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ON SMÍTAL PROPERTY

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ABSTRACT. In this paper, we define and study the property called Smítal property for the pair $(\mathcal{M}, \mathcal{I})$, where $\mathcal{M}$ denotes a σ-algebra of subsets of real line and $\mathcal{I} \subset \mathcal{M}$ is a σ-ideal. It is a generalization of Smítal lemma. We give some examples of pairs $(\mathcal{M}, \mathcal{I})$ having Smítal property and pairs which do not have this property.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers and $\mathbb{Q}$—the set of rational numbers. Lebesgue outer (resp. inner) measure on the real line will be denoted by λ^* (resp. λ_*), whereas λ will stand for the Lebesgue measure itself. Moreover, $\mathcal{L}$ will denote the σ-algebra of λ-measurable subsets of $\mathbb{R}$ and $\mathcal{N}$ will denote the σ-ideal of Lebesgue null sets. We will consider a natural topology on $\mathbb{R}$. Notation $\mathcal{B}$ will be adopted for the case of subsets of $\mathbb{R}$ having the Baire property and $\mathcal{K}$ will denote the σ-ideal of the sets of the first category. $\mathcal{I}_p$ will denote the σ-ideal of at most countable sets. The sign "+" indicates the operation of finding an algebraic sum of two sets A and B contained in $\mathbb{R}$, so the algebraic sum of this sets will be denoted by $A + B = \{a + b : a \in A, b \in B\}$.

For mathematicians working with Lebesgue measure on the real line, the following lemma become a part of the folklore.

Lemma 1 (Smítal's lemma for measure). *If a set $B \subset \mathbb{R}$ has positive Lebesgue measure and D is a dense set in $\mathbb{R}$ with natural topology, then the set $B+D$ has a full measure.*

The proof based on the Lebesgue Density Theorem can be found in [1] (for outer measure) or in [8]. For the convenience, we repeat a sketch of the proof. If we denote by D_0 a countable and dense subset of D, then the set $A = B+D_0 = \bigcup_{d\in D_0}(B+d)$ is measurable. Fix a number $x \in \mathbb{R}$. We will show that x is a density point of A, i.e.,

$$\lim_{h\to 0^+} \frac{\lambda\,(A \cap [x-h, x+h])}{2h} = 1.$$

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Since $\lambda(B) > 0$, from Lebesgue Density Theorem, it follows that there exists a density point x_0 of the set B. Because D_0 is a dense set, there exists an increasing sequence $(d_n)_{n\in\mathbb{N}} \in D_0$ tending to $x - x_0$. Then $(x_0 + d_n)_{n\in\mathbb{N}} \nearrow x$, and for $h > 0$ there is $n_0 \in \mathbb{N}$ such that $x - h < x_0 + d_n$ for all $n \geq n_0$. Hence, for any $n \geq n_0$,

$$\begin{aligned} \lambda\big(A \cap [x-h, x+h]\big) &\geq \lambda\big((B + d_n) \cap [x-h, x+h]\big) \\ &\geq \lambda\big((B + d_n) \cap [x-h, x_0 + d_n + h]\big) \\ &\geq \lambda\big((B + d_n) \cap [x_0 + d_n - h, x_0 + d_n + h]\big) \\ &\quad - \lambda\big([x_0 + d_n - h, x - h]\big) \\ &= \lambda\big(B \cap [x_0 - h, x_0 + h]\big) - \big(x - (x_0 + d_n)\big) \end{aligned}$$

Therefore,

$$\lambda\big(A \cap [x-h, x+h]\big) \geq \lambda\big(B \cap [x_0 - h, x_0 + h]\big)$$

and, since x_0 is a density point of B, x is a density point of A. Applying Lebesgue Density Theorem again, we obtain that $\lambda(\mathbb{R} \setminus A) = 0$.

Even easier, one can prove (compare [1, Th. 2, p. 66]):

Lemma 2 (Smítal's lemma for category)**.** *Let $B \subset \mathbb{R}$ be a second category set having the Baire property and D is dense in $\mathbb{R}$. Then the set $A = (B + D)$ is a residual set.*

Indeed, the second category set having the Baire property can be presented as a symmetric difference $U \triangle P$ of a nonempty open set U and the first category set P. Let D_0 be a countable dense subset of D. It is easy to check that $U + D_0 = \mathbb{R}$. The set $B + D_0$ differs from $U + D_0$ on a first category set and $(B + D) \supset (B + D_0)$.

Let $\mathcal{M}$ be a σ-algebra consisted of subsets of $\mathbb{R}$ and $\mathcal{I} \subset \mathcal{M}$ be a σ-ideal. We will say that the pair $(\mathcal{M}, \mathcal{I})$ has *Smítal property* if for any set $B \in \mathcal{M} \setminus \mathcal{I}$ and for any dense set D, the set $\mathbb{R} \setminus (B + D)$ belongs to $\mathcal{I}$.

As we have seen, pairs $(\mathcal{L}, \mathcal{N})$ and $(\mathcal{B}, \mathcal{K})$ have Smítal property. In the following theorem, we will examine Smítal property for a σ-algebra $\mathcal{S}_\mu \subset 2^{\mathbb{R}}$ consisted of sets measurable with respect to a Borel measure μ and a σ-ideal $\mathcal{I}_\mu$ of null sets with respect to μ. Recall that μ is a Borel measure if $\mathcal{S}_\mu$ contains all Borel subsets of $\mathbb{R}$.

A measure μ_a is absolutely continuous with respect to λ if $\lambda(A) = 0$ implies $\mu_a(A) = 0$ for any $A \subset \mathbb{R}$. We say that μ_s is singular with respect to λ, if there exists a set $A \subset R$, such that $\mu_s(A) = 0$ and $\lambda(\mathbb{R} \setminus A) = 0$.

A measure μ_d defined on the σ-algebra $2^{\mathbb{R}}$ is called a purely discontinuous measure if there exists a countable set $X = \{x_k : k \in \mathbb{N}\} \subset \mathbb{R}$ and a function $\varphi : X \to (0, \infty)$ satisfying the condition $\sum_{\{\varphi(x_k): |x_k| \leq n\} < \infty} \varphi(x_k)$ for any $n \in \mathbb{N}$, such that for any $A \subset \mathbb{R}$, we have $\mu_d(A) = \Sigma_{k=1}^{\infty}\varphi(x_k)\cdot\chi_A(x_k)$, where χ_A denotes the characteristic function of A.

Finally, recall that for any Borel measure μ on $\mathbb{R}$ there exist three measures: μ_a-absolutely continuous with respect to λ, μ_d-purely discontinuous, and μ_s--nonatomic and singular with respect to λ such that

$$\mu(A) = \mu_a(A) + \mu_s(A) + \mu_d(A)$$

for any Borel set $A \subset \mathbb{R}$ (compare [2, p. 337]).

Theorem 1.

a) *For any Borel measure μ_a absolutely continuous with respect to λ, the pair $(\mathcal{S}_{\mu_a}, \mathcal{I}_{\mu_a})$ has Smítal property.*

b) *For any Borel measure μ_d purely discontinuous, the pair $(\mathcal{S}_{\mu_d}, \mathcal{I}_{\mu_d})$ does not have Smítal property.*

c) *There exists a nonatomic Borel measure μ_s singular with respect to λ, such that the pair $(\mathcal{S}_{\mu_s}, \mathcal{I}_{\mu_s})$ does not have Smítal property.*

Proof.

a) Let $B \in \mathcal{S}_{\mu_a} \setminus \mathcal{I}_{\mu_a}$ and D be a dense set on the real line. Since μ_a is absolutely continuous with respect to λ, so $\lambda(B) > 0$. From Lemma 1 we have that $\lambda\big(\mathbb{R} \setminus (B + D)\big) = 0$. Using the definition of absolutely continuous measure again, we obtain that $\mu_a\big(\mathbb{R} \setminus (B + D)\big) = 0$.

b) Assume that the measure μ_d is concentrated on the set $\{x_1, x_2, \dots, x_n, \dots\}$ for $n \in N$. Then, $\mu_d\big(\{x_n\}\big) > 0$ for $n \in \mathbb{N}$ and

$$\mu_d\big(\mathbb{R} \setminus \{x_1, x_2, \dots, x_n, \dots\}\big) = 0.$$

Put $B = \{x_1\}$ and $D = \mathbb{R} \setminus \{0\}$. We have $\mathbb{R} \setminus (B + D) = \{x_1\}$, so $\mu_d\big(\mathbb{R} \setminus (B + D)\big) \geq \mu_d\big(\{x_1\}\big) > 0$. That means that $\mathbb{R} \setminus (B + D) \in \mathcal{S}_{\mu_d} \setminus \mathcal{I}_{\mu_d}$.

c) Let C denotes the ternary Cantor set and $g_0 : [0,1] \to [0,1]$ the Cantor function. Define a function

$$g(x) = \begin{cases} 0 & \text{for } x < 0, \\ g_0(x) & \text{for } x \in [0,1]\,, \\ 1 & \text{for } x > 1 \end{cases}$$

and λ_g the Lebesgue-Stieltjes measure generated by g. The algebra $\mathcal{S}_g$ of measurable sets with respect to λ_g contains Borel sets.

Since g is continuous, the measure λ_g vanishes on one-point sets. Moreover, if a set A is disjoint from C, then $\mu_g(A) = 0$. Therefore, λ_g is singular with respect to λ.

We will construct a dense set D such that for any $x \in D$, $\operatorname{card}\big(C \cap (C + D)\big) \leq \aleph_0$. Observe first that, for any natural number n and integer k, if the set $C \cap \left(\frac{k}{3^n}, \frac{k+1}{3^n}\right)$ is nonempty, then $C \cap \left(\frac{k+1}{3^n}, \frac{k+2}{3^n}\right) = \emptyset$. It follows that, for any k, the set $C \cap \left(C + \frac{k}{3^n}\right)$ is finite, containing only endpoints of finite many intervals of the form $\left[\frac{k}{3^n}, \frac{k+1}{3^n}\right]$, $k = 1, 2, \ldots, \frac{3^n+1}{2}$. Moreover, for any $x \in (-\infty, -1) \cup (1, \infty)$, $C \cap (C + x) = \emptyset$.

Clearly, the set

$$D = (-\infty, -1) \cup (1, \infty) \cup \bigcup_{n=1}^{\infty} \left(\left\{ z_k = \frac{2k-1}{3^n}, k = 1, 2, \ldots, \frac{3^n+1}{2} \right\} \right.$$
$$\left. \cup \left\{ z_k = -\frac{2k-1}{3^n}, k = 1, 2, \ldots, \frac{3^n+1}{2} \right\} \right)$$

is dense and $\operatorname{card}\big(C \cap (C + D)\big) \leq \aleph_0$. Therefore, $\lambda_g(C + D) = 0$ and a measure $\mu_s = \lambda_g$ fulfils c).

□

Example 1. The pair $(\beta, \mathcal{I}_p)$, composed of a σ-algebra of Borel sets and a σ-ideal of at most countable sets does not have Smítal property.

Indeed, it is sufficient to take a Cantor set C which belongs to $\beta \setminus \mathcal{I}_p$ and $D = \mathbb{Q}$.

Example 2. The pair $(\mathcal{L} \cap \mathcal{B}, \mathcal{N} \cap \mathcal{K})$ containing the σ-algebra of measurable sets having Baire property and σ-ideal of Lebesgue measure zero sets of the first category, does not have Smítal property.

There exists a residual set B of Lebesgue measure zero (compare [5, p. 15, Th. 1.6]). If we put $D = \mathbb{Q}$, the set $B + \mathbb{Q}$ has measure zero and $\mathbb{R} \setminus (B + \mathbb{Q})$ does not belong to $\mathcal{N} \cap \mathcal{K}$.

Example 3. We say that a set M has the (s)-Marczewski property if every nonempty perfect set has a nonempty perfect subset, which is a subset of M or does not intersect the set M. A set M has the (s_0)-Marczewski property if every nonempty perfect set has a nonempty perfect subset which does not intersect the set M. The family of sets having the (s)-Marczewski property is a σ-algebra and we will denote it by $\mathcal{S}$. The family of sets having the (s_0)-Marczewski property is a σ-ideal and we will denote it by $\mathcal{S}_0$ (compare [7]).

The pair $(\mathcal{S}, \mathcal{S}_0)$ does not have Smítal property. It is easy to observe that any nonempty perfect set belongs to σ-algebra $\mathcal{S}$ and does not belong to σ-ideal $\mathcal{S}_0$, whereas any at most countable set belongs to σ-ideal $\mathcal{S}_0$. Moreover, any uncountable Borel set contains the perfect set and belongs to $\mathcal{S}$. Let B be a Cantor set. B is a perfect set, so it belongs to $\mathcal{S} \setminus \mathcal{S}_0$. Put $D = \mathbb{Q}$. The set $B + D$ is the null F_σ set. The set $\mathbb{R} \setminus (B + D)$ is G_δ set of full measure, so it is a Borel set and it is uncountable. Then $\mathbb{R} \setminus (B + D) \in \mathcal{S} \setminus \mathcal{S}_0$.

If we consider, instead of the pair $(\mathcal{M},\mathcal{I})$, a pair which consists of the same σ-algebra and smaller σ-ideal, then the new pair does not have Smítal property.

Lemma 3. *Let $\mathcal{I}_1$, $\mathcal{I}_2$ be σ-ideals such that $\mathcal{I}_2$ is invariant under translation, $\mathcal{I}_1 \neq \mathcal{I}_2$ and $\mathcal{I}_1 \subset \mathcal{I}_2 \neq 2^{\mathbb{R}}$. Then for any σ-algebra $\mathcal{M}$ such that $\mathcal{I}_2 \subset \mathcal{M}$, the pair $(\mathcal{M},\mathcal{I}_1)$ does not have Smítal property.*

Proof. In fact, if $\mathcal{I}_1 \neq \mathcal{I}_2$ and $\mathcal{I}_1 \subset \mathcal{I}_2$ then there exists a set $B \in \mathcal{I}_2 \setminus \mathcal{I}_1$. Let D be a dense and countable set. Then, $(B+D) \in \mathcal{I}_2$ and $\mathbb{R} \setminus (B+D) \notin \mathcal{I}_2$. ☐

Corollary 1.

a) *The pairs $(\mathcal{L},\mathcal{N}\cap\mathcal{K})$ and $(\mathcal{B},\mathcal{N}\cap\mathcal{K})$ do not have Smítal property.*

b) *The pairs $(\mathcal{L},\mathcal{I}_p)$, $(\mathcal{B},\mathcal{I}_p)$ do not have Smítal property.*

c) *Put $\mathcal{I} = \{\emptyset\}$. The pairs $(\mathcal{L},\mathcal{I})$, $(\mathcal{B},\mathcal{I})$, $(\mathcal{L}\cap\mathcal{B},\mathcal{I})$ and $(\beta,\mathcal{I})$ do not have Smítal property.*

From the above consideration, it follows that decreasing a σ-ideal, we always lose Smítal property. On the other hand, increasing a σ-ideal (when the σ-algebra is established), we can (but we do not need to) obtain Smítal property. Indeed, the pair $(\mathcal{L},\mathcal{I}_p)$ does not have Smítal property and the σ-ideal $\mathcal{N}\cap\mathcal{K}$ is bigger than $\mathcal{I}_p$, but the pair $(\mathcal{L},\mathcal{N}\cap\mathcal{K})$ does not have Smítal property, too. However, the pair $(\mathcal{L},\mathcal{N})$ has this property.

Directly from definition of Smítal's property, we have

Lemma 4. *If the pair $(\mathcal{M},\mathcal{I})$ does not have Smítal property and $\mathcal{M}_1$ is a σ-algebra including $\mathcal{M}$, then the pair $(\mathcal{M}_1,\mathcal{I})$ does not have Smítal property.*

Proof. Indeed, if the pair $(\mathcal{M},\mathcal{I})$ does not have Smítal property then there exists a set $B \in \mathcal{M}\setminus\mathcal{I}$ and a dense set D such that the set $\mathbb{R}\setminus(B+D)$ does not belong to $\mathcal{I}$. Obviously, $B \in \mathcal{M}_1 \setminus \mathcal{I}$. ☐

Corollary 2. *The pairs $(2^{\mathbb{R}},\mathcal{N}\cap\mathcal{K})$, $(\mathcal{L},\mathcal{I}_p)$, $(\mathcal{B},\mathcal{I}_p)$, $(2^{\mathbb{R}},\mathcal{I}_p)$ and $(2^{\mathbb{R}},\mathcal{S}_0)$ do not have Smítal property.*

Example 4. The pair $(2^{\mathbb{R}},\mathcal{N})$ does not have Smítal property.

Denote by $Z(L)$ a linear space over the field $\mathbb{Q}$ generated by the set L, i.e.,

$$Z(L) = \{x : x = \kappa_1 \cdot l_1 + \dots + \kappa_n \cdot l_n;\ n \in \mathbb{N};\ i = 1,\dots,n;\ l_i \in L;\ \kappa_i \in \mathbb{Q}\}.$$

Let H be a Hamel basis and h_0 be a fixed point of H. Observe that the set $Z\bigl(H\setminus\{h_0\}\bigr)$ is nonmeasurable ([1, Th. 2, p. 255]) and $Z\bigl(H\setminus\{h_0\}\bigr)+\mathbb{Q}\cdot h_0 = \mathbb{R}$. Moreover, the sets $Z\bigl(H\setminus\{h_0\}\bigr)$ and $Z\bigl(H\setminus\{h_0\}\bigr)+\kappa\cdot h_0$ are disjoint for any $\kappa \in \mathbb{Q}\setminus\{0\}$. Putting $B = Z\bigl(H\setminus\{h_0\}\bigr)$ and $D = \bigcup_{\kappa\in\mathbb{Q}\setminus\{0\}}(\kappa\cdot h_0)$, we obtain that $\mathbb{R}\setminus(B+D) = B \notin \mathcal{N}$.

In the same way, we check that the pair $(2^{\mathbb{R}},\mathcal{K})$ does not have Smítal property.

Suppose now that a σ-ideal $\mathcal{I} \subset 2^{\mathbb{R}}$ is invariable with respect to translations, i.e., for any real number x, the fact that $A \in \mathcal{I}$ implies that $(x + A) \in \mathcal{I}$.

Theorem 2. *For any σ-ideal $\mathcal{I}$, which is invariant with respect to translation, there exists a σ-algebra $\mathcal{M}$ such that the pair $(\mathcal{M}, \mathcal{I})$ has Smítal property.*

P r o o f. If $\mathcal{I} = 2^{\mathbb{R}}$ then we can put $\mathcal{M} = \mathcal{I}$. Therefore, we can assume that $\mathcal{I} \neq 2^{\mathbb{R}}$. The family $\mathcal{M}_{\mathcal{I}} = \{A \subset \mathbb{R} : A \in \mathcal{I} \text{ or } \mathbb{R} \setminus A \in \mathcal{I}\}$ is the σ-algebra, which includes $\mathcal{I}$. It is easy to check that the pair $(\mathcal{M}_I, \mathcal{I})$ has Smítal property. Indeed, if the set $B \in \mathcal{M}_I \setminus \mathcal{I}$, then $\mathbb{R} \setminus B \in \mathcal{I}$. For any number $d \in \mathbb{R}$, the set $\mathbb{R} \setminus (B + d) = \big((\mathbb{R} \setminus B) + d\big) \in \mathcal{I}$. Therefore, $(B + d) \in \mathcal{M}_I \setminus \mathcal{I}$. Since $(B + D) \supset (B + d)$ for any dense set D and for any $d \in D$, so $B + D$ belongs to $\mathcal{M}_I \setminus \mathcal{I}$. From there, $\mathbb{R} \setminus (B + D) \in \mathcal{I}$. □

Note that the invariance of $\mathcal{I}$ is essential. Indeed, if $\mathcal{I} = 2^{\mathbb{R}\setminus\{1\}}$ then the only σ-algebra including $\mathcal{I}$ is the σ-algebra $2^{\mathbb{R}}$. Let $B = \{1\}$ and $D = \mathbb{Q} \setminus \{0\}$. Then $B + D = \mathbb{Q} \setminus \{1\}$ and $\mathbb{R} \setminus (B + D) \notin \mathcal{I}$.

The σ-algebra $\mathcal{M}_{\mathcal{I}}$ constructed in the last proof consists only of "small sets" belonging to $\mathcal{I}$ and "big sets" which complements belong to $\mathcal{I}$. We will show that for any pair $(\mathcal{M}, \mathcal{I})$ having Smítal property, the family $\mathcal{M} \setminus \mathcal{I}$ consists of "big sets" in some sense.

Proposition 1. *If the pair $(\mathcal{M}, \mathcal{I})$ has Smítal property and $A \in \mathcal{M} \setminus \mathcal{I}$, then the smallest σ-ideal invariable with respect to translations containing σ-ideal $\mathcal{I}$ and the set A is equal to $2^{\mathbb{R}}$.*

P r o o f. Assume that $(\mathcal{M}, \mathcal{I})$ has Smítal property, fix a set $A \in \mathcal{M} \setminus \mathcal{I}$ and denote by $\mathcal{I}(A)$ the smallest σ-ideal containing $\mathcal{I} \cup \{A\}$, and invariable with respect to translations. Let $D = \mathbb{Q}$. The set $\mathbb{R} \setminus (A + D) \in \mathcal{I} \subset \mathcal{I}(A)$. Moreover, for any $d \in D$, the set $(A + d) \in \mathcal{I}(A)$, so $A + D = \bigcup_{d\in\mathbb{Q}}(A + d) \in \mathcal{I}(A)$. Therefore, the set $(A + D) \cup \big(\mathbb{R} \setminus (A + D)\big) = \mathbb{R}$ and every subset of set $\mathbb{R}$ belongs to the σ-ideal $\mathcal{I}(A)$. □

In [1], using a measurable hull, there is proved the "outer version" of Smítal's lemma for measure.

Theorem 3 ([1, Th. 1, p. 65])**.** *If a set $B \subset \mathbb{R}$ has a positive outer Lebesgue measure and D is a dense set, then the set $\mathbb{R} \setminus (B + D)$ has inner measure zero.*

We will show that Theorem 3 is equivalent to Lemma 1. In fact, there is an other description of the same property.

We will say that the pair $(\mathcal{M}, \mathcal{I})$ has outer Smítal property if for any set $B \notin \mathcal{I}$ and a dense set D, the set $\mathbb{R} \setminus (B + D)$ does not include a set belonging to $\mathcal{M} \setminus \mathcal{I}$.

Theorem 4. *A pair $(\mathcal{M}, \mathcal{I})$ has Smítal property if and only if it has outer Smítal's property.*

P r o o f. Assume that the pair $(\mathcal{M},\mathcal{J})$ does not have outer Smítal's property. It means that there exists a set $B \notin \mathcal{J}$ and a dense set D, such that $\mathbb{R} \setminus (B + D)$ includes a set $A \in \mathcal{M} \setminus \mathcal{J}$. Consider the set $\big(A + (-D)\big)$. It is easy to check that $\big(A + (-D)\big) \cap B = \emptyset$ and, consequently, $\mathbb{R} \setminus \big(A + (-D)\big) \supset B \notin \mathcal{J}$. Since $A \in \mathcal{M} \setminus \mathcal{J}$ and $(-D)$ is a dense set, the pair $(\mathcal{M},\mathcal{J})$ does not have Smítal property.

Now assume, that the pair $(\mathcal{M},\mathcal{J})$ does not have Smítal property. Then there exists a set $B_0 \in \mathcal{M} \setminus \mathcal{J}$ and a dense set D_0 such that $\mathbb{R} \setminus (B_0 + D_0) \notin \mathcal{J}$.

Let $B_1 = \mathbb{R} \setminus (B_0 + D_0)$ and $D_1 = (-D_0)$. We show that $(B_1 + D_1) \cap B_0 = \emptyset$. For any $x \in B_1 + D_1$ there exist elements $b_1 \in B_1$ and $d \in D_0$ such that $x = b_1 - d$. Then $x + d \in B_1$, so $x + d \notin B_0 + D_0$. It means that $x \notin B_0$.

Therefore, the sets $(B_1 + D_1)$ and B_0 are disjoint and consequently, $\mathbb{R} \setminus (B_1 + D_1) \supset B_0 \in \mathcal{M} \setminus \mathcal{J}$. □

In the definition of outer Smítal's property, we consider the sets $B \notin \mathcal{J}$. If we demand that the set $\mathbb{R} \setminus (B + D)$ belongs to a σ-ideal $\mathcal{J}$, then we will get another, stronger condition, in which σ-algebra $\mathcal{M}$ does not occur. Thus, we will say, that σ-ideal $\mathcal{J}$ satisfies condition $(\star)$ when, for any set $B \notin \mathcal{J}$ and any dense set D, the set $\mathbb{R} \setminus (B + D)$ belongs to the σ-ideal $\mathcal{J}$.

Immediately from condition $(\star)$, we have

Proposition 2. *A σ-ideal $\mathcal{J}$ satisfies condition $(\star)$ if and only if the pair $(2^{\mathbb{R}}, \mathcal{J})$ has Smítal property.*

Corollary 2 yields the following fact.

Proposition 3. *The σ-ideals $\mathcal{N}, \mathcal{K}, \mathcal{J}_p$ do not satisfy condition $(\star)$.*

Theorem 5. *The only σ-ideal, which satisfies condition $(\star)$, is $\mathcal{J} = 2^{\mathbb{R}}$.*

P r o o f. Assume that σ-ideal $\mathcal{J}$ satisfies condition $(\star)$. Let H be the Hamel basis and $h_0 \in H$. Consider the set $Z\big(H \setminus \{h_0\}\big)$ described in Example 4.

Suppose that there exists a rational number $q_0 \in \mathbb{Q}$ such that

$$B = Z\big(H \setminus \{h_0\}\big) + q_0 \cdot h_0$$

does not belong to $\mathcal{J}$. Let

$$D = \{qh_0 : q \in \mathbb{Q} \wedge q \neq 0\}.$$

For any $q \neq 0$,

$$B + q \cdot h_0 = Z\big(H \setminus \{h_0\}\big) + q_0 \cdot h_0 + q \cdot h_0 = Z\big(H \setminus \{h_0\}\big) + (q_0 + q)h_0$$

and $B \cap (B + qh_0) = \emptyset$.

Therefore, we have

$$B + D = \bigcup_{q \in D} \Big(Z\big(H \setminus \{h_0\}\big) + q_0 \cdot h_0 + q \cdot h_0 \Big)$$

$$= \left(\bigcup_{q \in D \setminus \{0\}} \Big(Z\big(H \setminus \{h_0\}\big) + (q_0 + q)h_0 \Big) \right) + \Big(Z\big(H \setminus \{h_0\}\big) + q_0 \cdot h_0 \Big).$$

Hence, the set $\mathbb{R} \setminus (B + D) = Z\big(H \setminus \{h_0\}\big) + q_0 \cdot h_0$ does not belong to σ-ideal $\mathcal{I}$. We get a contradiction. Therefore, for any rational number q the set $Z\big(H \setminus \{h_0\}\big) + q \cdot h_o$ belongs to σ-ideal $\mathcal{I}$. Hence, $\mathbb{R} = \bigcup_{q \in \mathbb{Q}} \big(Z\,(H \setminus \{h_0\} + h_0 \cdot q) \big)$ belongs to $\mathcal{I}$, so $\mathcal{I} = 2^{\mathbb{R}}$. □

The notion of the pair $(\mathcal{M}, \mathcal{I})$ having Smítal property and having outer Smítal property can also be considered for σ-algebras and σ-ideals of sets of topological group. Most of the results obtained in this paper remain true for topological groups.

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