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wQN SPACES AND RELATED NOTIONS

Jaroslav Šupina

ABSTRACT. We discuss modifications of wQN and SSP properties for semicontinuous functions with values in some special sets of reals and we give the relations among these notions. We present direct proofs of $\mathrm{wQN}_* \to \mathrm{wQN}^*$ and $\mathrm{SSP}_* \to \mathrm{SSP}^*$.

1. Preliminaries

In this paper, a space X is an infinite perfectly normal topological space and all functions are functions on a space X to a set of reals A, where we will deal with A being one of the sets $\mathbb{R}$, $[-1,1]$, $[0,\infty)$, $[0,1]$. A function f is said to be lower semicontinuous (upper semicontinuous) if for every real number r the set

$$f^{-1}((r,\infty)) = \{x \in X : f(x) > r\} \qquad \Big(f^{-1}((-\infty,r)) = \{x \in X : f(x) < r\}\Big)$$

is open in a space X, respectively. By $\mathrm{C}_p(X,A)$, $\mathrm{LSC}_p(X,A)$, $\mathrm{USC}_p(X,A)$ (compare [3]) we denote the set of all continuous, lower semicontinuous, upper semicontinuous functions from X to A, respectively. All these sets are closed under minimum and maximum of finitely many functions. If $A \in \{\mathbb{R}, [-1,1]\}$, then $f \in \mathrm{LSC}_p(X,A)$ if and only if $-f \in \mathrm{USC}_p(X,A)$.

The notation "$f_n \to f$ (on X)" means that the sequence of functions $\langle f_n : n \in \omega\rangle$ converges on X to f pointwise. "$f_n \searrow f$ (on X)" means that $f_n \to f$ (on X) and $f_n \geq f_{n+1} \geq f$, $n \in \omega$. A sequence of functions $\langle f_n : n \in \omega\rangle$ converges quasi-normally to f on X, written $f_n \xrightarrow{QN} f$ (on X), if there exists a sequence of positive reals $\langle \varepsilon_n : n \in \omega\rangle$ converging to zero such that $|f_n(x) - f(x)| < \varepsilon_n$ holds for all but finitely many $n \in \omega$ for any $x \in X$.

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A space X has the property $\mathrm{QN}(A)$ if each sequence from $\mathrm{C}_p(X,A)$ converging to zero converges to zero quasi-normally. X has the property $\mathrm{wQN}(A)$, $\mathrm{wQN}_*(A)$, $\mathrm{wQN}^*(A)$ if each sequence from $\mathrm{C}_p(X,A)$, $\mathrm{LSC}_p(X,A)$, $\mathrm{USC}_p\big(X,A\big)$, respectively, converging to zero contains a subsequence that converges to zero quasi-normally. X has the property $\mathrm{SSP}(A)$, $\mathrm{SSP}_*(A)$, $\mathrm{SSP}^*(A)$ if for each sequence of sequences $\langle\langle f_{n,m} : m \in \omega\rangle \ : n \in \omega\rangle$ of functions from $\mathrm{C}_p(X,A)$, $\mathrm{LSC}_p(X,A)$, $\mathrm{USC}_p\big(X,A\big)$, respectively, such that $f_{n,m} \to 0$ for any $n \in \omega$, there exists a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{n,m_n} \to 0$. Argument A in $\mathrm{QN}(A)$, $\mathrm{wQN}(A)$ or $\mathrm{SSP}(A)$ properties is superfluous because

$$\mathrm{QN}(\mathbb{R}) \equiv \mathrm{QN}\big([-1,1]\big) \equiv \mathrm{QN}\big([0,\infty)\big) \equiv \mathrm{QN}\big([0,1]\big),$$

and similarly for $\mathrm{wQN}(A)$ or $\mathrm{SSP}(A)$. Therefore, we write only QN, wQN or SSP.

A family $\mathcal{U} \subseteq \mathcal{P}(X)$ is a cover of X if $X = \cup\mathcal{U}$ and $X \notin \mathcal{U}$. An infinite cover $\mathcal{U}$ is a γ-cover if every $x \in X$ lies in all but finitely many members of $\mathcal{U}$. A γ-cover $\mathcal{U}$ is shrinkable if there exists a closed γ-cover $\mathcal{V}$ which is a refinement of $\mathcal{U}$. $\Gamma(X)$ (or Γ) denotes the set of all countable open γ-covers and $\Gamma^{\mathrm{sh}}(X)$ (or Γ^{sh}) denotes the set of all countable open shrinkable γ-covers. Let $\mathcal{A}$, $\mathcal{B}$ be families of covers of the space X. X possesses the property $\mathrm{S}_1(\mathcal{A},\mathcal{B})$ if for every sequence $\langle\mathcal{U}_n : n \in \omega\rangle$ of covers from $\mathcal{A}$ there exist sets $U_n \in \mathcal{U}_n,\ n \in \omega$ such that $\{U_n;\ n \in \omega\} \in \mathcal{B}$.

2. Introduction

The main results related to relations mentioned above are summarized in [2]. If we add a result of [3], we can get the following Diagram of implications and equivalences:

$$\begin{array}{ccccc}
\mathrm{SSP}_*\big([0,1]\big) & \longrightarrow & \mathrm{SSP}^*\big([0,1]\big) & \longrightarrow & \mathrm{SSP} \\
\| & & \| & & \| \\
\mathrm{QN} & \longrightarrow & \mathrm{S}_1(\Gamma,\Gamma) & \longrightarrow & \mathrm{S}_1\big(\Gamma^{\mathrm{sh}},\Gamma\big) \\
\| & & \| & & \| \\
\mathrm{wQN}_*\big([0,1]\big) & \longrightarrow & \mathrm{wQN}^*\big([0,1]\big) & \longrightarrow & \mathrm{wQN}\,.
\end{array}$$

Moreover (see, e.g., [2]), there is a model of ZFC, where all of these properties are equivalent, and there is a model of ZFC with an $\mathrm{S}_1(\Gamma,\Gamma)$-set which is not a QN-set. Hence, $\mathrm{S}_1(\Gamma,\Gamma) \to \mathrm{QN}$ is undecidable in ZFC. We do not know if the second reversed implication, e.g., $\mathrm{S}_1(\Gamma^{\mathrm{sh}},\Gamma) \to \mathrm{S}_1(\Gamma,\Gamma)$ holds. The implication $\mathrm{wQN} \to \mathrm{S}_1(\Gamma,\Gamma)$ is known as Scheepers conjecture (see, e.g., [2]).

3. Direct proofs

The "semicontinuous" asymmetry in the Diagram was a motivating factor. By the Diagram, implications $\mathrm{wQN}_*\big([0,1]\big) \to \mathrm{wQN}^*\big([0,1]\big)$ and $\mathrm{SSP}_*\big([0,1]\big) \to \mathrm{SSP}^*\big([0,1]\big)$ hold true and the above mentioned model shows that one cannot prove the reversed implications in ZFC. However, none of these implications was proved directly, they are only consequences of implication $\mathrm{QN} \to \mathrm{S}_1(\Gamma,\Gamma)$ and equivalences $\mathrm{QN} \equiv \mathrm{SSP}_*([0,1]) \equiv \mathrm{wQN}_*([0,1])$ and $\mathrm{S}_1(\Gamma,\Gamma) \equiv \mathrm{SSP}^*\big([0,1]\big) \equiv \mathrm{wQN}^*\big([0,1]\big)$. We will present the proof using $\mathrm{USC}\big([0,1]\big)$ property investigated in [4].[1] The following lemmas are mentioned in a modified shape in [1].

The characteristic function χ_A of the set $A \subseteq X$ is defined as 1 for $x \in A$ and as 0 for $x \in X \setminus A$.

Lemma 3.1. *For $f \in \mathrm{USC}_p\big(X,[0,1]\big)$ there exists a sequence $\langle f_n : n \in \omega\rangle$ of functions from $\mathrm{USC}_p\big(X,[0,1]\big)$ such that $f_n \searrow f$ and f_n is a finite linear combination with positive coefficients of characteristic functions of closed sets.*

P r o o f. Let $f \in \mathrm{USC}_p\big(X,[0,1]\big)$ and, for $n \in \omega$ and $i \in \{0,\dots,2^n-1\}$,

$$A_{n,i} = \left\{x \in X \colon f(x) \geq \frac{i}{2^n}\right\},$$

then $A_{n,i+1} \subseteq A_{n,i}$ for $i \in \{0,\dots,2^n-2\}$. If we define $\langle f_n : n \in \omega\rangle$ by

$$f_n(x) = \frac{1}{2^n}\sum_{i=0}^{2^n-1} \chi_{A_{n,i}}(x),$$

then $\langle f_n : n \in \omega\rangle$ is a sequence of upper semicontinuous functions because for any $r \in [0,1]$ the set $f_n^{-1}\big([0,r)\big)$ is one of the sets $X \setminus A_{n,i}$ for some $i \in \{0,\dots,2^n-1\}$. Due to property that each real from unit interval $[0,1]$ is expressable by infinite sum of negative powers of 2, we also have $f_n \searrow f$. □

Lemma 3.2. *If a set $A \subseteq X$ is closed, then there is a sequence $\langle f_n : n \in \omega\rangle$ of continuous functions such that $f_n \searrow \chi_A$. Hence, if h is a finite linear combination with positive coefficients of characteristic functions of closed sets, then there is a sequence $\langle h_n : n \in \omega\rangle$ of continuous functions such that $h_n \searrow h$.*

P r o o f. Let $A \subseteq X$ be closed. Since X is perfectly normal, there are open sets B_n, such that $A = \cap_n B_n$. Setting $A_n = \cap_{i=0}^n B_i$, we obtain $A_{n+1} \subseteq A_n$ with $A = \cap_n A_n$. Again, X is perfectly normal, therefore there are continuous functions g_n such that $g_n(x) = 1$ for $x \in A$ and $g_n(x) = 0$ for $x \in X \setminus A_n$. For any $n \in \omega$ we can set

[1] Originally denoted as (USC).

$$f_0 = g_0,$$
$$f_{n+1} = \min\{f_n,\ g_{n+1}\},$$

and we obtain $\langle f_n : n \in \omega\rangle$ such that $f_n \geq f_{n+1}$, $f_n(x) = 1$ for $x \in A$, $f_n(x) = 0$ for $x \in X \setminus A_n$ for $n \in \omega$ and $f_n \searrow \chi_A$.

Let $i \in \{0, \dots, k\}$ and $k \in \omega$. If $h = c_0\chi_{A_0} + \cdots + c_k\chi_{A_k}$ for A_i closed and $c_i > 0$, then there are $h_{i,n} \in C_p\big(X, [0, \infty)\big)$ such that $h_{i,n} \searrow \chi_{A_i}$. The desired sequence is $h_n = c_0h_{0,n} + \cdots + c_kh_{k,n},\ n \in \omega$. □

For a set of functions $\Phi \subseteq {}^A\mathbb{R}$ we denote

$$\Phi^{\downarrow} = \big\{f \in {}^A\mathbb{R};\ (\exists\, f_n \in \Phi,\ n \in \omega)\ f_n \searrow f \text{ on } A\big\}.$$

If Φ is closed under minimum of finitely many functions, then $(\Phi^{\downarrow})^{\downarrow} = \Phi^{\downarrow}$.

Lemma 3.3. *For any $f \in \mathrm{USC}_p\big(X, [0, 1]\big)$ there are $f_n \in \mathrm{C_p}\big(\mathrm{X}, [0, 1]\big), \mathrm{n} \in \omega$ such that $f_n \searrow f$.*

Proof. By Lemma 3.1, there is a sequence $\langle h_n : n \in \omega\rangle$ of upper semicontinuous functions such that $h_n \searrow f$ and by Lemma 3.2, $h_n \in \mathrm{C_p}\big(\mathrm{X}, [0, 1]\big)^{\downarrow}$ for any $n \in \omega$. Therefore, there is a sequence $\langle h_{n,m} : m \in \omega\rangle$ of continuous functions such that $h_{n,m} \searrow h_n$ for $n \in \omega$. Finally, due to $\big(\mathrm{C}_p(X, [0, 1])^{\downarrow}\big)^{\downarrow} \subseteq \mathrm{C}_p\big(X, [0, 1]\big)^{\downarrow}$ there is a sequence $\langle f_n : n \in \omega\rangle$ of continuous functions such that $f_n \searrow f$. □

By [4], we say that a space X has a property $\mathrm{USC}\big([0, 1]\big)$ if, whenever a sequence $\langle f_n : n \in \omega\rangle$ of functions from $\mathrm{USC}_p\big(X, [0, 1]\big)$ converges to zero, there is a sequence $\langle g_n : n \in \omega\rangle$ of functions from $\mathrm{C}_p\big(X, [0, 1]\big)$ converging to zero such that $f_n \leq g_n,\ n \in \omega$. H. Ohta and M. Sakai [4] proved that every perfectly normal QN-space has property $\mathrm{USC}\big([0, 1]\big)$. As the Diagram shows, $\mathrm{QN} \equiv \mathrm{SSP}_*\big([0, 1]\big)$. This means that $\mathrm{SSP}_*\big([0, 1]\big) \to \mathrm{USC}\big([0, 1]\big)$. We can prove this fact by Lemma 3.3 as well.

Theorem 3.1. $\mathrm{SSP}_*\big([0, 1]\big)$ *implies* $\mathrm{USC}\big([0, 1]\big)$.

Proof. Let $\langle f_n : n \in \omega\rangle$ be a sequence of functions from $\mathrm{USC}_p\big(X, [0, 1]\big)$ such that $f_n \to 0$ on X. By Lemma 3.3 there are $g_{n,m} \in \mathrm{C}_p\big(X, [0, 1]\big),\ m \in \omega$ such that $g_{n,m} \searrow f_n$ for any $n \in \omega$. Therefore, functions $f_{n,m} = g_{n,m} - f_n \in \mathrm{LSC}_p\big(X, [0, 1]\big),\ n, m \in \omega$ and $f_{n,m} \searrow 0,\ n \in \omega$. By $\mathrm{SSP}_*\big([0, 1]\big)$, there is a sequence $\langle m_n : n \in \omega\rangle$ such that $f_{n,m_n} \to 0$. Since $g_{n,m_n} = f_{n,m_n} + f_n$, we obtain $g_{n,m_n} \to 0$. □

Now, it is easy to prove the implications mentioned in the introduction of this section.

Theorem 3.2. $\mathrm{wQN}_*\big([0, 1]\big)$ *implies* $\mathrm{wQN}^*\big([0, 1]\big)$.

P r o o f. $\mathrm{wQN}_*([0,1])$ implies $\mathrm{SSP}_*([0,1])$ by the Diagram and this implies $\mathrm{USC}([0,1])$ by Theorem 3.1. Let $f_n \in \mathrm{USC}_p(X,[0,1])$, $n \in \omega$ and $f_n \to 0$ on X. By $\mathrm{USC}([0,1])$, there is a sequence $\langle g_n : n \in \omega\rangle$ of functions from $\mathrm{C}_p(X,[0,1])$ such that $f_n \le g_n$ for any $n \in \omega$ and $g_n \to 0$. By $\mathrm{wQN}_*([0,1])$, there exists a sequence $\langle n_m : m \in \omega\rangle$ such that $g_{n_m} \overset{QN}{\to} 0$ with a control $\langle \varepsilon_m : m \in \omega\rangle$. Then, $f_{n_m} \overset{QN}{\to} 0$ with the control $\langle \varepsilon_m : m \in \omega\rangle$ as well. □

Theorem 3.3. $\mathrm{SSP}_*([0,1])$ *implies* $\mathrm{SSP}^*([0,1])$.

P r o o f. Let $\mathrm{SSP}_*([0,1])$ hold true and $f_{n,m} \in \mathrm{USC}_p(X,[0,1])$, $n,m \in \omega$ such that $f_{n,m} \to 0$ for $n \in \omega$. By Theorem 3.1, there are $g_{n,m} \in \mathrm{C}_p(X,[0,1])$ such that $f_{n,m} \le g_{n,m}$, $n,m \in \omega$ and $g_{n,m} \to 0$ for $n \in \omega$. By $\mathrm{SSP}_*([0,1])$, there is $\langle m_n : n \in \omega\rangle$ such that $g_{n,m_n} \to 0$. Since $f_{n,m_n} \le g_{n,m_n}$, $n \in \omega$, we have $f_{n,m_n} \to 0$. □

4. Range of functions

As we have noted, the considered properties related to continuous functions do not depend on which set from $\mathbb{R}$, $[-1,1]$, $[0,\infty)$ or $[0,1]$ is taken in their definitions as a range of functions. However, the modifications with semicontinuous functions do. In this section, we present the relations among such modified notions. Many of them are based on simple properties of semicontinuous functions.

Theorem 4.1. $\mathrm{wQN}_*(\mathbb{R})$ *if and only if* $\mathrm{wQN}^*(\mathbb{R})$.

P r o o f. $\langle f_n : n \in \omega\rangle$ is a sequence of functions from $\mathrm{USC}_p(X,\mathbb{R})$ converging to zero if and only if $\langle -f_n : n \in \omega\rangle$ is a sequence of functions from $\mathrm{LSC}_p(X,\mathbb{R})$ converging to zero and vice versa. Therefore, by $\mathrm{wQN}_*(\mathbb{R})$, there is a subsequence $\langle -f_{n_m} : m \in \omega\rangle$ converging to zero quasi-normally and finally also the subsequence $\langle f_{n_m} : m \in \omega\rangle$ converging to zero quasi-normally due to equility $|f_{n_m}| = |-f_{n_m}|$ for $m \in \omega$. Conversely, using $\mathrm{wQN}^*(\mathbb{R})$ on $\langle -f_n : n \in \omega\rangle$ from $\mathrm{USC}_p(X,\mathbb{R})$ converging to zero, we obtain a subsequence $\langle f_{n_m} : m \in \omega\rangle$ converging to zero quasi-normally. □

Similarly,

Theorem 4.2. $\mathrm{wQN}_*([-1,1])$ *if and only if* $\mathrm{wQN}^*([-1,1])$.

Theorem 4.3. $\mathrm{wQN}_*([0,1])$ *implies* $\mathrm{wQN}_*([0,\infty))$.

P r o o f. Let $\langle f_n : n \in \omega \rangle$ be a sequence of functions from $\mathrm{LSC}_p\big(X, [0, \infty)\big)$ converging to zero and let us define $\langle g_n : n \in \omega \rangle$ by $g_n = \min\{f_n, 1\}$ for $n \in \omega$. This sequence converges to zero due to the convergence of f_n and $g_n \in \mathrm{LSC}_p\big(X, [0, 1)\big)$ for $n \in \omega$. By $\mathrm{wQN}_*\big([0, 1]\big)$, there is a subsequence $\langle g_{n_m} : m \in \omega \rangle$ converging to zero quasi-normally with a control $\langle \varepsilon_m : m \in \omega \rangle$. Then, $\langle f_{n_m} : m \in \omega \rangle$ converges to zero quasi-normally with the same control $\langle \varepsilon_m : m \in \omega \rangle$. □

Similarly we can prove:

Theorem 4.4. $\mathrm{wQN}^*([0, 1])$ *implies* $\mathrm{wQN}^*([0, \infty))$.

Theorem 4.5. $\mathrm{wQN}_*([-1, 1])$ *implies* $\mathrm{wQN}_*(\mathbb{R})$.

P r o o f. Let $\langle f_n : n \in \omega \rangle$ be a sequence of functions from $\mathrm{LSC}_p(X, \mathbb{R})$ converging to zero. If $g_n = \min\{\max\{f_n, -1\}, 1\}$, then $g_n \in \mathrm{LSC}_p(X, [-1, 1])$ for any $n \in \omega$. By $\mathrm{wQN}_*([-1, 1])$ there is a subsequence $\langle g_{n_k} : k \in \omega \rangle$ converging to zero quasi-normally with a control $\langle \varepsilon_k : k \in \omega \rangle$. The subsequence $\langle f_{n_k} : k \in \omega \rangle$ is then convergent to zero quasi-normally with the same control $\langle \varepsilon_k : k \in \omega \rangle$. □

For a function f we denote $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then, $f = f^+ - f^-$. If $f \in \mathrm{LSC}_p(X, [-1, 1])$, then $f^+ \in \mathrm{LSC}_p(X, [0, 1])$ and $f^- \in \mathrm{USC}_p\big(X, [0, 1]\big)$.

Theorem 4.6. $\mathrm{wQN}_*([0, 1])$ *implies* $\mathrm{wQN}_*([-1, 1])$.

P r o o f. Let $\langle f_n : n \in \omega \rangle$ be a sequence of functions from $\mathrm{LSC}_p(X, [-1, 1])$ converging to zero. Then both sequences $\langle f_n^+ : n \in \omega \rangle$, $\langle f_n^- : n \in \omega \rangle$ converge to zero and $f_n^+ \in \mathrm{LSC}_p(X, [0, 1])$, $f_n^- \in \mathrm{USC}_p\big(X, [0, 1]\big)$ for $n \in \omega$. By $\mathrm{wQN}_*([0, 1])$, there is a subsequence $\langle f_{n_k}^+ : k \in \omega \rangle$ converging to zero quasi-normally with a control $\langle \delta_k : k \in \omega \rangle$. Since $\mathrm{wQN}_*([0, 1])$ implies $\mathrm{wQN}^*([0, 1])$ by the Diagram, there is a subsequence $\langle f_{n_{k_l}}^- : l \in \omega \rangle$ converging to zero quasi-normally with a control $\langle \varepsilon_l : l \in \omega \rangle$. If $\alpha_l = 2 \cdot \max\{\varepsilon_l, \delta_{k_l}\}$, then the subsequence $\langle f_{n_{k_l}} : l \in \omega \rangle$ converges to zero quasi-normally with the control $\langle \alpha_l : l \in \omega \rangle$. □

Summarizing results of proved Theorems, we get the following corollaries.

Corollary 4.1. *The following properties are equivalent:*
$\mathrm{wQN}_*([0, 1])$, $\mathrm{wQN}_*([0, \infty))$, $\mathrm{wQN}_*([-1, 1])$, $\mathrm{wQN}^*([-1, 1])$, $\mathrm{wQN}_*(\mathbb{R})$, $\mathrm{wQN}^*(\mathbb{R})$.

Corollary 4.2. *The following properties are equivalent:*
$\mathrm{wQN}^*([0, 1])$, $\mathrm{wQN}^*([0, \infty))$.

As we have already mentioned, a model of ZFC with an $\mathrm{S}_1(\Gamma, \Gamma)$-set which is not a QN-set, all of the above modifications of wQN_* and wQN^* cannot be equivalent.

Analogous relations hold true for modified properties of SSP_* and SSP^* as well.

Corollary 4.3. *The following properties are equivalent:*
$\mathrm{SSP}_*([0,1])$, $\mathrm{SSP}_*([0,\infty))$, $\mathrm{SSP}_*([-1,1])$, $\mathrm{SSP}^*([-1,1])$, $\mathrm{SSP}_*(\mathbb{R})$, $\mathrm{SSP}^*(\mathbb{R})$.

Corollary 4.4. *The following properties are equivalent:*
$\mathrm{SSP}^*([0,1])$, $\mathrm{SSP}^*([0,\infty))$.

Proofs of the previous corollaries are analogous to those of Corollary 4.1 and 4.2 using analogous theorems to Theorems 4.1–4.6 with SSP_* and SSP^* properties. Their proofs are very similar except for Theorem 4.6. We therefore present its proof.

Theorem 4.7. $\mathrm{SSP}_*([0,1])$ *implies* $\mathrm{SSP}_*([-1,1])$.

Proof. Let $f_{n,m} \in \mathrm{LSC}_p(X,[-1,1])$ such that $f_{n,m} \to 0,\ n \in \omega$. By Theorem 3.1, there are $g_{n,m} \in \mathrm{C}_p(X,[0,1]),\ n,m \in \omega$ such that $f^-_{n,m} \le g_{n,m},\ n,m \in \omega$ and $g_{n,m} \to 0$ for $n \in \omega$. Then, $f^+_{n,m}+g_{n,m} \in \mathrm{LSC}_p(X,[0,\infty))$ and $f^+_{n,m}+g_{n,m} \to 0$ for $n \in \omega$. By $\mathrm{SSP}_*([0,\infty))$ we obtain a sequence $\langle f^+_{n,m_n} + g_{n,m_n} : n \in \omega\rangle$ such that $f^+_{n,m_n}+g_{n,m_n} \to 0$. While $f^-_{n,m_n} \le f^+_{n,m_n}+g_{n,m_n}$ and $f^+_{n,m_n} \le f^+_{n,m_n}+g_{n,m_n}$ for any $n \in \omega$, $f^+_{n,m_n} \to 0$ and $f^-_{n,m_n} \to 0$ which finally entails $f_{n,m_n} \to 0$. ☐

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REFERENCES

[1] BUKOVSKÝ, L.: *Structure of the Real Line*, (to appear).
[2] BUKOVSKÝ, L.: *On* wQN_* *and* wQN^* *spaces*, Topology Appl. **156** (2008), 24–27.
[3] SAKAI, M.: *Selection principles and upper semicontinuous functions*, Colloq. Math. **117** (2009), 251–256.
[4] OHTA, H.—SAKAI, M.: *Sequences of semicontinuous functions accompanying continuous functions*, Topology Appl. **156** (2009), 2683–2906.

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Institute of Mathematics
University of P. J. Šafárik
Jesenná 5
SK–040-01 Košice
SLOVAKIA
E-mail: jaroslav.supina@student.upjs.sk