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A HELLY THEOREM IN METRIC SPACES AND MAJORED OPERATIONS

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ABSTRACT. We present a generalization of a Helly type theorem given in [Duchoň, M.—Maličký, P.: *A Helly theorem for functions with values in metric spaces,* Tatra Mt. Math. Publ. **44** (2009), 159–168.] for sequences of functions with values in metric spaces and apply it to representations of some majored mappings on the space of continuous functions. A generalization of the Riesz theorem is formulated and proved. In particular, a representation of certain majored linear operators on the space of continuous functions, into a Banach space.

1. Introduction

It is well known that the following theorem is true, [BDS], [Na], [W].

Riesz representation theorem. *Every continuous linear functional L on the set of continuous functions f defined on $[0,1]$ has the form*

$$Lf = \int_0^1 f(s)\, dg(s) \tag{R}$$

with a function g of bounded variation on $[0,1]$.

This theorem has many extensions and generalizations with various proofs. One of the possible proofs is based on the Helly theorem [Na] and also on the moment problem theorem [Na]. It can be shown that the problem of determining the general continuous linear functionals on the set of continuous functions is equivalent to that of determining the set of all moment sequences. It is our purpose to extend this result to majored linear operators from continuous functions to Banach spaces.

Helly's theorem had been of some importance a long time above all in the probability theory in connection with a problem of moments of distributions.

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Recall that in this connection, real-valued nondecreasing functions f on the interval $[a, b]$ of the real line are considered and that the following facts are true, [Na].

(1) (*First Helly's theorem*) Given a uniformly bounded sequence (f_n) of real--valued nondecreasing functions, there exists a subsequence (f_{n_k}) of (f_n) converging to a real-valued nondecreasing function f on $[a, b]$.

(2) (*Second Helly's theorem*) Given a sequence (f_n) of real-valued nondecreasing functions on $[a, b]$, converging to a real-valued nondecreasing function f, then, for every continuous function g on $[a, b]$, we have

$$\lim_{n\to\infty} \int_a^b g(t)\, df_n(t) = \int_a^b g(t)\, df(t).$$

More generally there are true the following facts.

(1) (*First Helly's theorem*) Given the sequence (f_n) of complex-valued functions of uniformly bounded variation on $[a, b]$ such that for some $x_0 \in [a, b]$ the sequence $\big(f_n(x_0)\big)$ is bounded then there exists a subsequence (f_{n_k}) of (f_n) converging to a some function f of bounded variation on $[a, b]$.

(2) (*Second Helly's theorem*) Given a sequence (f_n) of functions of uniformly bounded variation on $[a, b]$, converging to a some function f of bounded variation, then, for every continuous function g on $[a, b]$, we have

$$\lim_{n\to\infty} \int_a^b g(t)\, df_n(t) = \int_a^b g(t)\, df(t)\,.$$

2. A Helly type theorem in metric spaces

First we recall the definition of a bounded variation of the function with values in a metric space, [DM].

Definition. Let D be a subset of the real line, (X, d) be metric space and $h\colon D \to X$ be a function. If the set of all sums $\sum_{i=1}^{n} d\big(h(t_{i-1}), h(t_i)\big)$, where $(t_i)_{i=0}^{n}$ is an increasing sequence of elements of D, is bounded then g is said to be a function of bounded variation on D. The corresponding least upper bound is a variation of function h on a set D.

For the next we shall need the following proposition [DM].

Proposition 1. *Let D be a dense subset of the interval $[a, b]$, (X, d) be a complete metric space and $h\colon D \to X$ be a function of bounded variation on D.*

(i) *For any* $t \in (a,b]$ *(resp.* $t \in [a,b)$*) there exists limit* $h(t-0)$ *(resp.* $h(t+0)$*)*

(ii) *Function* $f\colon (a,b] \to X$ *(resp.* $g\colon [a,b) \to X$*) defined by* $f(t) = h(t-0)$ *(resp.* $g(t) = h(t+0)$*) are functions of bounded variation on* $(a,b]$ *(resp.* $[a,b)$*).*

(iii) $f(t-0) = g(t-0) = h(t-0)$ *for all* $t \in (a,b]$ *and* $f(t+0) = g(t+0) = h(t+0)$ *for all* $t \in [a,b)$*.*

(iv) *Function* f *(resp.* g*) is continuous at the point* $t \in (a,b)$ *if and only if* $h(t-0) = h(t+0)$*.*

(v) *For all* $t \in (a,b)$*, except possibly countable set,* $h(t-0) = h(t+0)$*.*

P r o o f. Let $(t_i)_{i=0}^{\infty}$ be an increasing (resp. decreasing) sequence elements of D. Since (X,d) is a complete metric space, it is sufficient to show that $\big(h(t_i)\big)_{i=0}^{\infty}$ is a Cauchy sequence. To see this, note that series

$$\sum_{i=1}^{\infty} d\big(h(t_{i-1}), h(t_i)\big)$$

is convergent and

$$d\big(h(t_n), h(t_m)\big) \le \sum_{i=n+1}^{m} d\big(h(t_{i-1}), h(t_i)\big) \le \sum_{i=n+1}^{\infty} d\big(h(t_{i-1}), h(t_i)\big)$$

for any $n < m$. It proves (i).

Let $(t_i)_{i=0}^{n}$ be an increasing sequence elements of $(a,b]$. Take $\varepsilon > 0$ and a sequence $(s_i)_{i=0}^{n}$ of D such that $s_0 < t_0 < s_1 < t_1 < \ldots < t_{n-1} < s_n < t_n$ and $d\big(f(t_i), h(s_i)\big) < \frac{\varepsilon}{2n}$. Then

$$\begin{aligned}\sum_{i=1}^{n} d\big(f(t_{i-1}), f(t_i)\big) &\le \sum_{i=1}^{n}\Big(d\big(f(t_{i-1}), h(s_{i-1})\big) \\ &\quad + d\big(h(s_{i-1}), h(s_i)\big) + d\big(h(s_i), f(t_i)\big)\Big) \\ &\le \sum_{i=1}^{n} d\big(h(s_{i-1}), h(s_i)\big) + \sum_{i=1}^{n} d\big(f(t_{i-1}), h(s_{i-1})\big) \\ &\quad + \sum_{i=1}^{n} d\big(f(t_i), h(s_i)\big) \\ &< \varepsilon + \sum_{i=1}^{n} d\big(h(s_{i-1}), h(s_i)\big).\end{aligned}$$

It means that f is of bounded variation. Bounded variation of g may be proved analogously. It proves (ii).

Now we prove

$$f(t-0) = h(t-0) \qquad \text{for} \quad t \in (a,b].$$

Let $(t_i)_{i=0}^{\infty}$ be an increasing sequence of elements of $(a,b]$ with $\lim_{n\to\infty} t_n = t$. Take a sequence $(s_i)_{i=0}^{\infty}$ of D such that $s_0 < t_0 < s_1 < t_1 < \ldots < t_{n-1} < s_n < t_n < \ldots$ and $d\big(f(t_i), h(s_i)\big) < \frac{1}{n}$. Clearly

$$f(t-0) = \lim_{n\to\infty} f(t_n) = \lim_{n\to\infty} h(s_n) = h(t-0).$$

Equalities

$$g(t-0) = h(t-0) \qquad \text{for} \quad t \in (a,b],$$
$$f(t+0) = h(t+0)$$

and

$$g(t+0) = h(t+0) \qquad \text{for} \quad t \in [a,b)$$

may be proved analogously. It proves (iii).

Part (iv) follows from (iii).

Let M be a variation of g on D and $n > 0$ be a fixed. Assume that there are m points $t_1, \ldots t_m$, where $m > nM$, such that inequality

$$d\big(f(t_i), h(t_i+0)\big) > \frac{1}{n}$$

is satisfied for all $i = 1, \ldots, m$. There is $\varepsilon > 0$ such that

$$d\big(f(t_i), h(t_i+0)\big) > 2\varepsilon + \frac{1}{n} \qquad \text{for all} \quad i = 1, \ldots, m.$$

We may assume that $(t_i)_{i=1}^{m}$ is an increasing sequence. There are sequences $(s_i)_{i=1}^{m}$ and $(u_i)_{i=1}^{m}$ in D such that $s_1 < t_1 < u_1 < s_2 < t_2 < u_2 < \ldots, t_{m-1} < u_{m-1} < s_m < t_m < u_m$,

$$d\big(f(t_i), h(s_i)\big) < \varepsilon \quad \text{and} \quad d\big(h(t_i+0), h(u_i)\big) < \varepsilon \qquad \text{for all} \quad i = 1, \ldots, m.$$

Now

$$\sum_{i=1}^{m} d\big(h(s_i), h(u_i)\big) > \sum_{i=1}^{m} \Big(d\big(f(t_i), h(t_i+0)\big) - 2\varepsilon \Big) > \frac{m}{n} > M$$

what is a contradiction. Therefore inequality

$$d\big(h(t-0), h(t+0)\big) > 0$$

may be satisfied only for countably many $t \in (a,b)$. □

Now we first formulate a Helly type theorem for functions taking values in a one relatively compact set and having uniformly bounded variations given in [DM]. We shall bring it with the proof since we shall need it in the proving the another type of Helly theorem.

Helly theorem 1. *Let (X, d) be a complete metric space and $(g_n)_{n\in\ N}$ a sequence of functions from $[a, b]$ into X such that*

a) *the set $g_n(x)$ is relatively compact for any $x \in [a, b]$,*

b) *the functions $(g_n)_{n\in N}$ have uniformly bounded variations.*

Then there exists a subsequence of the sequence $(g_n)_{n\in N}$ converging pointwise in X to a function $g\colon [a, b] \to X$ of bounded variation.

Proof. Let D be a countable set dense in $[a, b]$ and $b \in D$. By the diagonal procedure we obtain a subsequence $(h_n)_{n\in N}$ of $(g_n)_{n\in N}$ converging for every $t \in D$ to a function $h(t)$ which has bounded variation on D. Hence for every $t \in [a, b)$ there exists the limit $h(t + 0)$. Put

$$g(t) = h(t + 0), \qquad \text{for} \quad t \in [a, b)$$

and

$$g(b) = h(b).$$

Let $\delta > 0$ be given. We prove that

$$\limsup_{n\to\infty} d(h_n(t),\, g(t)) \le \delta$$

for every $t \in [a, b)$, except possible, finite number $m \le 2M/\delta$, where M is the upper bound of variations of all g_n. So, assume that for some $m > 2M/\delta$ there are points $t_1 < t_2 < \ldots < t_m < b$ such that

$$\limsup_{n\to\infty} d\big(h_n(t_i),\, g(t_i)\big) > \delta \qquad \text{for all} \quad i = 1, 2, \ldots, m\,.$$

There is a subsequence $(h_{n_k})_{k=1}^{\infty}$ such that

$$d\big(h_{n_k}(t_i),\, g(t_i)\big) > \delta \qquad \text{for all} \quad i = 1, 2, \ldots, m \quad \text{and} \quad k = 1, 2, \ldots$$

Since $g(t_i) = h(t_i + 0)$, there is a sequence $(s_i)_{i=1}^{m}$ in D such that

$$d\big(g(t_i),\, h(s_i)\big) < \frac{\delta}{4} \qquad \text{for all} \quad i = 1, 2, \ldots, m$$

and $t_1 < s_1 < t_2 < s_2 < \ldots, t_{m-1} < s_{m-1} < t_m < s_m$. Let k be so large that

$$d(h_{n_k}(s_i),\, h(s_i)) \le \frac{\delta}{4}, \qquad \text{for all} \quad i = 1, \ldots, m\,.$$

Then

$$\begin{aligned} M &\ge \sum_{i=1}^{m} d\big(h_{n_k}(t_i),\, h_{n_k}(s_i)\big) \\ &\geqq \sum_{i=1}^{m} \Big(d\big(h_{n_k}(t_i),\, g(t_i)\big) - d\big(g(t_i),\, h(s_i)\big) - d\big(h(s_i),\, h_{n_k}(s_i)\big)\Big) \\ &> \sum_{i=1}^{m} \left(\delta - \frac{\delta}{4} - \frac{\delta}{4}\right) = \frac{m\delta}{2} > M. \end{aligned}$$

which is not possible. So,

$$\limsup_{n\to\infty} d\big(h_n(t),\, g(t)\big) = 0$$

for every $t \in [a,b)$, except possible, countable set $A \subset [a,b)$. The last application of the diagonal procedure gives a sequence convergent to $\widetilde{g}(t)$ on A. For some $t \in A$ it may be $\widetilde{g}(t) \neq g(t)$. Therefore g has to be redefined on A by $g(t) = \widetilde{g}(t)$. □

3. A Helly type theorem in Banach spaces

In the following we shall use the next proposition for functions with values in Banach space.

Proposition 2. *Let $D \subset [a,b]$, X be a Banach space and $f_n\colon D \to X$ be a sequence of functions of uniformly bounded variations which converges weakly on D to a function f. Then f is of bounded variation on D.*

Proof. Note that for any sequence $(x_n)_{n=0}^{\infty}$ weakly converging in X to x_0 we have $\|x_0\| \le \liminf_{n\to\infty} \|x_n\|$. □

Now we shall prove Helly type theorem for functions with values in one weakly compact set of Banach space X and having uniform bounded variations.

Helly theorem 2. *Let X be a Banach space and $(g_n)_{n\in N}$ a sequence of functions from $[a,b]$ into X such that*

a) *the set $g_n(t)$ is relatively weakly compact for any $t \in [a,b]$,*

b) *the functions $(g_n)_{n\in N}$ have uniformly bounded variations.*

Then there exists a subsequence of the sequence $(g_n)_{n\in N}$ converging weakly pointwise in X to a function $g\colon [a,b] \to X$ of bounded variation.

Proof. Since all functions f_n have bounded variations, their ranges have (strongly) compact closures. Let X_0 be a Banach space generated by ranges of all f_n. Clearly, X_0 is separable. There is a sequence of $u_j \in X_0^*$ separating points of X_0 with $\|u_j\| = 1$. Linear functionals u_j may be extended to the whole space X (without increasing norm). Let $p\colon X \to X$ be a seminorm defined by

$$p(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} |u_j(x)|.$$

Clearly $p(x) \le \|x\|$ for any $x \in X$. The restriction of p onto X_0 is a norm. Let d be a metric on X_0 associated with p. Since X_0 is weakly closed, it contains weak closures of all its subsets, in particular, weak closures of $\big\{f_n(t) : n \in N\big\}$ for any

$t \in [a,b]$. These closures are assumed to be weakly compact. For any weakly compact $C \subset X_0$ the metric d is (weakly) continuous on $C \times C$. Therefore the weak topology on C is metrizable by the metric d. Let D be a countable set dense in $[a,b]$ and $b \in D$. By the diagonal procedure we obtain a subsequence $(h_n)_{n\in N}$ of $(g_n)_{n\in N}$ converging for every $t \in D$ to a function $h(t)$ which has bounded variation on D (with respect to $\|\,.\,\|$) and also with respect to the metric d. Hence for every $t \in [a,b)$ there exists the (strong) limit $h(t+0)$. This limit exists with respect to the metric d as well. Put

$$g(t) = h(t+0), \qquad \text{for} \quad t \in [a,b)$$

and

$$g(b) = h(b).$$

Now we can repeat the proof of Helly theorem 1, because the weak convergence of sequences may be reduced to the convergence with respect to the metric d. □

4. Integral representation for majored linear operators

From the preceding generalization of Helly theorem we will derive the result giving the representation of majored linear mapping. First we recall the definition of majored linear mapping [Di].

For each subset A of $[a,b]$, let $C\big([a,b],A\big)$ denote the space of continuous functions on $[a,b]$ vanishing outside A. Let X be a Banach space. If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum_i ||F(f_i)||,$$

where the supremum is over all finite families $\{f_i\}$ in $C\big([a,b],A\big)$ with

$$\sum_i |f_i(t)| \le c_A(t) \qquad \text{for all} \quad t \in [a,b].$$

If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, then (see [Di, § 19, pp. 380, 383]), the mapping F is called majored (also dominated), if

$$|||F_A||| < \infty \tag{M}$$

for all A in $\mathcal{B}\big([a,b]\big)$. An equivalent definition of majored mapping is as follows.

If $F\colon C\big([a,b]\big) \to X$ is a linear mapping, it is majored (dominated) if and only if there exists a nonnegative Borel measure μ on $\mathcal{B}([a,b])$ such that

$$||F(\psi)|| \le \int_a^b |\psi(t)|\, d\mu(t), \qquad \psi \in C\big([a,b]\big).$$

Since such F can be extended to all bounded Borel functions, it is easy to see that it is weakly compact linear mapping on $C[0,1]$ into X.

Let on the interval $[a,b]$ a function g of bounded variation with values in Banach space X be given. This function makes it possible to every continuous function f on $[a,b]$ to associate the element of X of the form

$$F(f) = \int_a^b f(x)\, dg(x). \tag{!}$$

The following properties are true. Let $f_1, f_2, f \in C[a,b]$. Then

$$F(f_1 + f_2) = F(f_1) + F(f_2), \tag{a}$$

$$\|F(f)\| \le V_a^b M(f),\ M(f) = \max|f(x)|,\ V_a^b = \mathrm{Var}_a^b(g). \tag{b}$$

We shall now prove a Riesz type representation theorem for majored linear mapping.

Theorem (Riesz type). *Let on the set $C[a,b]$ the majored linear mapping F with values in a Banach space X be given. Then there exists a function g of bounded variation with values X such that for every function $f \in C[a,b]$ we have*

$$F(f) = \int_a^b f(x)\, dg(x). \tag{1}$$

P r o o f. It is enough to consider the case $a = 1, b = 1$, because the general case can be reduced to this case by means of a linear transformation of argument.

Put

$$\varphi_{n,k} = \binom{n}{k} x^k (1-x)^{n-k}.$$

It is easy to see that for every $x \in [0,1]$ we have

$$\sum_{k=0}^{n} \varphi_{n,k}(x) = 1.$$

Moreover, for $x \in [0,1]$ every member of this sum is nonnegative. Hence, for complex numbers a_k such that

$$|a_k| \le 1, \qquad k = 0, 1, \dots, n,$$

we have

$$\left| \sum_{k=0}^{n} a_k \varphi_{n,k} \right| \le 1. \tag{2}$$

Note that the considered majored linear operator F is defined for functions f continuous on $[0,1]$. It is known that majored linear mapping is weakly compact, hence there exists a weakly compact subset W of X such that

$$F(f) \in M(f)W,$$

or, equivalently mapping the unit sphere in $C[0,1]$ into W. From this and (2) we obtain

$$\sum_{k=0}^{n} a_k F(\varphi_{n,k}) \in W.$$

Further from the majority of operator F we obtain

$$\sum_{k=0}^{n} \|F(\varphi_{n,k})\| \le \sup \sum_{i} \|F(f_i)\| \le |||F_A||| < \infty, \tag{3}$$

where the supremum is over all finite families $f_i \in C\big([a,b],A\big)$ with $\sum |f_i(t)| \le c_A(t)$ for all $t \in [a,b]$, for all $A \in \mathcal{B}\big([a,b]\big)$.

Let us define the step function g_n to put

$$\begin{aligned}
g_n(0) &= 0, \\
g_n(x) &= F(\varphi_{n,0}) && \left(0 < x < \frac{1}{n}\right), \\
g_n(x) &= F(\varphi_{n,0}) + F(\varphi_{n,1}) && \left(\frac{1}{n} \le x < \frac{2}{n}\right), \\
&\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots \\
g_n(x) &= \sum_{k=0}^{n-1} F(\varphi_{n,k}) && \left(\frac{n-1}{n} \le x < 1\right), \\
g_n(1) &= \sum_{k=0}^{n} F(\varphi_{n,k}).
\end{aligned}$$

By (3) the functions g_n have bounded (with one number) total variations. Moreover, because F is weakly compact operator on $C[0,1]$ into X, it takes unit sphere in $C[0,1]$ into a weakly compact set W in X. Hence the set $\big\{g_n(x)\big\}$, $n = 1,2,\dots$ is contained in W for all $x \in [0,1]$. Therefore on the base of Helly theorem 2 from the sequence $\big\{g_n(x)\big\}$ it is possible to choose the subsequence $\big\{g_{n_i}(x)\big\}$ which converges weakly in each point of $[0,1]$ to a function of the bounded variation.

If f is a continuous function on $[0,1]$, then on the base of [Na, Th 3, § 6] it can be shown that

$$\int_0^1 f(x)\,dg_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) F(\varphi_{n,k}),$$

from where

$$\int_0^1 f(x)\,dg_n(x) = F(B_n),$$

where

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

is the Bernstein polynomial for the function f.

By theorem of S. N. Bernstein [Na, § 5, ch. IV]

$$M(B_n - f) \to 0, \qquad n \to \infty,$$

and by the definition of continuous linear operator we have

$$\|F(B_n) - F(f)\| = \|F(B_n - f)\| \le KM(B_n - f).$$

This means that

$$F(B_n) \to F(f),$$

from where

$$\lim_{n\to\infty} \int_0^1 f(x)\,dg_n(x) = F(f).$$

But if $n \to \infty$, going through the values $n_1, n_2, \dots$, then by Helly theorem (cf. also [Na, § 7], and [DD1]), we obtain

$$\lim_{n\to\infty} \int_0^1 f(x)\,dg_n(x) = \int_0^1 f(x)\,dg(x).$$

Therefore we obtain

$$F(f) = \int_0^1 f(x)\,dg(x).$$

□

Remark 1. We have derived [DM] a representation theorem for majored operators using Helly theorem in linear metric spaces. To prove it by means of the Helly theorem from that paper [DM] we were able to do it for compact and majored mappings. In this paper it is done for Banach spaces without compactness condition. On the other hand, our approach based on Helly type theorem is more “constructive” and simpler than used, but in more general context, in [Di].

Remark 2. Let $T = [a, b]$ and let g be a function of bounded variation, [Di, p. 362]. We can always suppose that g is continuous on the left in all points of T except for b. In this case, if $u < b$ we have

$$m\big((a,u)\big) = g(u) - g(a+0),$$

$$m\big([a,u]\big) = g(u+0) - g(a),$$

$$m\big([a,u)\big) = g(u) - g(a),$$

$$m\big((a,u]\big) = g(u+0) - g(a+0).$$

If $u = b$, we put $b - 0$ instead of u, and u instead of $u + 0$.

$$m\big((a,b)\big) = g(b-0) - g(a+0),$$

$$m\big([a,b]\big) = g(b) - g(a),$$

$$m\big([a,b)\big) = g(b-0) - g(a),$$

$$m\big((a,b]\big) = g(b) - g(a+0).$$

Then m can be uniquely extended to a regular Borel measure with bounded variation, defined on the sigma ring of Borel sets $A \subset T = [a, b]$.

Let $g\colon T \to X$ be a function with bounded variation on $T = [a, b]$ and $v(t) = V_a^t(g)$ be variation of g. If $m\colon \mathcal{B}\big([a, b]\big) \to X$ is the measure corresponding to g and μ is the measure corresponding to v, and if g is continuous on the left in $[a, b)$ then μ is the variation of m, and if S is the semiring of the intervals of the form $[u, v)$ and $[u, b]$, then for all $I \in S$, we have $|m|(I) = \mu(I)$.

From the preceding we may derive the following result.

Theorem. *Let $L\colon C[a, b] \to X$ be a majored linear mapping. Then there exists a Borel measure $m\colon \mathcal{B}\big([a, b]\big) \to X$ with finite variation such that*

$$L(f) = \int_a^b f(x)\,dm(x), \qquad f \in C[a,b].$$

Now we give an example of continuous linear mapping on continuous functions into Banach space which is not majored (with finite variation) nor (weakly) compact.

Example ([Di, p. 401]). Let $K = [0, 1]$ and $C(K)$ be the space of the real continuous functions on K. Take $X = C(K)$ and consider the linear mapping $U\colon C(K) \to X$ defined by

$$U(f) = f, \qquad f \in C(K).$$

For every Borel set $A \subset K$ we have, evidently,

$$\|U_A\| = \sup\{\|U_A(f)\|, \|f\| \leq 1\} = 1,$$

the mapping U is not weakly compact. But for every open set $G \subset K$ we have

$$|||U_G||| = \sup \sum \|U(f_i)\| = \infty,$$

where the supremum is taken for all the finite families (f_i) of $C(K, A)$ with $\sum_i |f_i(t)| \leq c_A(t)$ for all t in $[a, b]$. Hence U is not majored.

In fact, for every n we can find a family of n functions $f_1, \dots, f_n$ of $C(K, G)$ with $\|f_i\| = 1$ and $|f_i|.|f_j| = 0$ if $i \neq j$. Then

$$\sum \|U(f_i)\| = \sum \|f_i\| = n,$$

therefore $|||U_G||| > n$, consequently $|||U_G||| = \infty$.

The conjugate space of $C(K)$ is isomorph to the space $M(K)$ of the regular real Borel measures on K, $C'(K) = M(K)$.

For every measure

$$\mu \in M(K) \quad \text{and} \quad U\colon C(K) \to C(K)$$

define a linear functional U_μ by

$$U_\mu(f) =< U(f), \mu >=< f, \mu >= \int f d\mu.$$

On the other hand, by a theorem, [Di, p. 401], or [BDS] there exists an additive set function $m\colon \mathcal{B} \to M' = C'^*(K)$ with finite semi-variation, such that for every $\mu \in M$, the scalar-valued set function $m_\mu, m_\mu(A) =< m(A), \mu >$ is a regular measure and we have

$$U_\mu(f) = \int f \, dm_\mu, \qquad f \in C(K)$$

It follows that

$$\int f d\mu = \int f \, dm_\mu, \qquad f \in C(K),$$

i.e., $m_\mu = \mu$, for every $\mu \in M$. For every set $A \in \mathcal{B}$ we have $m(A) \in M'$ and

$$\|m(A)\| = \sup_{\|\mu\| \leq 1} |m_\mu(A)| = \sup_{\|\mu\| \leq 1} |\mu(A)| = 1.$$

It follows that m cannot be regular and countably additive and that the variation of m on A, $|m|(A) = \infty$, i.e., m has not finite variation. On the other hand, the semivariation $\tilde{m}(A)$ of m on A,

$$\tilde{m}(A) = \sup_{\|\mu\| \leq 1} \bar{m}_\mu(A) = \sup_{\|\mu\| \leq 1} |\mu|(A) = 1.$$

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