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ON THE LUKASIEWICZ PROBABILITY THEORY ON IF-SETS

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ABSTRACT. A review of main methods of the probability theory on IF-events is presented in the case that the used connectives are Lukasiewicz

$$f \oplus g = (f + g) \wedge 1\,,$$
$$f \odot g = (f + g - 1) \vee 0\,,$$

(f, g are functions, $f, g : \Omega \to \langle 0, 1\rangle$). Representation theorem for probabilities on IF-events is given. For sequences of independent observables the central limit theorem is presented as well as basic results about conditional expectation. Finally the Lukasiewicz probability theory to the MV-algebra probability theory is embedded.

1. Lukasiewicz probability

Similarly as in the Kolmogorov probability theory we start with a measurable space $(\Omega, \mathcal{S})$, where Ω is a non-empty set and $\mathcal{S}$ is a σ-algebra of subsets of Ω (i.e., $\Omega \in \mathcal{S}; A \in \mathcal{S} \Rightarrow \Omega \setminus A \in \mathcal{S}; A_n \in \mathcal{S}(n = 1, 2, \ldots) \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$), Atanassov theory [1] will be work with the following IF-events, i.e., each pairs

$$A = (\mu_A, \nu_A)\,,$$

such that

$$\mu_A, \nu_A : \Omega \to \langle 0, 1\rangle\,, \quad \mu_A + \nu_A \le 1\,,$$

and μ_A, ν_A are measurable, i.e.,

$$I \subset R \text{ is an interval } \Rightarrow \ \mu_A^{-1}(I) \in \mathcal{S},\ \nu_A^{-1}(I) \in \mathcal{S}\,,$$

μ_A is called the membership function, ν_A is non membership function it is in a connective with the partial ordering.

We shall use the Lukasiewicz connectives:

$$\text{if } A = (\mu_A, \nu_A),\ B = (\mu_B, \nu_B),\ \text{then}$$

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$$\begin{aligned} A \oplus B &= ((\mu_A + \mu_B) \wedge 1,\ (\nu_A + \nu_B - 1) \vee 0)\,, \\ A \odot B &= ((\mu_A + \mu_B - 1) \vee 0,\ (\nu_A + \nu_B) \wedge 1)\,, \end{aligned}$$

a partial ordering in $\mathcal{F}$ is given by the

$$A \le B \ \Leftrightarrow\ \mu_A \le \mu_B,\ \nu_A \ge \nu_B\,.$$

Evidently,

$$\begin{aligned} (0_\Omega, 1_\Omega) &\ \text{ is the least element of } (\mathcal{F}, \le)\,, \\ (1_\Omega, 0_\Omega) &\ \text{ is the greatest element of } (\mathcal{F}, \le)\,. \end{aligned}$$

Denote by $\mathcal{F}$ the family of all IF-events. Probability is considered as a mapping

$$\mathcal{P} : \mathcal{F} \to \mathcal{J}\,,$$

(where $\mathcal{J} = \{\langle a, b\rangle\,;\ a, b \in R,\ a \le b\}$) satisfying the following conditions:

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = [1,1]$, $\mathcal{P}((0_\Omega, 1_\Omega)) = [0,0]$,

(ii) $A \odot B = (0_\Omega, 1_\Omega) \Rightarrow \mathcal{P}(A \oplus B) = \mathcal{P}(A) \oplus \mathcal{P}(B)$,

(iii) $A_n \nearrow A \Rightarrow \mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

Of course, $A_n \diagup A$ means (with respect to the ordering) that

$$\mu_{A_n} \nearrow \mu_A\,, \quad \nu_{A_n} \searrow \nu_A\,.$$

On the other hand, $\langle a_n, b_n\rangle \nearrow \langle a, b\rangle$ means that $a_n \nearrow a$, $b_n \nearrow b$.

Of course $\mathcal{P}(A)$ is a compact interval on R, denote it by

$$\mathcal{P}(A) = \left\langle \mathcal{P}^\flat(A), \mathcal{P}^\sharp(A) \right\rangle\,.$$

It is easy to see that the main results can be described by the mappings $A \mapsto \mathcal{P}^\flat(A)$, $A \mapsto \mathcal{P}^\sharp(A)$. We use the terminology from the quantum theory [3].

Definition 1.1. A mapping $m : \mathcal{F} \to \langle 0, 1\rangle$ is called a state if the following properties are satisfied:

(i) $m\big((1_\Omega, 0_\Omega)\big) = 1$, $m\big((0_\Omega, 1_\Omega)\big) = 0$,

(ii) $A \odot B = (0_\Omega, 1_\Omega) \Rightarrow m\big((A \oplus B)\big) = m(A) + m(B)$,

(iii) $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$.

It is easy to see that the following property holds.

Proposition 1.1. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be defined by $\mathcal{P}(A) = \langle \mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)\rangle$. Then $\mathcal{P}$ is a probability if and only if $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to \langle 0, 1\rangle$ are states.*

Hence, in the paper we shall be interested in states. In Section 2 we present a representation of states by integrals given by Kolmogorov probability measures $p : \mathcal{S} \to \langle 0, 1\rangle$. Evidently it gives also a representation theorem for probabilities on $\mathcal{F}$. As an example of the theory we present in Section 3 the central limit theorem. In Section 4 we present a way for to work with conditional property notions. Finally in Section 5 we embed our theory to the MV-algebra probability theory, hence we show that our theory is in the strong connectives with good developped probability theory on MV-algebras.

2. Representation

Theorem 2.1. *For any state $m : \mathcal{F} \to \langle 0, 1\rangle$ there exist probability measures $P, Q : \mathcal{S} \to \langle 0, 1\rangle$ and $\alpha \in \langle 0, 1\rangle$ such that*

$$m\big((\mu_A, \nu_A)\big) = \int_\Omega \mu_A dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) dQ\right).$$

P r o o f. The main instrument in our investigation is the following implication, a corrolary of (ii):

$$f, g \in \mathcal{F}, \quad f + g \le 1 \implies m(f, g) = m(f, 1 - f) + m(0, f + g) . \tag{1}$$

We shall define a mapping $P : \mathcal{S} \to \langle 0, 1\rangle$ by the formula $P(A) = m(\chi_A, 1 - \chi_A)$. Let $A, B \in \mathcal{S}$, $A \cap B = \varnothing$. Then $\chi_A + \chi_B \le 1$ hence $(\chi_A, 1 - \chi_A) \odot (\chi_B, 1 - \chi_B) = \big((\chi_A + \chi_B - 1) \vee 0, (1 - \chi_A + 1 - \chi_B) \wedge 1\big) = (0, 1)$. We obtain

$$\begin{aligned}
&P(A) + P(B) \\
&= m(\chi_A, 1 - \chi_A) + m(\chi_B, 1 - \chi_B) \\
&= m\big((\chi_A, 1 - \chi_A) \oplus (\chi_B, 1 - \chi_B)\big) \\
&= m\big((\chi_A + \chi_B) \wedge 1,\ (1 - \chi_A + 1 - \chi_B - 1) \vee 0\big) \\
&= m(\chi_A + \chi_B, 1 - \chi_A - \chi_B) \\
&= m(\chi_{A \cup B}, 1 - \chi_{A \cup B}) = P(A \cup B) ,
\end{aligned}$$

hence P is additive.

Let $A_n \in \mathcal{S}$ $(n = 1, 2, \ldots)$, $A_n \nearrow A$. Then

$$\chi_{A_n} \nearrow \chi_A, \ 1 - \chi_{A_n} \searrow 1 - \chi_A ,$$

hence by (iii)

$$P(A_n) = m(\chi_{A_n}, 1 - \chi_{A_n}) \nearrow m(\chi_A, 1 - \chi_A) = P(A) .$$

Evidently, $P(\Omega) = m(\chi_\Omega, 1 - \chi_\Omega) = m(\langle 1, 0\rangle) = 1$, hence $P : \mathcal{S} \to \langle 1, 0\rangle$ is a probability measure.

Now we prove two identities.

$$\begin{aligned} A_1,\dots,A_n\in\mathcal{S},\quad \alpha_i\in\langle 0,1\rangle \quad (i=1,2,\dots,n)\,,\\ A_i\cap A_j=\varnothing \quad (i\neq j) \ \Rightarrow\ m\left(\textstyle\sum_{i=1}^n \alpha_i\chi_{A_i}, 1-\sum_{i=1}^n \alpha_i\chi_{A_i}\right)\\ = \ \textstyle\sum_{i=1}^n m\left(\alpha_i\chi_{A_i}, 1-\alpha_i\chi_{A_i}\right)\,. \end{aligned} \tag{2}$$

The implication (2) can be proved by induction by the help of (ii).

$$m(\alpha\beta\chi_A, 1-\alpha\beta\chi_A)=\alpha\, m(\beta\chi_A, 1-\beta\chi_A)\,. \tag{3}$$

The identity (3) will be proved first for $n\in N$ such that $n\chi_A\le 1$ by induction. If $p,q\in N,\ p\le q$, then

$$m\left(q\left(\frac{1}{q}\beta\chi_A\right),\ 1-q\left(\frac{1}{q}\beta\chi_A\right)\right)=qm\left(\frac{1}{q}\beta\chi_A,\ 1-\frac{1}{q}\beta\chi_A\right),$$

$$m\left(\frac{1}{q}\beta\chi_A,\ 1-\frac{1}{q}\beta\chi_A\right)=\frac{1}{q}m\big(\beta\chi_A,\ 1-\beta\chi_A\big),$$

$$m\left(\frac{p}{q}\beta\chi_A,\ 1-\frac{p}{q}\beta\chi_A\right)=\frac{p}{q}m\big(\beta\chi_A,\ 1-\beta\chi_A\big),$$

hence (3) holds for rational $\alpha\in\langle 0,1\rangle$. Let $\alpha\in R$, $\alpha\in\langle 0,1\rangle$. Take $\alpha_n\in Q$ such that $\alpha_n\nearrow\alpha$. Then

$$\alpha_n\chi_{A_n}\nearrow\alpha\chi_A,\quad 1-\alpha_n\chi_{A_n}\searrow 1-\alpha\chi_A\,.$$

Therefore,

$$\begin{aligned} m(\alpha\beta\chi_A, 1-\alpha\beta\chi_A) &= \lim_{n\to\infty}(\alpha_n\beta\chi_A, 1-\alpha_n\beta\chi_A)\\ &= \lim_{n\to\infty}\alpha_n m(\beta\chi_A, 1-\beta\chi_A)=\alpha m(\beta\chi_A, 1-\beta\chi_A) \end{aligned}$$

hence, (3) is proved, too. Particulary, if we give $\beta=1$, then

$$m(\alpha\chi_A, 1-\alpha\chi_A)=\alpha m(\chi_A, 1-\chi_A)\,.$$

Let $f:\Omega\to\langle 0,1\rangle$ be simple $\mathcal{S}$ measurable i.e.,

$$f=\sum_{i=1}^n\alpha_i\chi_{A_i}\,,\quad A_i\in\mathcal{S}\ \ (i=1,2,\dots,n)\,,\quad A_i\cap A_j=\emptyset\ \ (i\neq j)\,.$$

Combining (2), (3), and the definition of P we obtain

$$\begin{aligned} m(f,1-f) &= \sum_{i=1}^n m(\alpha_i\chi_{A_i}, 1-\alpha_i\chi_{A_i})\\ &= \sum_{i=1}^n \alpha_i m(\chi_{A_i}, 1-\chi_{A_i})\\ &= \sum_{i=1}^n \alpha_i P(A_i)=\int_\Omega f\,dP\,, \end{aligned}$$

hence,

$$m(f, 1-f) \quad = \quad \int_\Omega f\, dP\,,$$

for any $f : \Omega \to \langle 0, 1\rangle$ simple.
If $f : \Omega \to \langle 0, 1\rangle$ is an arbitrary $\mathcal{S}$-measurable function, then there exists a sequence (f_n) of simple measurable functions such that $f_n \nearrow f$. Evidently, $1 - f_n \searrow 1 - f$. Therefore

$$m(f, 1-f) = \lim_{n\to\infty} m(f_n, 1-f_n) = \lim_{n\to\infty} \int_\Omega f_n\, dP = \int_\Omega f\, dP\,,$$

hence

$$m(f, 1-f) = \int_\Omega f\, dP\,,$$

for any measurable $f : \Omega \to \langle 0, 1\rangle$.

Now take our attention to the second term $m(0, f+g)$ in the right side of equality mentioned in (1). First define first $M : \Omega \to \langle 0, 1\rangle$ by the formula

$$M(A) = m(0, 1-\chi_A)\,.$$

As before, it is possible to prove that M is a measure. Of course,

$$M(\Omega) = m(0, 0) = \alpha \in \langle 0, 1\rangle\ .$$

Define $Q : \mathcal{S} \to \langle 0, 1\rangle$ by the formulas

$$\begin{aligned} Q(A) = \frac{1}{\alpha} M(A) \quad &= \quad \frac{1}{\alpha} m(0, 1-\chi_A)\,, \\ m(0, 1-\chi_A) \quad &= \quad \alpha Q(A)\,. \end{aligned}$$

As before, it is possible to prove

$$m(0, 1-f) \ = \ \alpha \int_\Omega f\, dQ\,,$$

for any $f : \Omega \to \langle 0, 1\rangle$ measurable, or

$$m(0, h) \ = \ \alpha \int_\Omega (1-h)\, dQ\,, \tag{4}$$

$h : \Omega \to \langle 0, 1\rangle$, $\mathcal{S}$ – measurable. Combining (1), (2), and (4) we obtain

$$\begin{aligned} m(A) &= m((\mu_A, \nu_A)) \\ &= m((\mu_A, 1-\mu_A)) + m((0, \mu_A + \nu_A)) \\ &= \int_\Omega \mu_A \, dP + \alpha \int_\Omega (1 - \mu_A - \nu_A) \, dQ \\ &= \int_\Omega \mu_A \, dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) \, dQ \right) . \end{aligned}$$

□

Corollary 2.1. ([12]). *If*

$$m(A) = f\left(\int_\Omega \mu_A \, dP, \int_\Omega \nu_A \, dP \right)$$

for some $f : \Omega \to R$ and $P : \mathcal{S} \to \langle 0, 1\rangle$, then $P = Q$, hence there exists $\alpha \in \langle 0, 1\rangle$ such that

$$\begin{aligned} m(A) &= m((\mu_A, \nu_A)) \\ &= \int_\Omega \mu_A \, dP + \alpha - \alpha \int_\Omega \mu_A \, dP - \alpha \int_\Omega \nu_A \, dP \\ &= (1-\alpha) \int_\Omega \mu_A \, dP + \alpha \left(1 - \int_\Omega \nu_A \, dP \right) . \end{aligned}$$

It was presented in [12].

Proposition 2.1. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be a probability measure. Then there exist the Kolmogorov probability measures $P, Q, R, S : \mathcal{S} \to \langle 0, 1\rangle$ and constants $\alpha, \beta \in \langle 0, 1\rangle$ such that*

$$\begin{aligned} &\mathcal{P}(\langle \mu_A, \nu_A \rangle) \\ &= \left\langle \int_\Omega \mu_A \, dP + \alpha \left(1 - \int_\Omega (\mu_A + \nu_A) \, dQ \right) , \right. \\ &\qquad \left. \int_\Omega \mu_A \, dR + \beta \left(1 - \int_\Omega (\mu_A + \nu_A) \, dS \right) \right\rangle . \end{aligned}$$

P r o o f. A consequence of Theorem 2.1. □

Example 2.1. ([7]). Define $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ by the formula

$$\mathcal{P}(A) = \left\langle \int_\Omega \mu_A \, dP, \; 1 - \int_\Omega \nu_A \, dP \right\rangle .$$

Then the mapping is a probability measure according to Proposition 2.1; it suffices to put $\alpha = 0$, $\beta = 1$. The example was given in [7].

Example 2.2. ([6]). Put $\alpha = \frac{1}{2} = \beta$, $P = Q = R = S$ in Proposition 2.1. Then

$$\begin{aligned} &\mathcal{P}(\langle \mu_A, \nu_A \rangle) \\ &= \left\langle \frac{1}{2} \int_\Omega (\mu_A + 1 - \nu_A) \, dP, \; \frac{1}{2} \int_\Omega (\mu_A + 1 - \nu_A) \, dP \right\rangle \\ &= \left\{ \frac{1}{2} \int_\Omega (\mu_A + 1 - \nu_A) \, dP \right\}. \end{aligned}$$

The definition was published in [6].

3. Central limit theorem

If we consider a sequence of independent measurements $\xi_1, \xi_2, \xi_3, \ldots$, then for sufficiently large n the arithmetic mean

$$\frac{1}{n} \sum_{i=1}^{n} \xi_i$$

has approximately normal distribution. We want to translate the assertion from the classical Kolmogorovian case to the IF-events probability theory.

3.1. Observable

In the Kolmogorov case a probability space $(\Omega, \mathcal{S}, P)$ is given. By a random variable an $\mathcal{S}$-measurable mapping is considered

$$\xi : \Omega \to R \,,$$

i. e., $I \subset R$ is an interval $\Longrightarrow \xi^{-1}(I) \in \mathcal{S}$.

To any random variable its distribution function

$$F : R \to \langle 0, 1 \rangle$$

is defined by the formula

$$F(u) = P\left(\xi^{-1}(-\infty, u)\right) .$$

The most frequently types of distributions are discrete and continuous. In the first case ξ has values

$$\alpha_1, \alpha_2, \alpha_3, \dots$$

with probabilities

$$p_1, p_2, p_3, \dots$$

$$\Big(p_i = P(\{\omega; \xi(\omega) = \alpha_i\}) = P\left(\xi^{-1}(\{\alpha_i\})\right)\Big)\,,$$

hence

$$F(u) = \sum_{\alpha_i < u} p_i\,.$$

In the case the mean value $E(\xi)$ can be expressed by the formula

$$E(\xi) = \sum_i \alpha_i p_i\,.$$

In the continuous case

$$F(u) = \int_{-\infty}^{u} f(v)\,dv$$

and

$$E(\xi) = \int_{-\infty}^{\infty} x f(x)\,dx\,.$$

Sometimes it is convenient in both cases to use the notion (Stieltjes integral)

$$\begin{aligned} E(\xi) &= \int_{-\infty}^{\infty} x\,dF(x), \\ &= \begin{cases} \sum x_i p_i, & \text{in the discrete case,} \\ \int_{-\infty}^{\infty} x f(x)\,dx, & \text{in the continuous case.} \end{cases} \end{aligned}$$

In our case, instead of random variables $\xi : \Omega \to R$, we consider mappings $x : \sigma(\mathcal{J}) \to \mathcal{F}$ we call it observables (the terminology in taken from the quantum structures).

Definition 3.1. An observable is a mapping

$$x : \sigma(\mathcal{J}) \to \mathcal{F}$$

satisfying the following conditions

(i) $x(R) = (1, 0)$, $x(\varnothing) = (0, 1)$,

(ii) $A \cap B = \varnothing \;\Rightarrow\; x(A) \odot x(B) = (0, 1)\,,\; x(A \cup B) = x(A) \oplus x(B)$,

(iii) $A_n \nearrow A \;\Rightarrow\; x(A_n) \nearrow x(A)$.

Proposition 3.1. *If $x : \sigma(\mathcal{J}) \to \mathcal{F}$ is an observable, and $m : \mathcal{F} \to \langle 0, 1\rangle$ is a state, then*

$$m_x = m \circ x : \sigma(\mathcal{J}) \to \langle 0, 1\rangle,$$

defined by

$$m_x(A) = m(x(A))$$

is a probability measure.

Proof. First

$$m_x(R) = m((x(R)) = m(1, 0) = 1\,.$$

If $A \cap B = \varnothing$, then $x(A) \odot x(B) = (0, 1)$, hence

$$\begin{aligned} m_x(A \cup B) &= m(x(A \cup B)) \\ &= m(x(A) \oplus x(B)) \\ &= m(x(A)) + m(x(B)) \\ &= m_x(A) + m_x(B)\,. \end{aligned}$$

Finally, $A_n \nearrow A$ implies $x(A_n) \nearrow x(A)$ hence

$$m_x(A_n) = m(x(A_n)) \nearrow m(x(A)) = m_x(A)\,.$$

□

Proposition 3.2. *Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable, $m : \mathcal{F} \to \langle 0, 1\rangle$ be a state. Define $F : R \to \langle 0, 1\rangle$ by the formula*

$$F(u) = m(x((-\infty, u)))\,.$$

Then F is non-decreasing, left continuous in any point $u \in R$,

$$\lim_{u\to\infty} F(u) = 1\,, \qquad \lim_{u\to-\infty} F(u) = 0\,.$$

Proof. If $u < v$, then

$$\begin{aligned} x((-\infty, v)) &= x((-\infty, u)) \oplus x((u, v)) \\ &\geq x((-\infty, u)), \end{aligned}$$

hence

$$F(v) = m((-\infty, u)) \geq m(x((-\infty, u))) = F(u)\,,$$

F is non decreasing. If $u_n \nearrow u$, then

$$x((-\infty, u_n)) \nearrow x((-\infty, u))\,,$$

hence

$$F(u_n) = m(x((-\infty, u_n))) \nearrow m(x((-\infty, u))) = F(u)\,,$$

F is left continuous in any $u \in R$. Similarly, $u_n \nearrow \infty$ implies

$$x((-\infty, u_n)) \nearrow x((-\infty, \infty)) = (1, 0)\,.$$

Therefore,

$$F(u_n) = m(x((-\infty, u_n))) \nearrow m((1,0))) = 1$$

for every $u_n \nearrow \infty$, hence $\lim_{u\to\infty} F(u) = 1$. Similarly we obtain

$$u_n \searrow -\infty \quad \text{implies} \quad -u_n \nearrow \infty,$$

hence

$$m(x((u_n, -u_n))) \nearrow m(x(R)) = 1\,.$$

$$\begin{aligned} 1 &= \lim_{n\to\infty} F(-u_n) \\ &= \lim_{n\to\infty} m(x((u_n, -u_n))) + \lim_{n\to\infty} F(u_n) \\ &= 1 + \lim_{n\to\infty} F(u_n)\,, \end{aligned}$$

hence $\lim_{n\to\infty} F(u_n) = 0$ for any $u_n \searrow -\infty$. □

Remark 3.1. It is very well known that to any distribution function $F : R \to \langle 0,1\rangle$ there exists exactly one probability measure $\lambda_F : \sigma(\mathcal{J}) \to \langle 0,1\rangle$ such that

$$\lambda_F(\langle a,b)) = F(b) - F(a) \quad \text{for any} \quad a,b \in R, a \le b.$$

Of course, our probability measure m_x from Proposition 3.1 has the property:

$$\begin{aligned} F(b) &= m(x((-\infty,b))) = m_x((-\infty,b)) \\ &= m_x((-\infty,a)) + m_x(\langle a,b)) \\ &= F(a) + m_x(\langle a,b))\,, \end{aligned}$$

hence

$$m_x(\langle a,b)) = F(b) - F(a)\,.$$

We have obtained two possibilities for the obtain the same notion. The way by the help of distribution function is very useful from the point of view of applications. Analogously with the classical case the notion of mean value can be defined.

Definition 3.2. An observable $x : \sigma(\mathcal{J}) \to \mathcal{F}$ is integrable if there exists

$$E(x) = \int_{-\infty}^{\infty} u\,dF(u) = \int_{-\infty}^{\infty} id\,dm_x\,,$$

it is square integrable, if there exists

$$\int_{-\infty}^{\infty} u^2\,dF(u)\,.$$

In the case we define dispersion $D(x) = \sigma^2(x)$ by the formula

$$D(x) = \int_{-\infty}^{\infty} u^2 \, dF(u) - E(x)^2.$$

Of course, for the formulation of central limit theorem, we need also the notion of sum of observables

$$x_1 + \cdots + x_n \,.$$

It will be realized in the next section.

3.2. Joint observable

If we want to define the sum $\xi + \eta$ of two observables, one of possibilities is the following formulation. Put

$$T = (\xi, \eta) : \ \Omega \to R^2 \,, \quad g : R^2 \,, \quad g(u,v) = u + v \,, \quad \xi + \eta = g \circ T : \ \Omega \to \Omega \,.$$

Namely, it is convenient for the constructing of preimages

$$(\xi + \eta)^{-1}(A) = T^{-1}\left(g^{-1}(A)\right).$$

In our IF-case, we have two observables

$$x, y : \sigma(\mathcal{J}) \to \mathcal{F},$$

hence $x + y$ could be defined as a morphism

$$(x + y)(A) = h\left(g^{-1}(A)\right),$$

where $h : \sigma(\mathcal{J}_2) \to \mathcal{F}$ is a morphism connecting with x, y. In the classical case it was realized by the formula

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D).$$

In our IF-case, instead of intersection, we shall use the product of IF-sets.

$$\begin{aligned} A \cdot B &= (\mu_A, \nu_A) \cdot (\mu_B, \nu_B) \\ &= \left(\mu_A \cdot \mu_B, \ 1 - (1 - \nu_A)(1 - \nu_B)\right) \\ &= \left(\mu_A \cdot \mu_B, \ \nu_A + \nu_B - \nu_A \cdot \nu_B\right). \end{aligned}$$

Definition 3.3. Let $x_1, \dots, x_n : \sigma(\mathcal{J}) \to \mathcal{F}$ be observables. By the joint observable of $x_1, \dots, x_n$ we consider a mapping $h : \sigma\left(\mathcal{J}^n\right) \to \mathcal{F}$ ($\mathcal{J}^n$ being the set all intervals of R^n) satisfying the following conditions:

(i) $h(R^n) = (1, 0)$,

(ii) $A \cap B = \varnothing \Rightarrow h(A \cup B) = h(A) \oplus h(B)$, and $h(A) \odot h(B) = (0, 1)$,

(iii) $A_n \nearrow A \Rightarrow h(A_n) \nearrow h(A)$,

(iv) $h(C_1 \times C_2 \times \cdots \times C_n) = x_1(C_1) \cdot x_2(C_2) \cdot \ldots \cdot x_n(C_n)$, for any $C_1, C_2, \ldots \ldots, C_n \in \mathcal{J}$.

Theorem 3.1. *For any observables $x_1, \dots, x_n : \sigma(\mathcal{J}) \to \mathcal{F}$ there exists their joint observable $h : \sigma\big(\mathcal{J}^n\big) \to \mathcal{F}$.*

P r o o f. We shall prove it for $n = 2$. Consider two observables $x, y : \sigma(\mathcal{J}) \to \mathcal{F}$. Since $x(A) \in \mathcal{F}$, we shall write

$$x(A) = \big(x^\flat(A),\ 1 - x^\sharp(A)\big)$$

and similarly,

$$y(B) = \big(y^\flat(B),\ 1 - y^\sharp(B)\big)\,.$$

By the definition of product $x(C) \cdot y(D)$, we have

$$\begin{aligned} &x(C) \cdot y(D) \\ &= \big(x^\flat(C),\ 1 - x^\sharp(C)\big) \cdot \big(y^\flat(D),\ 1 - y^\sharp(D)\big) \\ &= \Big(x^\flat(C) \cdot y^\flat(D),\ 1 - \big(1 - \big(1 - x^\sharp(C)\big) \cdot \big(1 - \big(1 - y^\sharp(D)\big)\big)\big)\Big) \\ &= \big(x^\flat(C) \cdot y^\flat(D),\ 1 - x^\sharp(C) \cdot y^\sharp(D)\big)\,. \end{aligned}$$

Therefore, we shall construct similarly

$$\big(h^\flat(K),\ 1 - h^\sharp(K)\big).$$

Fix $\omega \in \Omega$ and put

$$\begin{aligned} \mu(A) &= x^\flat(A)(\omega)\,, \\ \nu(B) &= y^\flat(B)(\omega)\,, \\ h^\flat(K)(\omega) &= \mu \times \nu(K)\,. \end{aligned}$$

$\mu \times \nu$ is the product of probability measures μ, ν.

Then

$$\begin{aligned} h^\flat((C \times D)(\omega) = \mu \times \nu(C \times D) &= \mu(C) \cdot \nu(D) \\ &= x^\flat(C) \cdot y^\flat(D)(\omega), \end{aligned}$$

hence

$$h^\flat(C \times D) = x^\flat(C) \cdot y^\flat(D)\,.$$

Analogously,

$$h^\sharp(C \times D) = x^\sharp(C) \cdot y^\sharp(D)\,.$$

If we define

$$h(A) = \big(h^\flat(A),\ 1 - h^\sharp(A)\big)\,, \quad A \in \sigma\big(\mathcal{J}^2\big)\,,$$

then

$$h(C \times D) = \big(x^\flat(C) \cdot y^\flat(D), 1 - x^\sharp(C) \cdot y^\sharp(D)) = x(C) \cdot y(D)\,.$$

□

The previous theorem can be applied for obtaining sum

$$x_1 + \cdots + x_n = h \circ g^{-1} \quad \text{with} \quad g(u_1, \ldots, u_n) = u_1 + \cdots + u_n\,,$$

or for the arithmetic means

$$\frac{1}{n}(x_1 + \cdots + x_n) = h \circ g^{-1} \quad \text{with} \quad g(u_1, \ldots, u_n) = \frac{1}{n}(u_1 + \cdots + u_n)\,.$$

3.3. Central limit theorem

Consider again a probability measure space $(\Omega, \mathcal{S}, P)$ and a sequence $(\xi_n)_n$ of square integrable, equally distributed variables with $E(\xi_n) = a$, $D(\xi_n) = \sigma^2$ $(n = 1, 2, \ldots)$. Then

$$\lim_{n\to\infty} P\left(\omega \in \Omega;\ \frac{\frac{1}{n}\sum_{i=1}^{n} \xi_i(\omega) - a}{\frac{\sigma}{\sqrt{n}}} < t\right) = \Phi(t)$$

for any $t \in R$. $\left(\text{Here } \Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{u^2}{2}}\, du.\right)$

We shall translate the theorem in our IF-case.

Definition 3.4. Let $m : F \to \langle 0, 1\rangle$ be a state, $(x_n)_{n=1}^{\infty}$ be a sequence of observables, $h_n : \sigma(\mathcal{J}_n) \to \mathcal{F}$ be the joint observable of $x_1, \ldots, x_n$ $(n = 1, 2, \ldots)$. Then $(x_n)_n$ is called independent, if

$$m\big(h_n(C_1 \times C_2 \times \cdots \times C_n)\big) = m(x_1(C_1)) \cdot m(x_2(C_2)) \cdots m(x_n(C_n))$$

for any $n \in N$ and any $C_1, \ldots, C_n \in \sigma(\mathcal{J})$.

Theorem 3.2. *Let $(x_n)_{n=1}^{\infty}$ be a sequence of square integrable, equally distributed, independent observables, with*

$$E(x_n) = a,\ D(x_n) = \sigma^2\ (n = 1, 2, \ldots).$$

Then

$$\lim_{n\to\infty} m\left(\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} x_i - a\right)\big((-\infty, t)\big)\right) = \Phi(t)$$

for any $t \in R$.

Proof. Put $g_n : R^n \to R$

$$g_n(u_1, \ldots, u_n) = \frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} u_i - a\right)$$

and

$$\frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^{n} x_i - a\right)(-\infty, t) = h_n \circ g_n^{-1}(-\infty, t)\,.$$

Now consider a sequence $(m \circ h_n)_n$ of probability measures

$$m \circ h_n : \sigma\big(\mathcal{J}^n\big) \to \langle 0, 1\rangle.$$

By the definition of h_n we have

$$m \circ h_{n+1}(A \times R) = m \circ h_n(A),\ A \in \sigma\big(\mathcal{J}^n\big)\,.$$

Therefore $(m \circ h_n)_n$ forms a consisting system of probability measures

$$m \circ h_n : \sigma\big(\mathcal{J}^n\big) \to \langle 0, 1\rangle\,.$$

Consider the space R^N, the projections $\Pi_n :\ R^N \to R^n$

$$\Pi_n\Big(\big(u_i\big)_{i=1}^{\infty}\Big) = (u_1, \dots, u_n)$$

and the family of all cylinders in R^N, i. e., sets of the form

$$\mathcal{E} = \Big\{\Pi_n^{-1}(A);\ n \in N,\ A \in \sigma\big(\mathcal{J}^n\big)\Big\}.$$

By the Kolmogorov consistency theorem there exists a probability measure

$$P : \sigma\big(\mathcal{E}\big) \to \langle 0, 1\rangle \quad \text{such that} \quad P \circ \Pi_n^{-1} = m \circ h_n \quad \text{for any} \quad n \in N.$$

Now return to our sequence $(x_n)_{n=1}^{\infty}$ of observables. Define on R^N the sequence $(\xi_n)_{n=1}^{\infty}$ by the formula

$$\xi_n\Big(\big(u_i\big)_{i=1}^{\infty}\Big) = u_n$$

Then

$$\begin{aligned} m\big(x_n(C)\big) &= m\big(h_n(R \times \dots \times R \times C \times R \times \dots \times R)\big) \\ &= P\big(\Pi_n^{-1}(R \times \dots \times R \times C \times R \times \dots \times R)\big) \\ &= P\big(\xi_n^{-1}(C)\big)\,. \end{aligned}$$

Therefore,

$$\begin{aligned} E\big(\xi_n\big) &= \int_{-\infty}^{\infty} t\, dm_{\xi_n}(t) = \int_{-\infty}^{\infty} t\, dm_{x_n}(t) \\ &= E(x_n) \end{aligned}$$

and similarly,

$$D\big(\xi_n\big) = D\big(x_n\big)\,.$$

Moreover,

$$\begin{aligned} P\Big(\xi_1^{-1}(C_1) \cap \ldots \cap \xi_n^{-1}(C_n)\Big) &= P\Big(\Pi_n^{-1}\big(C_1 \times \ldots \times C_n\big)\Big) \\ &= m\Big(h_n\big(C_1 \times \ldots \times C_n\big)\Big) \\ &= m\big(x_1(C_1)\big) \cdot \ldots \cdot m\big(x_n(C_n)\big) \\ &= P\big(\xi_1^{-1}(C_1)\big) \cdot \ldots \cdot P\big(\xi_n^{-1}(C_n)\big), \end{aligned}$$

hence $\xi_1, \xi_2, \dots$ are independent. Put $g_n : R^n \to R$ by the formula

$$g_n(t_1, \dots, t_n) = \frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} t_i - a \right) ,$$

$$\eta_n = g_n(\xi_1, \dots, \xi_n) = \frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} \xi_i - a \right) = g_n \circ \Pi_n .$$

Then

$$\lim_{n\to\infty} P\Big(\eta_n^{-1}\big((-\infty, t)\big)\Big) = \Phi(t) \quad \text{for any} \quad t \in R.$$

But

$$\frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} x_i - a \right) = h_n \circ g_n^{-1}.$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} m \left(\frac{\sqrt{n}}{\sigma} \left(\frac{1}{n} \sum_{i=1}^{n} x_i - a \right) \right) (-\infty, t) &= \lim_{n\to\infty} m\Big(h_n\big(g_n^{-1}(-\infty, t)\big)\Big) \\
&= \lim_{n\to\infty} P\bigg(\Pi_n^{-1}\Big(g_n^{-1}\big((-\infty, t)\big)\Big)\bigg) \\
&= \lim_{n\to\infty} P\Big(\eta_n^{-1}\big((-\infty, t)\big)\Big) = \Phi(t) .
\end{aligned}$$

□

4. Conditional probability

Conditional probability (of A with respect to B) is the real number $P(A|B)$ such that

$$P(A \cap B) = P(B)\, P(A|B) .$$

When A, B are independent then $P(A|B) = P(A)$, the event A does not depend on the occuring of event B. Another point of view:

$$P(A \cap B) = \int_B P(A|B)\, dP .$$

The number $P(A|B)$ can be regarded as a constant function. Constant functions are measurable with respect to the σ-algebra $\mathcal{S}_0 = \{\varnothing, \Omega\}$,

$$\big\{\omega \in \Omega;\ f(\omega) \in C\big\} = \begin{cases} \varnothing \\ \Omega \end{cases}$$

Generally $P(A|\mathcal{S}_0)$ can be defined for any σ-algebra $\mathcal{S}_0 \subset \mathcal{S}$, as an $\mathcal{S}_0$ measurable function such that

$$P(A \cap C) = \int\limits_C P(A|\mathcal{S}_0)\, dP\,, \quad C \in \mathcal{S}_0\,.$$

If $\mathcal{S}_0 = \mathcal{S}$, then we can put $P(A|\mathcal{S}_0) = \chi_A$, since χ_A is $\mathcal{S}_0$-measurable, and

$$\begin{aligned}\int\limits_\Omega V\chi_A\, dP &= \int\limits_\Omega \chi_C \chi_A\, dP \\ &= \int\limits_\Omega V\chi_{A\cap C}\, dP = P(A \cap C)\,.\end{aligned}$$

An important example of $\mathcal{S}_0$ is the family of all pre-images of a random variable $\xi : \Omega \to R$

$$\mathcal{S}_0 = \left\{\xi^{-1}(B);\ B \in \sigma(\mathcal{J})\right\}.$$

In this case we shall write $P(A|\mathcal{S}_0) = P(A|\xi)$, hence

$$\int\limits_C \big(P(A|\xi)\, dP = P(A \cap C)\,,$$

$$C = \xi^{-1}(B),\ B \in \sigma(\mathcal{J})\,.$$

By the transformation formula

$$\begin{aligned}P\big(A \cap \xi^{-1}(B)\big) &= \int\limits_{\xi^{-1}(B)} g_0 \xi\, dP \\ &= \int\limits_B g\, dP_\xi\,, \quad B \in \sigma(\mathcal{J})\,.\end{aligned}$$

And exactly this formulation will be used in our IF-case

$$m\big(A \cdot x(B)\big) = \int\limits_B p(A|x)\, dm_x = \int\limits_B p(A|x)\, dF\,.$$

Of course, we must first prove the existence of such a mapping $p(A|x)\colon\ R \to R$. Recall that the product of IF-events is defined by the formula

$$K \cdot L = (\mu_K \cdot \mu_L,\ \nu_K + \nu_L - \nu_K \cdot \nu_L)\,.$$

Proposition 4.1. *If $L \odot M = (0,1)$, then*

$$K \cdot (L \oplus M) = (K \cdot L) \oplus (K \cdot M)$$

and

$$(K \cdot L) \odot (K \cdot M) = (0,1)\,.$$

P r o o f.

$$(0,1) = L \odot M = \big((\mu_L + \mu_M - 1) \vee 0,\ (\nu_L + \nu_M) \wedge 1\big)$$

means that

$$\mu_L + \mu_M \le 1\,, \quad \nu_L + \nu_M \ge 1\,.$$

Therefore,

$$\begin{aligned} L \oplus M &= \big((\mu_L + \mu_M) \wedge 1,\ (\nu_L + \nu_M - 1) \vee 0\big) \\ &= (\mu_L + \mu_M,\ \nu_L + \nu_M - 1)\,. \end{aligned}$$

$$\begin{aligned} K \cdot (L \oplus M) &= \big(\mu_K \cdot (\mu_L + \mu_M),\ \nu_K + \nu_L + \nu_M - 1 - \nu_K(\nu_L + \nu_M - 1)\big) \\ &= (\mu_K\mu_L + \mu_K\mu_M,\ \nu_K + \nu_L - \nu_K\nu_L + \nu_K + \nu_M - \nu_K\nu_M - 1)\,. \end{aligned}$$

On the other hand,

$$K \cdot L = (\mu_K\mu_L,\ \nu_K + \nu_L - \nu_K\nu_L)$$
$$K \cdot M = (\mu_K\mu_M,\ \nu_K + \nu_M - \nu_K\nu_M)$$

$$\begin{aligned} K \cdot L \oplus K \cdot M = \big(&(\mu_K\mu_L + \mu_K\mu_M) \wedge 1, \\ &(\nu_K + \nu_L - \nu_K\nu_L + \nu_K + \nu_M + \nu_K\nu_M - \nu_K\nu_M - 1) \vee 0\big) \end{aligned}$$

Of course,

$$\begin{aligned} (\mu_K\mu_L + \mu_K\mu_M) &= \big(\mu_K(\mu_L + \mu_M)\big) \\ &\le (\mu_K \cdot 1) \le 1\,, \\ (\mu_K\mu_L + \mu_K\mu_M) \wedge 1 &= \mu_K\mu_L + \mu_K\mu_M\,. \end{aligned}$$

Similarly,

$$\begin{aligned} &\nu_K + \nu_L - \nu_K\nu_L + \nu_K + \nu_M - \nu_K\nu_M - 1 \\ &= (\nu_K + \nu_M - 1) + (\nu_K + (1 - \nu_M) + \nu_K(1 - \nu_M) \ge 0\,. \end{aligned}$$

Therefore,

$$\begin{aligned} K \cdot L \oplus K \cdot M &= (\mu_K\mu_L + \mu_K\mu_M, \nu_K + \nu_L - \nu_K\nu_L + \nu_K + \nu_M - \nu_K\nu_M - 1) \\ &= K \cdot (L \oplus M)\,. \end{aligned}$$

Moreover,

$$\begin{aligned} K \cdot L \odot K \cdot M &= (\mu_K\mu_L,\ \nu_K + \nu_L - \nu_K\nu_L) \odot (\mu_K\mu_M,\ \nu_K + \nu_M - \nu_K\nu_M) \\ &= \big((\mu_K\mu_L + \mu_K\mu_M - 1) \vee 0, \\ &\qquad (\nu_K + \nu_L - \nu_K\nu_L + \nu_K + \nu_M - \nu_K\nu_M) \wedge 1\big) \\ &= \big((\mu_K(\mu_L + \mu_M) - 1\big) \vee 0, \\ &\qquad \big((\nu_L + \nu_M) + \nu_K(1 - \nu_L) + \nu_K(1 - \nu_L)\big) \wedge 1\big) \\ &= (0,1)\,. \end{aligned}$$

□

Proposition 4.2. *If $C_n \nearrow C$, then $A \cdot C_n \nearrow A \cdot C$.*

Proof. We have $\mu_{C_n} \nearrow \mu_C$, $\nu_{C_n} \searrow \nu_C$. Therefore,

$$\begin{aligned} A \cdot C_n &= \big(\mu_A \mu_{C_n},\ 1-(1-\nu_A)(1-\nu_{C_n})\big) \nearrow \big(\mu_A \mu_C, 1-(1-\nu_A)(1-\nu_C)\big) \\ &= A \cdot C\,. \end{aligned}$$

□

Theorem 4.1. *Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable $m : \mathcal{F} \to \langle 0,1\rangle$ be a state and let $A \in \mathcal{F}$. Define $\nu : \sigma(\mathcal{J}) \to \langle 0,1\rangle$ by the equality*

$$\nu(B) = m\big(A \cdot x(B)\big)\,.$$

Then ν is a measure.

Proof. Let $B \cap C = \varnothing$, $B, C \in \sigma(\mathcal{J})$. Then

$$x(B) \odot x(C) = (0,1)\,,$$

hence by Proposition 4.1

$$A \cdot \big(x(B) \oplus x(C)\big) = \big(A \cdot x(B)\big) \oplus \big(A \cdot x(C)\big)\,,$$

and therefore,

$$\begin{aligned} \nu(B \cup C) &= m\big(A \cdot x(B \cup C)\big) \\ &= m\big(A \cdot \big(x(B) \oplus x(C)\big)\big) \\ &= m\big(A \cdot x(B)\big) \oplus \big(A \cdot x(C)\big)\big) \\ &= m\big(A \cdot x(B)\big) + m\big(A \cdot x(C)\big) \\ &= \nu(B) + \nu(C)\,. \end{aligned}$$

Let $B_n \nearrow B$. Then $x(B_n) \nearrow x(B)$, and by Proposition 4.2 $A \cdot x(B_n) \nearrow A \cdot x(B)$. Therefore,

$$\nu(B_n) = m\big(A \cdot x(B_n)\big) \nearrow m\big(A \cdot x(B)\big) = \nu(B)\,.$$

□

Theorem 4.2. *Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable, $m : \mathcal{F} \to \langle 0,1\rangle$ be a state, and let $A \in \mathcal{F}$. Then there exists a Borel measurable function $f : R \to R$ (i.e., $B \in \mathcal{J} \Rightarrow f^{-1}(B) \in \sigma(\mathcal{J})$) such that*

$$m\big(A \cdot x(B)\big) = \int_B f\, dm_x \tag{5}$$

for any $B \in \sigma(\mathcal{J})$. If g is another function satisfying (5), then

$$m_x\Big(\big\{u \in R;\ f(x) \neq g(x)\big\}\Big) = 0\,.$$

P r o o f. Define $\mu, \nu : \sigma(\mathcal{J}) \to \langle 0, 1 \rangle$ by the formulas

$$\mu(B) = m_x(B) = m\big(x(B)\big), \quad \nu(B) = m\big(A \cdot x(B)\big).$$

Then $\mu, \nu : \sigma(\mathcal{J}) \to \langle 0, 1 \rangle$ are measures and $\nu \le \mu$.

By the Radon-Nikodym theorem there exists exactly one function $f : R \to R$ (with respect to the equality μ-almost everywhere) such that

$$\begin{aligned} m\big(A \cdot x(B)\big) = \nu(B) &= \int_B f \, d\mu \\ &= \int_B f \, dm_x , \quad B \in \sigma(\mathcal{J}) . \end{aligned}$$

□

Definition 4.1. Let $x : \sigma(\mathcal{J}) \to \mathcal{F}$ be an observable $A \in \mathcal{F}$. Then the conditional probability $p(A|x) = f$ is a Borel measurable function (i.e., $B \in \mathcal{J} \Rightarrow f^{-1}(B) \in \mathcal{J}$) such that

$$\int_B p(A|x) \, dm_x = m\big(A \cdot x(B)\big)$$

for any $B \in \sigma(\mathcal{J})$.

5. Embedding

The aim of the chapter is to show that in our IF-probability theory it can be used very well developed MV-algebra probability theory. We shall show that any IF-events space $\mathcal{F}$ can be embedded to a convenient MV-algebra.

By the Mundici theorem any MV-algebra can be defined by the help of an l-group.

Definition 5.1. By an l-group we consider a triple $(G, +, \le)$, where

(i) $(G, +)$ is an Abelian group;

(ii) $(G, \le)$ is a partially ordered set being a lattice;

(iii) $a \le b \Rightarrow a + c \le b + c$.

Definition 5.2. An MV-algebra is an algebraic system $(M, \oplus, \odot, 0, u)$ satisfying the following conditions

(i) there exists an l-group $(G, +, \le)$ such that

$$M = \big\{v \in G, \ 0 \le v \le u\big\},$$

where 0 is the neutral element of $(G, +)$ and u is a strong unit (i.e., to any $a \in G$ there exists $n \in N$ such that $n \le nu$);

(ii) $\oplus, \odot$ are binary operations of M satisfying the following identities:

$$a \oplus b = (a + b) \wedge u\,,$$
$$a \odot b = (a + b - u) \vee 0\,.$$

Theorem 5.1. *On the space $(R^2)^\Omega$ define a binary operation $\uplus$ by the help of the equality*

$$(f,g) \uplus (k,l) = (f + k, g + l - 1_\Omega)$$

and a relation $\leq$ by

$$(f,g) \leq (k,l) \Leftrightarrow f \leq k,\ \ g \geq l\,.$$

Then $\Big((R^2)^\Omega, +, \leq\Big)$ is an l-group.

Proof. It is easy to see that $\big((R^2)^\Omega, \uplus\big)$ is commutative and associative. Moreover $(0_\Omega, 1_\Omega)$ is the neutral element

$$(f,g) \uplus (0_\Omega, 1_\Omega) = (f + 0_\Omega,\ g + 1_\Omega - 1_\Omega) = (f,g)\,,$$

and $(-f, 2-g)$ is the inverse element to (f,g),

$$(f,g) + (-f, 2-g) = (f - f, g + 2 - g - 1) = (0,1)\,.$$

If

$$(f,g) \leq (k,l),$$

then

$$f + p \leq k + p, \quad g + \xi \geq l + \xi,$$

hence

$$(f,g) \uplus (p,\xi) \leq (k,l) \uplus (p,\xi)\,.$$

□

Return now to the family $\mathcal{F}$ of all IF-events on the measurable space $(\Omega, \mathcal{S})$:

$$\mathcal{F} = \big\{A = (\mu_A, \nu_A);\ \mu_A, \nu_A : \Omega \to \langle 0,1\rangle, \\ \mu_A, \nu_A \text{ are } \mathcal{S} \text{ measurable},\ \mu_A + \nu_A \leq 1\big\}\,.$$

Theorem 5.2. *Put*

$$\mathcal{M} = \big\{A = (\mu_A, \nu_A);\ \mu_A, \nu_A :\ \Omega \to \langle 0,1\rangle, \mathcal{S}\ measurable\big\},$$

$$\begin{aligned} A \oplus B &= (A \uplus B) \wedge (1,0) \\ &= (\mu_A + \mu_B, \nu_A + \nu_B - 1) \wedge (1,0) \\ &= \big((\mu_A + \mu_B) \wedge 1,\ (\nu_A + \nu_B - 1) \vee 0\big), \end{aligned}$$

$$\begin{aligned} A \odot B &= \big(A \uplus B - (1,0)\big) \vee (0,1) \\ &= \big((\mu_A + \mu_B,\ \nu_A + \nu_B - 1) \uplus (-1,2)\big) \vee (0,1) \\ &= (\mu_A + \mu_B - 1,\ \nu_A + \nu_B - 1 + 2 - 1) \vee (0,1) \\ &= \big((\mu_A + \mu_B - 1) \vee 0,\ (\nu_A + \nu_B) \wedge 1\big). \end{aligned}$$

Then $\big(\mathcal{M}, \oplus, \odot, (0,1), (1,0)\big)$ *is an MV-algebra,* $\mathcal{F} \subset \mathcal{M}$ *and to any state* $m : \mathcal{F} \to \langle 0,1\rangle$ *there exists a state* $\overline{m} : \mathcal{M} \to \langle 0,1\rangle$ *such that* $\overline{m}|\mathcal{F} = m$.

P r o o f. Let $\overline{A} = (\mu_A, \nu_A) \in \mathcal{M}$. Then $\langle \mu_A, 0) \in \mathcal{F}$, $(0, 1 - \nu_A) \in \mathcal{F}$. Define

$$\overline{m}(\mu_A, \nu_A) = m\big((\mu_A, 0)\big) - m\big((0, 1 - \nu_A)\big)\,.$$

Then $\overline{m}$ satisfies conditions state above. □

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