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A HELLY THEOREM FOR FUNCTIONS WITH VALUES IN METRIC SPACES

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ABSTRACT. We present a Helly type theorem for sequences of functions with values in metric spaces and apply it to representations of some mappings on the space of continuous functions. A generalization of the Riesz theorem is formulated and proved. More concretely, a representation of certain majored linear operators on the space of continuous functions into a complete metric space.

1. Introduction

It is well known that the following theorem is true [BDS], [Na], [W].

Riesz representation theorem. *Every continuous linear functional L on the set of continuous functions f defined on $[0,1]$ has the form*

$$Lf = \int_0^1 f(s)\,dg(s) \tag{R}$$

with a function g of bounded variation on $[0,1]$.

This theorem has many extensions and generalizations with various proofs. One of the possible proofs is based on the Helly theorem [Na] and also on the moment problem theorem. It can be shown that the problem of determining the general continuous linear functionals on the set of continuous functions is equivalent to that of determining the set of all moment sequences. It is our purpose to extend this result for majored linear operators from continuous functions to Banach spaces.

Helly's theorem had been of some importance a long time above all in the probability theory in connection with a problem of moments of distributions. Recall that in this connection, real-valued nondecreasing functions f on the interval $[a,b]$ of the real line are considered and that the following facts are true.

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(1) (*First Helly's theorem*) Given a uniformly bounded sequence (f_n) of real-valued nondecreasing functions, there exists a subsequence (f_{n_k}) of (f_n) converging to a real-valued nondecreasing function f on $[a,b]$.

(2) (*Second Helly's theorem*) Given a sequence (f_n) of real-valued nondecreasing functions on $[a,b]$, converging to a real-valued nondecreasing function f, then, for every continuous function g on $[a,b]$, we have

$$\lim_{n\to\infty}\int_a^b g(t)\,df_n(t)=\int_a^b g(t)\,df(t).$$

More generally there are true the following facts [BDS], [Di], [Na], [W].

(1) (*First Helly's theorem*) Given the sequence (f_n) of complex-valued functions of uniformly bounded variation on $[a,b]$ such that for some $x_0\in[a,b]$ the sequence $\big(f_n(x_0)\big)$ is bounded then there exists a subsequence (f_{n_k}) of (f_n) converging to a some function f of bounded variation on $[a,b]$.

(2) (*Second Helly's theorem*) Given a sequence (f_n) of functions of uniformly bounded variation on $[a,b]$, converging to a some function f of bounded variation, then, for every continuous function g on $[a,b]$, we have

$$\lim_{n\to\infty}\int_a^b g(t)\,df_n(t)=\int_a^b g(t)\,df(t).$$

2. A Helly theorem in metric spaces

In this section we shall prove some facts concerning Helly's theorem in the context of metric spaces.

Definition. Let D be a subset of the real line, (X,d) be metric space and $h : D\to X$ be a function. If the set of all sums $\sum_{i=1}^{n} d\big(h(t_{i-1}),h(t_i)\big)$, where $(t_i)_{i=0}^{n}$ is an increasing sequence elements of D, is bounded then g is said to be a function of bounded variation on D. The corresponding least upper bound is a variation of function h on set D.

Proposition. *Let D be a dense subset of the interval $[a,b]$, (X,d) be a complete metric space and $h : D\to X$ be a function of bounded variation on D.*

(i) *For any $t\in(a,b]$ (resp. $t\in[a,b)$) there exists limit $h(t-0)$ (resp. $h(t+0)$)*

(ii) *Function $f\colon(a,b]\to X$ (resp. $g\colon[a,b)\to X$) defined by $f(t)=h(t-0)$ (resp. $g(t)=h(t+0)$) are functions of bounded variation on $(a,b]$ (resp. $[a,b)$).*

(iii) *$f(t-0) = g(t-0) = h(t-0)$ for all $t \in (a,b]$ and $f(t+0) = g(t+0) = h(t+0)$ for all $t \in [a,b)$.*

(iv) *Function f (resp. g) is continuous at the point $t \in (a,b)$ if and only if $h(t-0) = h(t+0)$.*

(v) *For all $t \in (a,b)$ except possibly countable set $h(t-0) = h(t+0)$.*

Proof. Let $(t_i)_{i=0}^{\infty}$ be an increasing (resp. decreasing) sequence of elements of D. Since (X,d) is a complete metric space, it is sufficient to show that $\big(h(t_i)\big)_{i=0}^{\infty}$ is a Cauchy sequence. To see this, note that series

$$\sum_{i=1}^{\infty} d\big(h(t_{i-1}), h(t_i)\big)$$

is convergent and

$$\begin{aligned} d\big(h(t_n), h(t_m)\big) &\le \sum_{i=n+1}^{m} d\big(h(t_{i-1}), h(t_i)\big) \\ &\le \sum_{i=n+1}^{\infty} d\big(h(t_{i-1}), h(t_i)\big) \qquad \text{for any} \quad n < m. \end{aligned}$$

It proves (i). Let $(t_i)_{i=0}^{n}$ be an increasing sequence of elements of $(a,b]$. Take $\varepsilon > 0$ and a sequence $(s_i)_{i=0}^{n}$ of D such that $s_0 < t_0 < s_1 < t_1 < \ldots < t_{n-1} < s_n < t_n$ and $d\big(f(t_i), h(s_i)\big) < \frac{\varepsilon}{2n}$. Then

$$\begin{aligned} \sum_{i=1}^{n} d\big(f(t_{i-1}), f(t_i)\big) &\le \sum_{i=1}^{n} \Big(d\big(f(t_{i-1}), h(s_{i-1})\big) \\ &\quad + d\big(h(s_{i-1}), h(s_i)\big) + d\big(h(s_i), f(t_i)\big) \Big) \\ &\le \sum_{i=1}^{n} d\big(h(s_{i-1}), h(s_i)\big) \\ &\quad + \sum_{i=1}^{n} d\big(f(t_{i-1}), h(s_{i-1})\big) + \sum_{i=1}^{n} d\big(f(t_i), h(s_i)\big) \\ &< \varepsilon + \sum_{i=1}^{n} d\big(h(s_{i-1}), h(s_i)\big). \end{aligned}$$

It means that f is of bounded variation. Bounded variation of g may be proved analogously. It proves (ii). Now we prove

$$f(t-0) = h(t-0) \qquad \text{for} \quad t \in (a,b].$$

Let $(t_i)_{i=0}^{\infty}$ be an increasing sequence of elements of $(a,b]$ with $\lim_{n\to\infty} t_n = t$. Take a sequence $(s_i)_{i=0}^{\infty}$ of D such that $s_0 < t_0 < s_1 < t_1 < \ldots < t_{n-1} < s_n <$

$t_n < \dots$ and $d\big(f(t_i), h(s_i)\big) < \frac{1}{n}$. Clearly

$$f(t-0) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} h(s_n) = h(t-0).$$

Equalities

$$g(t-0) = h(t-0) \qquad \text{for} \quad t \in (a,b],$$
$$f(t+0) = h(t+0)$$

and

$$g(t+0) = h(t+0) \qquad \text{for} \quad t \in [a,b)$$

may be proved analogously. It proves (iii). Part (iv) follows from (iii). Let M be a variation of g on D and $n > 0$ be a fixed. Assume that there are m points $t_1, \dots t_m$, where $m > nM$, such that inequality

$$d\big(f(t_i), h(t_i+0)\big) > \frac{1}{n}$$

is satisfied for all $i = 1, \dots, m$. There is $\varepsilon > 0$ such that

$$d\big(f(t_i), h(t_i+0)\big) > 2\varepsilon + \frac{1}{n} \qquad \text{for all} \quad i = 1, \dots, m.$$

We may assume that $(t_i)_{i=1}^m$ is an increasing sequence. There are sequences $(s_i)_{i=1}^m$ and $(u_i)_{i=1}^m$ in D such that $s_1 < t_1 < u_1 < s_2 < t_2 < u_2 < \dots, t_{m-1} < u_{m-1} < s_m < t_m < u_m$,

$$d\big(f(t_i), h(s_i)\big) < \varepsilon \quad \text{and} \quad d\big(h(t_i+0), h(u_i)\big) < \varepsilon \qquad \text{for all} \quad i = 1, \dots, m.$$

Now

$$\sum_{i=1}^m d\big(h(s_i), h(u_i)\big) > \sum_{i=1}^m \Big(d\big(f(t_i), h(t_i+0)\big) - 2\varepsilon\Big) > \frac{m}{n} > M$$

what is a contradiction. Therefore inequality

$$d\big(h(t-0), h(t+0)\big) > 0$$

may be satisfied only for countably many $t \in (a,b)$. □

Helly theorem. *Let (X,d) be a complete metric space and $(g_n)_{n\in N}$ a sequence of functions from $[a,b]$ into X such that*

a) *the set $g_n(x)$ is relatively compact for any $x \in [a,b]$,*

b) *the functions $(g_n)_{n\in N}$ have uniformly bounded variations.*

Then there exists a subsequence of the sequence $(g_n)_{n\in N}$ converging pointwise in X to a function $g\colon [a,b] \to X$ of bounded variation.

Proof. Let D be a countable set dense in $[a,b]$ and $b \in D$. By the diagonal procedure we obtain a subsequence $(h_n)_{n\in N}$ of $(g_n)_{n\in N}$ converging for every

$t \in D$ to a function $h(t)$ which has bounded variation on D. Hence for every $t \in [a,b)$ there exists the limit $h(t+0)$. Put

$$g(t) = h(t+0) \qquad \text{for} \quad t \in [a,b)$$

and

$$g(b) = h(b).$$

Let $\delta > 0$ be given. We prove that

$$\limsup_{n\to\infty} d\big(h_n(t), g(t)\big) \le \delta$$

for every $t \in [a,b)$, except possible, finite number $m \le 2M/\delta$, where M is the upper bound of variations of all g_n. So, assume that for some $m > 2M/\delta$ there are points $t_1 < t_2 < \ldots < t_m < b$ such that

$$\limsup_{n\to\infty} d\big(h_n(t_i), g(t_i)\big) > \delta \qquad \text{for all} \quad i = 1,2,\ldots,m.$$

There is a subsequence $(h_{n_k})_{k=1}^{\infty}$ such that

$$d\big(h_{n_k}(t_i), g(t_i)\big) > \delta \qquad \text{for all} \quad i = 1,2,\ldots,m \ \text{ and } \ k = 1,2,\ldots$$

Since $g(t_i) = h(t_i+0)$, there is a sequence $(u_i)_{i=1}^{m}$ in D such that

$$d\big(g(t_i), h(u_i)\big) < \frac{\delta}{4} \qquad \text{for all} \quad i = 1,2,\ldots,m$$

and $t_1 < u_1 < t_2 < u_2 < \ldots, t_{m-1} < u_{m-1} < t_m < u_m$. Let k be so large that

$$d\big(h_{n_k}(u_i), h(u_i)\big) \le \delta/4, \qquad \text{for all} \quad i = 1,\ldots,m\,.$$

Then

$$\begin{aligned}
&\sum_{i=1}^{m} d\big(h_{n_k}(t_i), h_{n_k}(u_i)\big)\\
&\geqq \sum_{i=1}^{m}\Big(d\big(h_{n_k}(t_i), g(t_i)\big) - d\big(g(t_i), h(u_i)\big) - d\big(h(u_i), h_{n_k}(u_i)\big)\Big)\\
&> \sum_{i=1}^{m}\left(\delta - \frac{\delta}{4} - \frac{\delta}{4}\right)\\
&= \frac{m\delta}{2} > M\,.
\end{aligned}$$

which is not possible. So,

$$\limsup_{n\to\infty} d\big(h_n(t), g(t)\big) = 0$$

for every $t \in [a,b)$, except possible, countable set $A \subset [a,b)$. The last application of the diagonal procedure gives sequence convergent to $\widetilde{g}(t)$ on A. For some $t \in A$ it may be $\widetilde{g}(t) \ne g(t)$. Therefore g has to be redefined on A by $g(t) = \widetilde{g}(t)$. $\square$

Corollary of Helly theorem. *Let X be a complete linear metric space and $(g_n)_{n\in N}$ a sequence of functions from $[a,b]$ into X such that*

a) *the set $\{g_n(x)\}$ is relatively compact for any $x \in [a,b]$,*

b) *the functions $(g_n)_{n\in N}$ have uniformly bounded variations.*

Then there exists a subsequence of the sequence $(g_n)_{n\in N}$ converging pointwise in X to a function $g\colon [a,b] \to X$ of bounded variation.

3. Integral representation theorem for majored linear operator

It is well known that every continuous linear form on the space of $C(I)$, $I = [0,1]$, is presentable in the form

$$f \to \int_0^1 f(t)dq(t), \tag{1}$$

where q is a function of bounded variation.

From the preceding generalization of Helly theorem we obtain the result giving the representation of majored linear mapping [Di]. First we recall the definition of majored linear operator.

For each subset A of $[a,b]$, let $C\big([a,b],A\big)$ denote the space of continuous functions on $[a,b]$ vanishing outside A. Let X be a Banach space. If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum_i ||F(f_i)||,$$

where the supremum is over all finite families $\{f_i\}$ in $C\big([a,b],A\big)$ with $\sum_i |f_i| \le \chi_A(t)$ for all t in $[a,b]$. If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, then (see [Di, §19, pp. 380, 383]), the mapping F is called majored (also dominated, [Di]), if

$$|||F_A||| < \infty \qquad \text{for all} \quad A \text{ in } \mathcal{B}\big([a,b]\big). \tag{M}$$

4. Riesz type theorem in some linear metric spaces

Let on the interval $[a,b]$ a function g of bounded variation with values in Banach X be given. This function may it possible to every continuous function

$f(x)$ on $[a, b]$ to associate the element of X of the form

$$F(f) = \int_a^b f(x)dg(x). \qquad (!)$$

The following properties are true. Let $f_1, f_2, f \in C[a, b]$. Then

$$F(f_1 + f_2) = F(f_1) + F(f_2), \qquad (a)$$

$$\|F(f)\| \le V_a^b M(f),\ M(f) = \max_{x\in[a,b]} |f(x)|,\ V_a^b = \mathrm{Var}_a^b(g). \qquad (b)$$

Theorem (F. Riesz). *Let on the set $C[a, b]$ the majored and compact linear mapping $F(f)$ with values in a Banach space X be given. Then there exists a function $g(x)$ of bounded variation with values X such that for every function $f(x) \in C[a, b]$ we have*

$$F(f) = \int_a^b f(x)\, dg(x). \qquad (2)$$

P r o o f. It is enough to consider the case $a = 1, b = 1$, because the general case can be reduced to this case by means of a linear transformation of argument. Put

$$\varphi_{n,k} = \binom{n}{k} x^k (1 - x)^{n-k}.$$

It is easy to see that for every x it is true

$$\sum_{k=0}^{n} \varphi_{n,k}(x) = 1.$$

Moreover, for $x \in [0, 1]$ every member of this sum in nonnegative. Hence, if

$$a_k = \pm 1,\ k = 0, 1, \ldots, n\,,$$

then

$$\left| \sum_{k=0}^{n} a_k \varphi_{n,k} \right| \le 1. \qquad (3)$$

Note that the considered linear operator $F(f)$ is defined for continuous on $[0, 1]$ functions $f(x)$. According to definition of compact linear operator there exists a compact subset C of X such that

$$F(f) \in M(f)C,$$

or, equivalently, mapping the unit sphere in $C[0, 1]$ into C. From this and (3) we obtain

$$\sum_{k=0}^{n} a_k F(\varphi_{n,k}) \in C.$$

□

Further from the majority of operator F we obtain

$$\sum_{k=0}^{n} \|F[\varphi_{n,k}]\| \leq \sup \sum_{i} \|F(f_i)\| \leq \||F_A\|| < \infty, \tag{4}$$

where the supremum is over all finite families $f_i \in C\big([a,b], A\big)$ with $\sum |f_i(t)| \leq \chi_A(t)$ for all $t \in [a,b]$, for all $A \in \mathcal{B}([a,b])$.

Let us define the step function $g_n(x)$ to put

$$\begin{aligned}
&g_n(0) = 0, \\
&g_n(x) = F[\varphi_{n,0}] && \left(0 < x < \frac{1}{n}\right), \\
&g_n(x) = F[\varphi_{n,0} + \varphi_{n,1}] && \left(\frac{1}{n} \leq x < \frac{2}{n}\right), \\
&\quad \cdots\cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots\cdots \\
&g_n(x) = F\left[\sum_{k=0}^{n-1} \varphi_{n,k}\right] && \left(\frac{n-1}{n} \leq x < 1\right), \\
&g_n(1) = F\left[\sum_{k=0}^{n} \varphi_{n,k}\right].
\end{aligned}$$

By (4) the functions $g_n(x)$ and their total variations are bounded with the one number. Moreover, because by assumption $F(f)$ is compact operator on $C[0,1]$ into X, it takes unit sphere in $C[0,1]$ into a compact set C into X. Hence the set $\{g_n(x)\}, n = 1, 2, \ldots$ is contained in C for all $x \in [0,1]$.

Hence on the base of Helly principle of choice from the sequence $\big\{g_n(x)\big\}$ it is possible to choose the subsequence $\{g_{n_i}(x)\}$ which converges in each point of $[0,1]$ to a function of the bounded variation.

If $f(x)$ is a continuous function on $[0,1]$, then on the base of [Na, Th. 3] it can be shown that

$$\int_0^1 f(x)\, dg_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) F(\varphi_{n,k}),$$

from where

$$\int_0^1 f(x)\, dg_n(x) = F\big[B_n(x)\big],$$

where

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$$

is the Bernstein polynom for the function $f(x)$.

By the theorem of S. N. B e r n s t e i n [Na, §5, Ch. IV]

$$M(B_n - f) \to 0, \qquad n \to \infty,$$

and by the definition of continuous linear operator we have

$$\|F(B_n) - F(f)\| = \|F(B_n - f)\| \le KM(B_n - f).$$

This means that

$$F(B_n) \to F(f),$$

from where

$$\lim_{n\to\infty} \int_0^1 f(x)\, dg_n(x) = F(f).$$

But if $n \to \infty$, going through values $n_1, n_2, \dots$, then by Helly theorem (cf. also [Na, §7] and [DD1]), we obtain

$$\lim_{n\to\infty} \int_0^1 f(x)\, dg_n(x) = \int_0^1 f(x)\, dg(x).$$

Remark. We have derived here a representation theorem for majored operators using Helly theorem in linear metric spaces. To prove it by means of the Helly theorem from this paper we were able to do it for compact majored mappings. On the other hand, our approach is more "constructive". In [Di] this is done for Banach spaces without such an assumption.

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