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# ON REPRESENTATION OF MULTIMEASURE

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ABSTRACT. We consider a multimeasure with the Radon-Nikodym derivative and apply its Castaing representation to get a representation of the multimeasure.

We assume that $X$ is a real normed space; we denote by $P_o(X)$ the family of nonempty subsets of $X$, by $P_f(X)$, $P_{fc}(X)$ the families of closed and closed convex members of $P_o(X)$, respectively. In the sequel, $(T,\mathcal{A})$ stands for measurable space, $\mu$ for positive real measure.

We say that a multifunction $F\colon T \to P_o(X)$ is measurable if for every open set $U \subset X$

$$F^-(U) := \{t \in T:\ F(t) \cap U \neq \emptyset\} \in \mathcal{A}.$$

The function $f\colon T \to X$ is a selection of $F$ if $f(t) \in F(t)$ for $t \in T$.

For measurable multifunctions we have the following characterization (see [3], [2], [5]).

**Theorem 1.** *Let $X$ be a complete separable metric space. A multifunction $F\colon T \to P_f(X)$ is measurable if and only if there exists a sequence $\{f_n : n \in \mathbb{N}\}$ of measurable selections of $F$ such that*

$$F(t) = \operatorname{cl}\{f_n(t):\ n \in \mathbb{N}\}, \qquad \textit{for} \quad t \in T.$$

The sequence $\{f_n : n \in \mathbb{N}\}$ is called Castaing representation of $F$.

A measurable multifunction $F\colon T \to P_0(X)$ is Aumann integrable if $S_F$, the set of all Bochner integrable selections of $F$, is nonempty. Then Aumann integral $\int_A F d\mu := \{\int_A f d\mu : f \in S_F\}$ is the set of all Bochner integrals of fuctions of $S_F$.

We say that a multifunction $M\colon \mathcal{A} \to P_f(X)$ is a weak multimeasure if for every $x^* \in X^*$, a mapping $A \mapsto \sup\{x^*(x) : x \in M(A)\}$ is a signed measure. $M$ is additive if and only if $M(A \cup B) = \operatorname{cl}\big(M(A) + M(B)\big)$ for disjoint sets $A, B \in \mathcal{A}$.

Example 1 ([6, Ex. 4.6]). If $F\colon T \to P_f(X)$ is an Aumann integrable multifunction, then a multifunction $A \mapsto \operatorname{cl}\int_A F d\mu$, $A \in \mathcal{A}$, is a weak multimeasure.

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A multifunction $F\colon T \to P_f(X)$ is the Radon-Nikodym derivative of a multimeasure $M\colon \mathcal{A} \to P_f(X)$ with respect to $\mu$ if

$$\int_A F d\mu = M(A) \qquad \text{for every} \quad A \in \mathcal{A}.$$

Since the Radon-Nikodym derivative of a multimeasure $M$ has Castaing representation, we can look for a similar representation for $M$.

A vector measure $m\colon \mathcal{A} \to X$ is called a measure selection of $M$ if $m(A) \in M(A)$ for every $A \in \mathcal{A}$. The set of all measure selections of multimeasure $M$ will be denoted by $S_M$.

Let $\mathcal{M} := \{m\colon \mathcal{A} \to X : m \text{ is a vector measure}\}$.

We say that $D \subset \mathcal{M}$ is decomposable if and only if for every measurable set $A \in \mathcal{A}$ and $m, n \in D$

$$m\chi_A + n\chi_{T\setminus A} \in D,$$

where $m\chi_A(B) = m(B \cap A)$.

We say that $A_1, \ldots, A_n \in \mathcal{A}$ is a partition of $T$ if the sets are mutually disjoint and $\bigcup_{i=1}^n A_i = T$.

A decomposable hull of $S \subset \mathcal{M}$ is the smallest, in the sense of inclusion, decomposable set containing $S$. Moreover, (cf. [7]),

$$\begin{aligned} \operatorname{dec} S &:= \bigcap \{D : S \subset D,\ D \text{ is decomposable}\} \\ &= \left\{ \sum_{i=1}^k m_i \chi_{A_i} : k \in \mathbb{N},\ m_i \in S,\ \{A_1, \ldots, A_k\} \text{ is a partition of } T \right\}. \end{aligned}$$

**Lemma 1.** *Let $(T, \mathcal{A}, \mu)$ be a measure space, $X$ be a normed space. Let $M\colon \mathcal{A} \to P_f(X)$ be a multimeasure with the Radon-Nikodym derivative $F\colon T \to P_f(X)$. Then $M$ is an additive weak multimeasure with a decomposable set $S_M$.*

P r o o f. According to Example 1, $M$ is a weak multimeasure, since $M$ has closed values and $M(A) = \int_A F d\mu$ for $A \in \mathcal{A}$.

To prove the additivity of $M$, take disjoint sets $A, B \in \mathcal{A}$.

$$x = \int_{A\cup B} f\, d\mu = \int_A f\, d\mu + \int_B f\, d\mu \in M(A) + M(B) \subset \operatorname{cl}\big(M(A) + M(B)\big).$$

If $x \in M(A) + M(B)$, then there exist $f, g \in S_F$ such that

$$x = \int_A f\, d\mu + \int_B g\, d\mu = \int_{A\cup B} (\chi_A f + \chi_B g)\, d\mu = \int_{A\cup B} (\chi_A f + \chi_{T\setminus A} g)\, d\mu.$$

Since $S_F$ is decomposable (see [5]), $\chi_A f + \chi_{T\setminus A} g \in S_F$ and we have

$$x \in \int_{A\cup B} F\, d\mu = M(A).$$

Therefore,

$$M(A) + M(B) \subset M(A \cup B)$$

and, by the closedness of values of $M$, we have

$$\mathrm{cl}\big(M(A) + M(B)\big) \subset M(A \cup B).$$

Finally, on account of Lemma 4 in [7], the set $S_M$ is decomposable. □

Let $m_n \colon T \to X$, $n \in \mathbb{N}$ be vector measures. It is natural to ask whether a multifunction

$$A \mapsto M(A) := \mathrm{cl}\,\big\{m_n(A) : n \in \mathbb{N}\big\}, \qquad A \in \mathcal{A}$$

is a weak multimeasure with a decomposable set $S_M$. The answer is in the following example (see [7]).

Example 2. Let $T = [0,1] \subset \mathbb{R}$, $\mathcal{A}$ be $\sigma$-algebra of Lebesgue measurable subsets of $T$. Let $m$ be Lebesgue measure on $\mathcal{A}$. Then multifunction given by

$$M(A) = \big\{m(A), 2m(A)\big\}, \qquad A \in \mathcal{A}$$

is a weak multimeasure but $S_M$ is not decomposable. Consequently, $M$ has not the Radon-Nikodym derivative.

**Theorem 2.** *Let $(T, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space, $X$ be a separable Banach space. Let $M \colon \mathcal{A} \to P_f(X)$ be a multimeasure with the Radon-Nikodym derivative $F \colon T \to P_f(X)$. Then there exists a sequence $\{m_n : n \in \mathbb{N}\}$ of measure selections of $M$ such that*

$$M(A) = \mathrm{cl}\,\big\{m(A) : m \in \mathrm{dec}\,\{m_n : n \in \mathbb{N}\}\big\}, \quad A \in \mathcal{A}.$$

Proof. Let $\{f_n : n \in \mathbb{N}\}$ be Castaing representation of the measurable multifunction $F$. Define vector measures

$$m_n(A) := \int_A f_n \, d\mu, \qquad \text{for} \quad n \in \mathbb{N}, \quad A \in \mathcal{A}$$

and let $S := \{m_n : n \in \mathbb{N}\}$.

It is clear that for every $A \in \mathcal{A}$, $m_{k_1}, \dots, m_{k_n} \in S$ and a partition $A_1, \dots, A_n$ of $T$

$$\left(\sum_{i=1}^{n} m_{k_i} \chi_{A_i}\right)(A) = \sum_{i=1}^{n} m_{k_i}(A \cap A_i) = \sum_{i=1}^{n} \int_{A \cap A_i} f_{k_i} \, d\mu = \int_A \left(\sum_{i=1}^{n} \chi_{A_i} f_{k_i}\right) d\mu,$$

where $\sum_{i=1}^{n} \chi_{A_i} f_{k_i} \in \mathrm{dec}\,\{f_n : n \in \mathbb{N}\}$. On the other hand, by the above equalities, we see that any function $g \in \mathrm{dec}\,\{f_n : n \in \mathbb{N}\}$ generates measure from $\mathrm{dec}\, S$.

Therefore,

$$\operatorname{dec}\{m_n : n \in \mathbb{N}\} = \left\{ \int_{(\cdot)} g\, d\mu : g \in \operatorname{dec}\{f_n : n \in \mathbb{N}\} \right\} \tag{1}$$

and consequently, for every $A \in \mathcal{A}$,

$$\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\} = \left\{ \int_A g\, d\mu : g \in \operatorname{dec}\{f_n : n \in \mathbb{N}\} \right\}. \tag{2}$$

We will show that for every $A \in \mathcal{A}$,

$$M(A) = \operatorname{cl}\big\{m(A) : m \in \operatorname{dec} S\big\}.$$

Observe that $S \subset S_M$ and on account of Lemma 1 $S_M$ is decomposable. Therefore, $\operatorname{dec} S \subset S_M$ and

$$\big\{m(A) : m \in \operatorname{dec} S\big\} \subset M(A), \qquad A \in \mathcal{A}.$$

Consequently, by the closedness of values of $M$,

$$\operatorname{cl}\big\{m(A) : m \in \operatorname{dec} S\big\} \subset M(A).$$

To prove the converse inclusion, take $A \in \mathcal{A}$ and $x \in M(A)$. Then there exists $f \in S_F$ such that $x = \int_A f\, d\mu$. Fix $\epsilon > 0$. According to Lemma 1.3 in [5], there exist $f_{k_1}, \dots, f_{k_n} \in \{f_n : n \in \mathbb{N}\}$ and $A_1, \dots, A_n$ a partition of $T$ such that

$$\int_T \left\| f - \sum_{i=1}^n \chi_{A_i} f_{k_i} \right\| d\mu < \epsilon.$$

Thus, the function $g := \sum_{i=1}^n \chi_{A_i} f_{k_i} \in \operatorname{dec}\{f_n : n \in \mathbb{N}\}$ and, moreover,

$$\left\| x - \int_A g\, d\mu \right\| = \left\| \int_A f\, d\mu - \int_A g\, d\mu \right\| \leq \int_A \|f - g\|\, d\mu \leq \int_T \|f - g\|\, d\mu < \epsilon.$$

Consequently,

$$x \in \operatorname{cl}\left\{ \int_A g\, d\mu : g \in \operatorname{dec}\{f_n : n \in \mathbb{N}\} \right\}$$

which together with (2) completes the proof. □

According to (1) we have a reformulation of the above theorem.

**Corollary 1.** *Let $(T, \mathcal{A}, \mu)$ be a $\sigma$-finite measure space, $X$ be a separable Banach space. Let $M\colon \mathcal{A} \to P_f(X)$ be a multimeasure with the Radon-Nikodym* *derivative* $F\colon T \to P_f(X)$. *Then*

$$M(A) = \operatorname{cl}\left\{ m(A) : m \in \operatorname{dec}\left\{ \int_{(\cdot)} f_n \, d\mu : n \in \mathbb{N} \right\}\right\},$$

*where* $\{f_n : n \in \mathbb{N}\}$ *is Castaing representation of* $F$.

A partial converse of Theorem 2 is following.

**Theorem 3.** *Let* $(T, \mathcal{A}, \mu)$ *be a finite measure space,* $X$ *be a separable Banach space with the Radon-Nikodym property. If* $\{m_n : n \in \mathbb{N}\}$ *is a sequence of vector measures of bounded variation absolutely continuous with respect to* $\mu$*, then there exists Aumann integrable multifunction* $F\colon T \to P_f(X)$ *such that*

$$M(A) := \operatorname{cl}\big\{ m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\} = \operatorname{cl}\int_A F d\mu, \qquad A \in \mathcal{A}.$$

P r o o f. Let $S := \{m_n : n \in \mathbb{N}\}$ and $M(A) := \operatorname{cl}\big\{m(A) : m \in \operatorname{dec} S\big\}$ for $A \in \mathcal{A}$. Since $X$ has the Radon-Nikodym property and $m_n \ll \mu$ for $n \in \mathbb{N}$ there exists $f_n := \frac{dm_n}{d\mu}$ for every $n \in \mathbb{N}$. Observe that analogously to the previous theorem, conditions (1) and (2) are satisfied.

Define $F(t) := \operatorname{cl}\big\{f_n(t) : n \in \mathbb{N}\big\}$. Observe that each $f_n$ is integrable, therefore $S_F \neq \emptyset$. We will show that $M(A) = \operatorname{cl}\int_A F \, d\mu$ for $a \in \mathcal{A}$.

Take $A \in \mathcal{A}$. By (2)

$$\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\} = \left\{ \int_A g \, d\mu : g \in \operatorname{dec}\{f_n : n \in \mathbb{N}\}\right\} \subset \int_A F \, d\mu$$

hence,

$$\operatorname{cl}\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\} \subset \operatorname{cl}\int_A F \, d\mu.$$

Now, let $A \in \mathcal{A}$ and $x \in \int_A F \, d\mu$. There exists $f \in S_F$ such that $x = \int_A f \, d\mu$.

Fix $\epsilon > 0$. According to Lemma 1.3 in [5], there exist $f_{k_1}, \dots, f_{k_n} \in \{f_n : n \in \mathbb{N}\}$ and $A_1, \dots, A_n$ a partition of $T$ such that

$$\int_T \left\| f - \sum_{i=1}^{n} \chi_{A_i} f_{k_i} \right\| d\mu < \epsilon.$$

Define $g := \sum_{i=1}^{n} \chi_{A_i} f_{k_i}$. It follows that $g \in \operatorname{dec}\{f_n : n \in \mathbb{N}\}$ and

$$\left\| x - \int_A g \, d\mu \right\| = \left\| \int_A f \, d\mu - \int_A g \, d\mu \right\| \leq \int_T \|f - g\| \, d\mu < \epsilon$$

which means that (see (2))

$$x \in \operatorname{cl}\left\{\int_A g\,d\mu : g \in \operatorname{dec}\{f_n : n \in \mathbb{N}\}\right\} = \operatorname{cl}\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\}.$$

Thus, $\int_A F\,d\mu \subset M(A)$ which with closedness of $M(A)$ completes the proof. □

By the above proof and by Theorem 4.5 in [5], we have what follows.

**Corollary 2.** *Let $(T, \mathcal{A})$ be a measurable space and $\mu$ be finite measure with no atoms, let $X$ be a separable reflexive Banach space. If $\{m_n : n \in \mathbb{N}\}$ is a sequence of vector measures of bounded variation absolutely continuous with respect to $\mu$, then there exists Aumann integrable multifunction $F\colon T \to P_{fc}(X)$ such that*

$$\operatorname{cl}\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\} = \int_A F\,d\mu, \qquad A \in \mathcal{A}.$$

*Moreover, $\left\{\frac{dm_n}{d\mu} : n \in \mathbb{N}\right\}$ is Castaing represntation of F.*

P r o o f. According to the above theorem, there exists $F\colon T \to P_f(X)$ such that

$$\operatorname{cl}\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\} = \operatorname{cl}\int_A F\,d\mu, \qquad A \in \mathcal{A}.$$

By Corollary 4.3 in [5], for $A \in \mathcal{A}$

$$\operatorname{cl}\int_A F\,d\mu = \operatorname{cl}\int_A \operatorname{cl}\operatorname{conv} F\,d\mu.$$

However, on account of Theorem 4.5 in [5], the integral $\int_A \operatorname{cl}\operatorname{conv} F d\mu$ is closed, so $M(A) = \int_A \operatorname{cl}\operatorname{conv} F d\mu$ for, $A \in \mathcal{A}$ and the proof is complete. □

Finally, we have a characterization of multimeasures with a representation generated by Castaing representation of their Radon-Nikodym derivative.

**Theorem 4.** *Let $(T, \mathcal{A})$ be a measurable space and $\mu$ be finite measure with no atoms, let $X$ be a separable reflexive Banach space. A multimeasure $M\colon \mathcal{A} \to P_f(X)$ has the Radon-Nikodym derivative $F\colon T \to P_{fc}(X)$ if and only if there exists a sequence of vector measures $\{m_n : n \in \mathbb{N}\}$ of bounded variation absolutely continuous with respect to $\mu$ such that*

$$M(A) = \operatorname{cl}\big\{m(A) : m \in \operatorname{dec}\{m_n : n \in \mathbb{N}\}\big\}.$$

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