$  has a Borel base.
- (3) For each Borel  $B \subset \mathbb{R}^2$ ,  $\{x \in \mathbb{R} : B_x \notin \mathcal{I}\}$  is a Borel set.
- (4)  $\sigma$ -algebra  $\mathcal{K}_2$  and  $\sigma$ -ideal  $\mathcal{I} \otimes \mathcal{I}$  both are invariant with respect to isometry of  $\mathbb{R}^2$ .

It is easy to verify that from properties (1)-(4) it follows that:

- (a)  $\mathcal{I} \otimes \mathcal{I} = (\mathcal{I} \otimes \mathcal{I}) \cap \mathcal{K}_2$ .
- (b) If  $E \in \mathcal{K}_2$ , then  $\{x \in \mathbb{R} : E_x \notin \mathcal{K}\} \in \mathcal{I}$ .

It is well-known that  $\sigma$ -ideals  $\mathcal{N}$  and  $\mathcal{M}$  satisfy conditions (1)–(4). On the other hand, there exist  $\sigma$ -ideals which do not satisfy all the conditions from (1) to (4).

EXAMPLE 1. The  $\sigma$ -ideal  $\mathcal{J}$  of all countable subsets of  $\mathbb{R}$  does not satisfy condition (3). Suppose that  $A \subset \mathbb{R}$  is an analytic but not a Borel set. Then, according to [6, Proposition 4.3.7], there exists a Borel set  $B \subset \mathbb{R}^2$  such that  $A = \{x \in \mathbb{R} : B_x \notin \mathcal{J}\}.$ 

EXAMPLE 2. If  $\mathcal{I} = \mathcal{N} \cap \mathcal{M}$ , then  $\mathcal{I} \otimes \mathcal{I}$  is not invariant with respect to rotations of the plane. Indeed, let consider  $E = A \times B$ , where A and B are Borel subsets of  $\mathbb{R}$  such that  $A \in \mathcal{N} \setminus \mathcal{M}$  and  $B \in \mathcal{M} \setminus \mathcal{N}$ . Then, E does not belong to  $\sigma$ -ideal  $\mathcal{I} \otimes \mathcal{I}$ . On the other hand, if  $\alpha \in (0, 2\pi) \setminus \{k \cdot \frac{\pi}{2} : k \in \mathbb{Z}\}$  and  $\varphi_{\alpha}$  is the rotation of the plane by angle  $\alpha$ , then  $\varphi_{\alpha}(E)$  belongs to  $\mathcal{I} \otimes \mathcal{I}$ . We will prove this fact for  $\alpha = \frac{\pi}{4}$ .

Let  $E^* = \varphi_{\frac{\pi}{4}}(E).$ 

Then,

$$E^* = \left\{ \frac{\sqrt{2}}{2} \left\langle a + b, -a + b \right\rangle : a \in A \land b \in B \right\}$$

and for each  $x \in \mathbb{R}$ , we have

$$E_x^* = \left\{ y \colon \exists_{a \in A, b \in B} \frac{\sqrt{2}}{2} (a+b) = x \land \frac{\sqrt{2}}{2} (-a+b) = y \right\}$$
$$= \left\{ y \colon \frac{x-y}{\sqrt{2}} \in A \land \frac{x+y}{\sqrt{2}} \in B \right\}.$$

Thus,

$$E_x^* \subseteq \left(x - \sqrt{2}A\right) \cap \left(\sqrt{2}B - x\right) \in \mathcal{N} \cap \mathcal{M} = \mathcal{I}.$$

In [1], the authors showed that the Mendez  $\sigma$ -ideals  $\mathcal{N} \otimes \mathcal{M}$  and  $\mathcal{M} \otimes \mathcal{N}$  are not invariant under nonzero rotations of the plane. A natural question is if there exists a nontrivial  $\sigma$ -ideal  $\mathcal{I}$  of subsets of the real line ( $\mathcal{I} \neq \mathcal{N}$  and  $\mathcal{I} \neq \mathcal{M}$ ) such that  $\sigma$ -ideal  $\mathcal{I} \otimes \mathcal{I}$  is invariant with respect to rotations of  $\mathbb{R}^2$ .

**DEFINITION 2.** We say that A is upward- $\mathcal{K}$ -full if for each  $P \subset A^c$  if P is in  $\mathcal{K}$  then P is in  $\mathcal{I}$ .

Kirchheim and Natkaniec proved under A(m) that if  $S \subset [0, \pi)$  is a Sierpiński set (i.e., set S such that S has cardinality  $\mathfrak{c}$  and  $S \cap A$  has cardinality less than  $\mathfrak{c}$  for each  $A \subset \mathbb{R}$  with Lebesgue measure zero), then the class  $\bigcap_{\theta \in S} \mathcal{G}_2(\theta) \cap$  $\bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$  contains nonmeasurable sets. They also observed that if A(m)holds, then there exists  $D \subset [0,\pi)$  of full outer measure for which  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap$  $\bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$  contains nonmeasurable sets. We will prove a similar fact in ZFC.

**THEOREM 1.** There exists  $D \subset [0, \pi)$  which is upward- $\mathcal{K}$ -full and for which the class  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap \bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$  contains  $\mathcal{K}_2$ -nonmeasurable sets.

Proof. Let  $(B_{\alpha})_{\alpha < \mathfrak{c}}$  be an enumeration of all Borel subsets of  $\mathbb{R}^2$  which are not in  $\sigma$ -ideal  $\mathcal{I} \otimes \mathcal{I}$  and let  $(F_{\alpha})_{\alpha < \mathfrak{c}}$  be an enumeration of all perfect subsets of  $[0, \pi)$ .

By induction, we will choose the points  $x_{\alpha} \in \mathbb{R}^2$ ,  $t_{\alpha} \in \mathbb{R}$ ,  $\alpha < \mathfrak{c}$  such that

$$\begin{cases} t_{\alpha} \in F_{\alpha} \setminus \left\{ \operatorname{dir} \left( \ell(x_{\beta}, x_{\gamma}) \right) \colon \gamma < \beta < \alpha \right\}; \\ x_{\alpha} \in B_{\alpha} \setminus \left( \bigcup_{\gamma < \beta < \alpha} \ell(x_{\beta}, x_{\gamma}) \cup \bigcup_{\beta < \alpha} (x_{\beta} + C_{D_{\alpha}}) \right), \end{cases}$$

where  $D_{\alpha} = \{t_{\beta} : \beta \leq \alpha\}.$ 

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Assume that for some  $\alpha < \mathfrak{c}$  and for all  $\beta < \alpha$ , the points  $x_{\alpha}$  and  $t_{\alpha}$  have already been defined. Since  $F_{\alpha}$  has cardinality  $\mathfrak{c}$  and  $\{\operatorname{dir}(\ell(x_{\beta}, x_{\gamma})): \gamma < \beta < \alpha\}$  has cardinality less than  $\mathfrak{c}$ , the set  $F_{\alpha} \setminus \{\operatorname{dir}(\ell(x_{\beta}, x_{\gamma})): \gamma < \beta < \alpha\}$  has cardinality  $\mathfrak{c}$ . We select  $t_{\alpha}$  from this set and define  $D_{\alpha} = \{t_{\beta}: \beta \leq \alpha\}$ .

Let us choose  $\theta \in [0, \pi) \setminus D_{\alpha}$ . Since  $\mathcal{K}_2$  is invariant with respect to rotations of the plane, we may assume that  $\theta = \frac{\pi}{2}$ . Observe that there exists a line in the direction  $\theta$  such that the set  $\ell \cap B_{\alpha}$  is not in  $\sigma$ -ideal  $\mathcal{I}$  in  $\ell$  and  $x_{\beta} \notin \ell$ , for each  $\beta < \alpha$ . Indeed, since  $B_{\alpha}$  is a Borel set which is not in  $\mathcal{I} \otimes \mathcal{I}$ , hence from (3) and (a) it follows that  $\{x \in \mathbb{R} : (B_{\alpha})_x \notin \mathcal{I}\}$  is a Borel set which is not in  $\mathcal{I}$ . Consequently,  $\{x \in \mathbb{R} : (B_{\alpha})_x \notin \mathcal{I}\}$  has cardinality  $\mathfrak{c}$  and hence there exist  $\mathfrak{c}$  lines  $\ell \in L(\frac{\pi}{2})$  such that  $\ell \cap B_{\alpha}$  is not in  $\mathcal{I}$  in  $\ell$ . Finally, there exists a line  $\ell \in L(\frac{\pi}{2})$ such that  $x_{\beta} \notin \ell$ , for each  $\beta < \alpha$ , and  $\ell \cap B_{\alpha}$  is not in  $\mathcal{I}$  in  $\ell$ . One easily observes that  $\ell \cap B_{\alpha}$  has cardinality  $\mathfrak{c}$  and hence the set

$$\ell \cap B_{\alpha} \setminus \left( \bigcup_{\beta < \gamma < \alpha} \ell(x_{\beta}, x_{\gamma}) \cup \bigcup_{\beta < \alpha} (x_{\beta} + C_{D_{\alpha}}) \right)$$

is nonempty. We select  $x_{\alpha}$  from this set.

Finally, we define  $E = \{x_{\alpha} : \alpha < \mathfrak{c}\}$  and  $D = \bigcup_{\alpha < \mathfrak{c}} D_{\alpha}$ .

We will show that E is  $\mathcal{K}_2$ -nonmeasurable. Observe that E is not in  $\mathcal{I} \otimes \mathcal{I}$ . Assume by way of contradiction that E is in  $\mathcal{I} \otimes \mathcal{I}$ . Then there exists a Borel set  $B \in \mathcal{I} \otimes \mathcal{I}$  such that  $E \subset B$ . However,  $B^c$  is a Borel set which is not in  $\mathcal{I} \otimes \mathcal{I}$ and thus  $B^c = B_\alpha$  for some  $\alpha < \mathfrak{c}$ . Moreover,  $B_\alpha \subset E^c$  and consequently,  $B_\alpha \subset \mathbb{R}^2 \setminus \{x_\beta : \beta < \mathfrak{c}\}$ . This clearly contradicts the fact that  $x_\alpha$  was chosen from the set  $B_\alpha$ . Hence, E is not in  $\mathcal{I} \otimes \mathcal{I}$ . On the other hand, every line intersects Ein at most two points, thus  $E_x$  is in  $\mathcal{I}$ , for each  $x \in \mathbb{R}$ . Finally, E is in  $\mathcal{I} \otimes \mathcal{I}$  and from (a) it follows that E is  $\mathcal{K}_2$ -nonmeasurable.

Now, we show that  $E^c$  is in

$$\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap \bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta).$$

Suppose that

$$E^c \notin \bigcap_{\theta \in D} \mathcal{G}_2(\theta)$$

Then, there exist  $\theta \in D$  and  $\ell \in L(\theta)$  such that  $E \cap \ell$  contains at least two different points. Hence, there exist  $\beta < \alpha < \mathfrak{c}$  such that  $x_{\beta}, x_{\alpha} \in E \cap \ell$ . Since  $\theta \in D$ , there exists  $\gamma < \mathfrak{c}$  such that  $\theta = t_{\gamma} \in D_{\gamma}$ . However, since

$$\ell \subset (x_{\beta} + C_{D_{\gamma}})$$
 and  $x_{\alpha} \notin \bigcup_{\beta < \alpha} (x_{\beta} + C_{D_{\alpha}}),$ 

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we have  $\gamma > \alpha$ . On the other hand, we have chosen  $t_{\gamma}$  from the set

$$F_{\gamma} \setminus \left\{ \operatorname{dir} \left( \ell(x_{\alpha}, x_{\beta}) \right) \colon \beta < \alpha < \gamma \right\},$$

which contradicts the fact that  $x_{\alpha}, x_{\beta} \in \ell$ . It is easy to verify that  $E^c$  is in  $\bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$ .

In the end, we will show that D is upward- $\mathcal{K}$ -full. Assume, by way of contradiction, that there exists a  $\mathcal{K}$ -measurable set  $A \subset [0,\pi) \setminus D$  such that Ais not in  $\mathcal{I}$ . Then, there exists a Borel set  $B \subset A$  such that  $B \notin \mathcal{I}$ . In fact, B contains some perfect subset F. There exists  $\alpha < \mathfrak{c}$  such that  $F = F_{\alpha}$ . But then,  $F_{\alpha} \subset [0,\pi) \setminus D$ , which contradicts the fact that we have chosen  $t_{\alpha}$  from the set  $F_{\alpha}$ .

**COROLLARY 2.** There exists  $D \subset [0, \pi)$  of full outer Lebesgue measure in  $[0, \pi)$  such that the class  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap \bigcap_{\theta \in [0,\pi)} \mathcal{G}_3(\theta)$  contains a Lebesgue nonmeasurable set.

**COROLLARY 3.** There exists  $D \subset [0, \pi)$  which is of the second category at each point of  $[0, \pi)$  such that the class  $\bigcap_{\theta \in D} \mathcal{G}_2(\theta) \cap \bigcap_{\theta \in [0, \pi)} \mathcal{G}_3(\theta)$  contains sets without the Baire property.

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