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ON HOMEOMORPHISMS OF DENSITY TYPE TOPOLOGIES GENERATED BY FUNCTIONS

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ABSTRACT. The paper is concerned with homeomorphisms on topological spaces $(\mathbb{R}, \mathcal{T}_f)$, where $\mathcal{T}_f$ is a generalization of the density topology generated by a function f. It is shown that the density topology is not homeomorphic with any other topology $\mathcal{T}_f$ and that, under some assumptions, homeomorphic f-density topologies have to be comparable.

We denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{N}$ the set of positive integers, by $\mathcal{L}$ the family of Lebesgue measurable subsets of $\mathbb{R}$ and by $|E|$ the Lebesgue measure of a measurable set E. The family of all functions $f\colon (0;\infty) \to (0;\infty)$ such that

(A1) $\lim_{x\to 0^+} f(x) = 0$,

(A2) $\liminf_{x\to 0^+} \frac{f(x)}{x} < \infty$,

(A3) f is nondecreasing

is denoted by $\mathcal{A}$. Let $f \in \mathcal{A}$. We say that x is a *right-hand f-density point* of a measurable set E if

$$\lim_{h\to 0^+} \frac{|(x;x+h) \setminus E|}{f(h)} = 0.$$

By $\Phi_f^+(E)$ we denote the set of all right-hand f-density points of E. In the same way, one can define *left-hand f-density points* of E and the set $\Phi_f^-(E)$. We say that x is an *f-density point* of E if it is a right and a left-hand f-density point of E. By $\Phi_f(E)$ we denote the set of all f-density points of E, i.e., $\Phi_f(E) = \Phi_f^+(E) \cap \Phi_f^-(E)$. For any $f \in \mathcal{A}$, the family

$$\mathcal{T}_f = \big\{E \in \mathcal{L};\ E \subset \Phi_f(E)\big\}$$

forms a topology stronger than the natural topology on the real line (see [1, Th. 7] and [3, Th. 1]). It is called the *f-density topology.* Properties of Lebesgue measure imply that any f-density topology is invariant with respect to translations and symmetries. In [2], it has been shown that properties of f-density operator Φ_f

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and f-density topology $\mathcal{T}_f$ depend on value of $\liminf_{x\to 0^+} \frac{f(x)}{x}$. The family of all functions $f \in \mathcal{A}$ with $\liminf_{x\to 0^+} \frac{f(x)}{x} > 0$ is denoted by $\mathcal{A}^1$. For any f from $\mathcal{A}^1$, $\mathcal{T}_f$ has properties similar to properties of the density topology. In particular, it is completely regular but not normal. Topologies generated by functions from $\mathcal{A} \setminus \mathcal{A}^1$ are even not regular (see [2, Th. 7 and Th. 9]).

The family of all increasing sequences tending to infinity is denoted by $\mathcal{S}$. Let $\langle s\rangle \in \mathcal{S}$. We say that x is an $\langle s\rangle$-*density point* of a measurable set E if

$$\lim_{n\to\infty} \frac{\left|E \cap \left[x - \frac{1}{s_n}; x + \frac{1}{s_n}\right]\right|}{\frac{2}{s_n}} = 1.$$

The set of all $\langle s\rangle$-density points of E is denoted by $\Phi_{\langle s\rangle}(E)$. In [6], it was proved that $\Phi_{\langle s\rangle}$ is a lower density operator and the family

$$\mathcal{T}_{\langle s\rangle} = \left\{E \in \mathcal{L};\ E \subset \Phi_{\langle s\rangle}(E)\right\}$$

is a topology containing the density topology $\mathcal{T}_d$.

It is clear that $\mathcal{T}_f = \mathcal{T}_{\langle s\rangle} = \mathcal{T}_d$ for $f(x) = x$ and $s_n = n$. Any $\langle s\rangle$-density topology is an f-density topology for some $f \in \mathcal{A}^1$. It is sufficient to set

$$f(x) = \frac{1}{s_n} \qquad \text{for} \quad x \in \left[\frac{1}{s_n}; \frac{1}{s_{n-1}}\right).$$

However, there is a function in $\mathcal{A}^1$ which generates the topology different from each $\mathcal{T}_{\langle s\rangle}$ (see [3, Th. 5 and Th. 6]).

Proposition 1. *For any function $f \in \mathcal{A}$, there is a sequence $\langle s\rangle \in \mathcal{S}$ such that $\mathcal{T}_f \subset \mathcal{T}_{\langle s\rangle}$.*

P r o o f. Since $M = \liminf_{x\to 0+} \frac{f(x)}{x} < \infty$, there is a decreasing sequence (a_n) tending to 0 such that

$$\frac{f(a_n)}{a_n} < M + 1.$$

It is sufficient to show that condition $0 \in \Phi_f^+(E)$ implies $0 \in \Phi^+_{\frac{1}{\langle a\rangle}}(E)$. Suppose that

$$\lim_{x\to 0+} \frac{|(0;x) \setminus E|}{f(x)} = 0.$$

Then, we have

$$0 \le \frac{|(0;a_n) \setminus E|}{a_n} < \frac{|(0;a_n) \setminus E|}{f(a_n)} (M+1) \underset{n\to\infty}{\longrightarrow} 0,$$

and consequently, $0 \in \Phi^+_{\frac{1}{\langle a\rangle}}(E)$. ☐

Theorem 1. *The families of connected sets in the natural topology and any f-density topology are equal.*

P r o o f. In [7, Th. 10], it was proved that the families of connected sets in the natural topology $\mathcal{T}_0$ and any $\langle s\rangle$-density topology are equal. Since $\mathcal{T}_0 \subset \mathcal{T}_f \subset \mathcal{T}_{\langle s\rangle}$ for some $\langle s\rangle \in \mathcal{S}$, we conclude our claim. $\square$

Lemma 1. *If $\mathcal{T}_{f_1} \setminus \mathcal{T}_{f_2} \neq \emptyset$ for some f_1, f_2 from $\mathcal{A}$, then there are a positive number η and sequences (a_n), (b_n) satisfying $0 < b_{n+1} < a_n < b_n$, $\lim_{n\to\infty} b_n = 0$ such that 0 is a right-hand f_1-density point of the complement of the set*

$$\bigcup_{n=1}^{\infty} [a_n; b_n]$$

and

$$\frac{b_n - a_n}{f_2(b_n)} > \eta \tag{1}$$

for each n.

P r o o f. Let A be a set from $\mathcal{T}_{f_1} \setminus \mathcal{T}_{f_2}$. There exists an f_1-density point x of A which is not an f_2-density point of A. Replacing A with $A - x$ or with $-A + x$, we may assume that $0 \in \Phi^+_{f_1}(A) \setminus \Phi^+_{f_2}(A)$. Thus, there are a positive number η and a decreasing sequence (h_n) tending to 0 such that

$$\frac{|[0; h_n] \setminus A|}{f_2(h_n)} > 2\eta.$$

We will define sequences (b_n) and (c_n). Let $b_1 = h_1$ and c_1 be a point from $(0; b_1)$ such that

$$\frac{|[c_1; b_1] \setminus A|}{f_2(b_1)} > \eta.$$

Suppose that we have defined b_k, c_k with $c_k < b_k$. Let n_k be a number satisfying $h_{n_k} < c_k$. We set $b_{k+1} = h_{n_k}$ and define c_{k+1} as a point from $(0; b_k)$ such that

$$\frac{|[c_{k+1}; b_{k+1}] \setminus A|}{f_2(b_{k+1})} > \eta.$$

Thus, we have defined (b_n) and (c_n). Let

$$a_n = b_n - |[c_n; b_n] \setminus A|$$

and

$$B = \bigcup_{n=1}^{\infty} [a_n; b_n].$$

The inequality (1) is evident. To finish the proof, we have to show that 0 is a right-hand f_1-density point of $\mathbb{R} \setminus B$. Since

$$\lim_{x\to 0+} \frac{|[0; x] \setminus A|}{f_1(x)} = 0.$$

it suffices to prove that for $x \in (0; b_1]$

$$|B \cap [0; x]| \leq |[0; x] \setminus A|. \tag{2}$$

For every n, we have

$$|B \cap [0; b_n]| = \sum_{k=n}^{\infty} (b_k - a_k) = \sum_{k=n}^{\infty} |[c_k; b_k] \setminus A| \leq |[0; b_n] \setminus A| .$$

Consequently, for $x \in (b_{n+1}; a_n]$

$$|B \cap [0; x]| = |B \cap [0; b_{n+1}]| \leq |[0; b_{n+1}] \setminus A| \leq |[0; x] \setminus A| ,$$

and for $x \in (a_n; b_n]$

$$|B \cap [0; x]| = |B \cap [0; b_n]| - |B \cap [x; b_n]| \leq |[0; b_n] \setminus A| - (b_n - x) \leq |[0; x] \setminus A| .$$

This gives (2), and completes the proof. $\square$

Theorem 2. *If $f_1, f_2 \in \mathcal{A}$ and $h\colon (\mathbb{R}, \mathcal{T}_{f_1}) \to (\mathbb{R}, \mathcal{T}_{f_2})$ is a homeomorphism, then*

(1) *h and h^{-1} are continuous (in a usual sense), strictly monotonic and satisfy Lusin's condition (N),*

(2) *the sets*

$$\begin{aligned} A &:= \{x;\ \textit{there exists derivative}\, h'(x)\}, \\ B &:= \Big\{x;\ \textit{there exists derivative}\, (h^{-1})'\big(h(x)\big)\Big\} \end{aligned}$$

have full measure,

(3) *if $h'(x) = 1$ for every $x \in A \cap B$, then $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$.*

P r o o f. We will show that h is continuous, strictly monotonic and satisfies condition (N) (the proof for h^{-1} is similar). Theorem 1 implies that, for any open interval $(a; b)$, the set $J = h^{-1}((a; b))$ is an interval, too. Obviously, interval $(a; b)$ is $\mathcal{T}_{f_2}$-open, hence J is $\mathcal{T}_{f_1}$-open, and consequently, $J \subset \Phi_{f_1}(J)$. Since no end of an interval can be its f_1-density point, the interval J has to be open. Thus, h is continuous. Since h is also an injection, it is strictly monotonic. Let P be a null set. Then P and all subsets of P are closed in $\mathcal{T}_{f_1}$. Consequently, $h(P)$ and all its subsets are closed in $\mathcal{T}_{f_2}$, and so they are measurable. Hence, $h(P)$ is of measure zero, which finishes the proof of (1).

Any monotonic function is almost everywhere differentiable, and so, A has full measure. In the same manner, we conclude that the set

$$C := \Big\{y;\ \text{there exists derivative } \left(h^{-1}\right)'(y)\Big\}$$

has full measure. From (1), it follows that $B = h^{-1}(C)$ is of full measure.

Suppose that $h'(x) = 1$ for $x \in A \cap B$. By (1) and Banach-Zarecki theorem we deduce that $h(x)$ and $h(x) - x$ are absolutely continuous on any interval $[a; b]$ (see [8]). Consequently, $h(x) = x$, which gives $\mathcal{T}_{f_1} = \mathcal{T}_{f_2}$. $\square$

Theorem 3. *Let f_1, f_2 be in $\mathcal{A}^1$. If topological spaces $(\mathbb{R}, \mathcal{T}_{f_1})$ and $(\mathbb{R}, \mathcal{T}_{f_2})$ are homeomorphic, then topologies $\mathcal{T}_{f_1}$ and $\mathcal{T}_{f_2}$ are comparable, i.e., $\mathcal{T}_{f_1} \subset \mathcal{T}_{f_2}$ or $\mathcal{T}_{f_2} \subset \mathcal{T}_{f_1}$.*

Proof. Suppose, contrary to our claim, that $\mathcal{T}_{f_1}$ and $\mathcal{T}_{f_2}$ are not comparable. Let h be a homeomorphism from $(\mathbb{R}, \mathcal{T}_{f_1})$ onto $(\mathbb{R}, \mathcal{T}_{f_2})$. By Theorem 2, h is strictly monotonic and for some x_0, there exist derivatives $h'(x_0)$, $(h^{-1})'\big(h(x_0)\big)$ with $h'(x_0) = c \neq 1$. Since f-density topologies are invariant with respect to translations and symmetries, we can assume that h is increasing and $h(x_0) = x_0 = 0$. Moreover, we can also assume that $0 < c < 1$ (we replace h with h^{-1}, if necessary).

Since $\liminf_{x \to 0+} \frac{f_2(x)}{x} > 0$, there are positive numbers β, δ such that

$$\frac{x}{f_2(x)} < \beta \qquad \text{for} \quad x \in (0; \delta).$$

From Lemma 1, it follows that there is a positive number η and a set

$$B = \bigcup_{n=1}^{\infty} [a_n; b_n] \subset (0; \delta)$$

such that $B' \in \mathcal{T}_{f_1}$ and

$$\frac{b_n - a_n}{f_2(b_n)} > \eta.$$

The proof will be completed by showing that $0 \notin \Phi_{f_2}\big(h(B')\big)$. Let $\varepsilon = \frac{c\eta}{4\beta}$. Since $0 < h'(0) = c < 1$, we have

$$\left|\frac{h(a_n)}{a_n} - c\right| < \varepsilon, \quad \left|\frac{h(b_n)}{b_n} - c\right| < \varepsilon \quad \text{and} \quad h(b_n) < b_n$$

for sufficiently large n. Hence,

$$(c-\varepsilon)\, a_n \,<\, h(a_n) < (c+\varepsilon)\, a_n,$$
$$(c-\varepsilon)\, b_n \,<\, h(b_n) < (c+\varepsilon)\, b_n,$$

and

$$h(b_n) - h(a_n) \,>\, c(b_n - a_n) - 2\varepsilon b_n.$$

Thus,

$$h(B) \cap [0; b_n] = \bigcup_{k=1}^{\infty} \big[h(a_k); h(b_k)\big] \cap [0; b_n] \supset \big[h(a_n); h(b_n)\big],$$

and consequently,

$$\begin{aligned}\frac{|[0;b_n]\setminus h(B')|}{f_2(b_n)} &\geq \frac{h(b_n)-h(a_n)}{f_2(b_n)}\\ &\geq c\frac{(b_n-a_n)}{f_2(b_n)}-2\varepsilon\frac{b_n}{f_2(b_n)}\\ &> c\eta-2\varepsilon\beta\\ &= \frac{c\eta}{2}>0,\end{aligned}$$

which gives $0\notin\Phi_{f_2}\big(h(B')\big)$. $\square$

There exist functions $f_1, f_2\in\mathcal{A}^1$ such that topological spaces $(\mathbb{R},\mathcal{T}_{f_1})$ and $(\mathbb{R},\mathcal{T}_{f_2})$ are homeomorphic and $\mathcal{T}_{f_1}\neq\mathcal{T}_{f_2}$. An easy example can be found between topologies generated by sequences. Let $\alpha\in(0;1)$, $\langle s\rangle=(n!)_{n\in\mathbb{N}}$ and $\langle\alpha s\rangle=(\alpha n!)_{n\in\mathbb{N}}$. From [6, Th. 4] and [5, Remark 13], it follows that $\mathcal{T}_{\langle s\rangle}\subsetneq\mathcal{T}_{\langle\alpha s\rangle}$ and the function $h(x)=\alpha x$ is a homeomorphism from $\big(\mathbb{R},\mathcal{T}_{\langle s\rangle}\big)$ onto $\big(\mathbb{R},\mathcal{T}_{\langle\alpha s\rangle}\big)$.

There are also homeomorphic topologies generated by functions from $\mathcal{A}^1$ and different from topologies generated by sequences. Let $f_\alpha(x)=f\left(\frac{\alpha}{x}\right)$. It is easy to check that $h(x)=\alpha x$ is a homeomorphism from $(\mathbb{R},\mathcal{T}_f)$ onto $(\mathbb{R},\mathcal{T}_{f_\alpha})$ for $\alpha\neq 0$. In [3, Th. 6], there is defined a function $f^*\in\mathcal{A}^1$ such that $\mathcal{T}_{f^*}\notin\big\{\mathcal{T}_{\langle s\rangle};\langle s\rangle\in\mathcal{S}\big\}$. From [4], it follows that $\mathcal{T}_{f^*_\alpha}\notin\big\{\mathcal{T}_{\langle s\rangle};\langle s\rangle\in\mathcal{S}\big\}$ and $\mathcal{T}_{f^*_\alpha}\subsetneq\mathcal{T}_{f^*_\beta}$ for any $\alpha>\beta>1$.

Theorem 4. *The density topology $\mathcal{T}_d$ is not homeomorphic with any other topology $\mathcal{T}_f$.*

P r o o f. Suppose on the contrary that for some $f\in\mathcal{A}$, the topology $\mathcal{T}_f$ is different but homeomorphic with $\mathcal{T}_d$. Since topologies generated by functions from $\mathcal{A}\setminus\mathcal{A}^1$ are not regular, $f\in\mathcal{A}^1$, and consequently, $\mathcal{T}_f\supsetneq\mathcal{T}_d$ (compare [3, Th. 3]).

Let h be a homeomorphism from $(\mathbb{R},\mathcal{T}_f)$ onto $(\mathbb{R},\mathcal{T}_d)$. Following the proof of Theorem 3, we can assume that $h(0)=0$, h and h^{-1} are differentiable at 0 and $h'(0)>0$. Since the density topology is invariant with respect to homothetic transformations, the function $h_\alpha(x)=\alpha h(x)$ is a homeomorphism from $(\mathbb{R},\mathcal{T}_f)$ onto $(\mathbb{R},\mathcal{T}_d)$ for each positive α. Replacing h with h_α if necessary, we can assume that $h'(0)<1$. Thus, it is sufficient to repeat the proof of Theorem 3 to get a contradiction. $\square$

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