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A CHARACTERISTIC OF i-CONNECTED SPACES

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ABSTRACT. A theorem characterizing i-connected spaces, which are completely Hausdorff, separable, locally and arcwise connected, is presented.

1. Introduction

In [3], we introduced definitions of i-connected sets and i-connected spaces. Roughly speaking, a connected set is said to be i-connected if it has a nonempty connected interior, the space X is i-connected if there are no connected sets which have disconnected interiors. In this paper, we give some properties of these sets and spaces.

Of course, we see at once that the straight line with the natural topology is i-connected, however the Euclidean plane is not. Moreover, in the space with a topology, which has at most countable complements, every connected set with a nonempty interior is i-connected. A similar fact is true for topological spaces defined by ordering relations which are dense and have no gap. Another examples of such spaces form closed curves on the Euclidean plane.

In view of the above examples, a natural question arises: what are the conditions for topological spaces in which connectedness is equivalent to i-connectedness? The aim of this paper is to characterize such spaces.

2. Main result

Let X be a nonempty set and let $(X, \mathcal{T})$ stand for a topological space. For any $A \subset X$, the closure of A will be denoted by $\operatorname{cl} A$ and the interior of A by $\operatorname{int} A$.

We start with the following definitions:

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Definition 1 ([3, p. 51]). Let $(X, \mathcal{T})$ be a topological space. A subset A of the space X is called i-connected if it has a nonempty interior and both A and $\operatorname{int} A$ are connected.

Definition 2 ([3, p. 52]). A topological space $(X, \mathcal{T})$ is said to be *i-connected* if every connected set in $(X, \mathcal{T})$ which has a nonempty interior is i-connected.

Definition 3. A topological space $(X, \mathcal{T})$ is said to be a completely Hausdorff space if for every two distinct points x and y in X there exist neighborhoods U_x and U_y of the points x, y, respectively, such that $\operatorname{cl} U_x \cap \operatorname{cl} U_y = \emptyset$.

Theorem 1. *Let $(X, \mathcal{T})$ be a completely Hausdorff and locally connected topological space consisting of at least two elements. If the space $(X, \mathcal{T})$ is i-connected then every connected set which is not a singleton has a nonempty interior.*

P r o o f. Let us suppose, to the contrary, that there exists a connected subset M of X, which is not a singleton, such that $\operatorname{int} M = \emptyset$. Let x and y be two arbitrary distinct points from M.

Since $(X, \mathcal{T})$ is completely Hausdorff space, then there exist two neighborhoods U_x, U_y of the points x, y, respectively, such that

$$\operatorname{cl} U_x \cap \operatorname{cl} U_y = \emptyset. \tag{1}$$

Since $(X, \mathcal{T})$ is locally connected, we conclude that there exist two connected neighborhoods C_x, C_y of the points x, y such that

$$C_x \subset U_x, \quad C_y \subset U_y \tag{2}$$

and, by (1),

$$\operatorname{cl} C_x \cap \operatorname{cl} C_y = \emptyset. \tag{3}$$

Put

$$A = M \cup C_x \cup C_y.$$

From the fact that M, C_x and C_y are connected and M is not separated from C_x and C_y we see at once that A is connected.

Moreover, we claim that

$$\operatorname{int} A \subset \operatorname{cl} C_x \cup \operatorname{cl} C_y. \tag{4}$$

To see this, observe that $\operatorname{int} A \setminus (\operatorname{cl} C_x \cup \operatorname{cl} C_y)$ is open and contained in M. Therefore,

$$\operatorname{int} A \setminus (\operatorname{cl} C_x \cup \operatorname{cl} C_y) = \emptyset$$

and (4) is proven.

By (3) and (4), it follows that $\operatorname{int} A$ is contained in the union of two separated sets in $(X, \mathcal{T})$ having points in both of them, therefore $\operatorname{int} A$ is disconnected, that contradicts the i-connectivity of the space $(X, \mathcal{T})$. □

Remark 1. The assumption that $(X, \mathcal{T})$ is locally connected is essential.

This can be shown in the following example.

EXAMPLE 1. Put

$$X = A \cup \Big(\big([-1,1] \cap Q\big) \times \big([-1,1] \cap Q\big)\Big),$$

where

$$A = \big\{(x,0) \in \mathbb{R}^2 \colon x \in [-1,1]\big\} \cup \big\{(0,y) \in \mathbb{R}^2 : y \in [-1,2]\big\}$$

and let $\mathcal{T}$ be the topology in X generated from the Euclidean plane.

Observe that the only nonempty open and connected sets are of the form $\{0\} \times I$, where I is a subinterval of interval $(1,2]$.

Then $(X,\mathcal{T})$ is completely Hausdorff and i-connected space, however there exists a connected set $[-1,1] \times \{0\}$ which has empty interior.

Remark 2. The assumption in Theorem 1 that the space $(X,\mathcal{T})$ is completely Hausdorff is also necessary.

In fact, the set of real numbers endowed by the topology of countable complements is locally connected and i-connected (see [4, Corollary 3]), however, there exists a connected set, which has at least two points and empty interior (for instance, the set of real positive numbers).

Let us recall the following definition.

DEFINITION 4. Let $(X,\mathcal{T})$ be a topological space and let $x \in X$, $y \in X$, $x \neq y$. The image of unit interval $I = [0,1]$ in $(X,\mathcal{T})$ by a homeomorphism h such that $h(0) = x$ and $h(1) = y$ is called an *arc* with the end points x, y and is denoted by S_{xy}.

A topological space in which any two points can be joined by an arc is called arcwise connected.

Remark 3. The metrizable locally connected continuum is arcwise connected.

From Theorem 1, we immediately have the following corollary.

COROLLARY 1. *Let $(X,\mathcal{T})$ be a completely Hausdorff, locally and arcwise connected topological space. If $(X,\mathcal{T})$ is i-connected, then*

$$\bigwedge_{a\in X} \bigwedge_{b\in X} (a \neq b \Rightarrow \operatorname{int} S_{ab} \neq \emptyset).$$

THEOREM 2. *Let $(X,\mathcal{T})$ be a completely Hausdorff, locally and arcwise connected topological space. If $(X,\mathcal{T})$ is i-connnected, then there is no set, which is a union of three arcs each two of which intersect only at a single common end point of the third arc (i.e., there is no simple triode).*

P r o o f. Suppose, to the contrary, that there exist arcs S_{ab}, S_{bc} and S_{bd} such that

$$S_{ab} \cap S_{bc} \cap S_{bd} = S_{ab} \cap S_{bc} = S_{bc} \cap S_{bd} = S_{ab} \cap S_{bd} = \{b\}.$$

Put

$$M = S_{ab} \cup S_{bc}.$$

Of course, M is connected.

For every neighborhood U of the point b there exists a point x such that

$$x \in S_{bd} \qquad \text{and} \qquad x \neq b.$$

Therefore, we infer that $b \notin \operatorname{int} M$, and finally that

$$\operatorname{int} M \subset \big(S_{ab} \setminus \{b\}\big) \cup \big(S_{bc} \setminus \{b\}\big).$$

By Corollary 1, both $\operatorname{int}\big(S_{ab} \setminus \{b\}\big)$ and $\operatorname{int}\big(S_{ab} \setminus \{b\}\big)$ are nonempty and contained in $\operatorname{int} M$. Therefore, $\operatorname{int} M$ is contained in the union of two separated sets in $(X, \mathcal{T})$ and have common points with both of them which contradicts that $(X, \mathcal{T})$ is i-connected. □

Let us observe that the above theorem states that if a topological space fulfills the assumptions of this theorem and is separable then this space is a union of a countable class of arcs which have the only common point at the end points of those arcs. Hence, we have the following corollaries:

Corollary 2. *Every i-connected topological space which is completely Hausdorff, separable, locally and arcwise connected is either submersible in the real line or is a closed curve (i.e., is homeomorphic to the unit circle in $\mathbb{R}^2$), (see* [2, Theorem]).

Corollary 3. *Every separable, metrizable i-connected and locally connected continuum is either an arc or is a closed curve.*

Corollary 4. *There exist only one dimensional (in the Urysohn sense, see* [1]*) i-connected topological spaces which are completely Hausdorff, separable, locally and arcwise connected.*

Corollary 5. *The assumption that the space has to be separable is essential. For instance, the so called "long line"* [5] *fulfills the remained assumptions of* Corollary 2 *but is not submersible in the real line.*

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