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THE DIESTEL-FAIRES THEOREM ON SERIES

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ABSTRACT. We give a proof of an Orlicz-Pettis Theorem of Diestel and Faires on weak* subseries convergent series in the dual of a Banach space using an elementary theorem on real valued matrices.

The Orlicz-Pettis Theorem on subseries convergence has proven to be one of the most useful theorems in functional analysis with applications to Banach space theory, vector measures and vector integration. The version of the theorem for normed spaces asserts that a series which is subseries convergent in the weak topology is subseries convergent in the norm topology (for the history of the Orlicz-Pettis Theorem, see [FL], [DU], [Ka]). Simple examples show that the analogue of the Orlicz-Pettis Theorem fails for the weak* topology of dual spaces (see Example 1), and, in fact, Diestel and Faires have shown that a Banach space X has the property that series in the dual X' are weak* subseries convergent if and only if they are norm subseries convergent $\iff$ the space X' contains no subspace isomorphic to l^∞. This result of Diestel/Faires is actually a corollary of a much more general result concerning vector valued measures. There have been a number of additional proofs of the Diestel/Faires result, but all of the proofs, including the original, use non-trivial properties of vector measures. For example, the proof in [DU, Diestel and Uhl] uses a lemma of Rosenthal on vector measures and the proof in [Sw2] uses a lemma of Drewnowski on finitely additive set functions. Since the statement of the Diestel/Faires result for series involves only series, it would seem to be desirable to give a proof which only involves basic properties of series and does not invoke properties of vector valued measures. In this brief note we will show that a simple theorem about real valued infinite matrices given in [AS] can be employed to give a proof of the Diestel/Faires result which involves only basic properties of series in normed spaces (actually we consider only one part of the Diestel/Faires result).

First, we give an example showing a straightforward analogue of the Orlicz-Pettis Theorem fails for the weak* topology.

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Example 1. Let e^j be the sequence with 1 in the jth coordinate and 0 in the other coordinates. Then the series $\sum_j e^j$ is weak* subseries convergent in l^∞, the dual of l^1, but is not norm, $\|\cdot\|_\infty$, subseries convergent.

We begin by giving a statement of the matrix result which will be used to derive the Diestel/Faires result.

Theorem 2. *Let $t_{ij} \in \mathbb{R}$ for $i, j \in \mathbb{N}$. If every increasing sequence $\{m_j\}$ has a subsequence $\{n_j\}$ such that the series $\sum_{j=1}^\infty t_{n_i n_j}$ converges and the sequence $\left\{\sum_{j=1}^\infty t_{n_i n_j}\right\}_i$ is bounded, then for every $\epsilon > 0$ there exists a subsequence $\{p_j\}$ such that*

$$\sum_{j=1, j\neq i}^{\infty} \left| t_{p_i p_j} \right| < \epsilon.$$

See [AS] for the proof which is entirely elementary and where the proof is compared to Rosenthal's Lemma.

Let X be a Banach space with dual X' throughout. A series $\sum_j x_j$ in a topological vector space (E, τ) is subseries convergent if the subseries $\sum_{j=1}^\infty x_{n_j}$ is τ convergent for every subsequence $\{n_j\}$ and the series is l^∞ multiplier (bounded multiplier) convergent if the series $\sum_{j=1}^\infty t_j x_j$ is τ convergent for every $\{t_j\} \in l^\infty$. We consider series which are weak* subseries convergent in X' and develop some of the properties of these series which will be needed.

First, since the weak* topology of X' is sequentially complete ([Sw1, 9.11], [Wi, 3.3.13]), any series $\sum_j x'_j$ in X' which is subseries convergent in the weak* topology is also l^∞ multiplier convergent in the weak* topology (see [Day, IV.1] or [Sw1, 16.20], [Sw2, 8.2.1]; the proof in [Sw2, 8.2.1] is based on an interesting inequality of McArthur and Rutherford). We also require the following result. In what follows if E, F are two vector spaces in duality, the weak (strong) topology on E induced by F will be denoted by $\sigma(E, F)$ $(\beta(E, F))$ (see [Sw1], [Wi]).

Proposition 3. *Let $\sum_j x'_j$ be subseries convergent in the weak* topology of X'. The linear operator $U : l^\infty \to X'$ defined by $U\{t_j\} = \sum_{j=1}^\infty t_j x'_j$ [weak* sum] is continuous with respect to the norm topologies of l^∞ and X'.*

Proof. Let $\{t_j\} \in l^\infty, x \in X$. Then

$$\left(\sum_{j=1}^{\infty} t_j x'_j \right) x = \sum_{j=1}^{\infty} t_j x'_j(x)$$

so $\{x'_j(x)\} \in l^1$. This equality implies that U is $\sigma(l^\infty, l^1) - \sigma(X', X)$ continuous and, therefore, U is $\beta(l^\infty, l^1) = \|\cdot\|_\infty - \beta(X', X) = \|\cdot\|$ continuous ([Sw1, 26.15], [Wi, 11.2.6]). □

Finally, we have a result often used in establishing Orlicz-Pettis theorems.

Lemma 4. *If each series $\sum_j x'_j$ in X' which is subseries convergent in the weak* topology satisfies $\|x'_j\| \to 0$, then every series in X' which is weak* subseries convergent is norm subseries convergent.*

Proof. For the proof, see [Day, IV.1.1], [Sw1, 16.20], [Sw2, 10.2.8]. □

Now we state and prove the Diestel/Faires result using Theorem 2.

Theorem 5. *If X' contains a series which is weak* subseries convergent but not norm subseries convergent, then X' contains a subspace isomorphic to l^∞ (note* Example 1).

Proof. By Lemma 4 there exists a series $\sum_j x'_j$ in X' which is weak* subseries convergent and satisfies $\|x'_j\| > \delta > 0$ for some δ and all j. For each j pick $x_j \in X, \|x_j\| = 1$, such that $|x'_j(x_j)| > \delta$. Consider the matrix $M = [x'_j(x_i)]$. For every subsequence $\{m_j\}$,

$$\left|\sum_{j=1}^{\infty} x'_{m_j}(x_i)\right| = \left|\left(\sum_{j=1}^{\infty} x'_{m_j}\right)(x_i)\right| \le \left\|\sum_{j=1}^{\infty} x'_{m_j}\right\|,$$

where $\sum_{j=1}^{\infty} x'_{m_j}$ is the weak* sum of the series. Therefore, the matrix M satisfies the conditions of Theorem 2. Let $\{p_j\}$ be the subsequence of Theorem 2 with $\epsilon = \delta/2$. Now define a linear operator

$$U : l^\infty \to X' \quad \text{by} \quad U\{t_j\} = \sum_{j=1}^{\infty} t_j x'_{p_j} \qquad \text{[weak* sum]}.$$

By Proposition 3, U is norm continuous. We show U has a continuous inverse and this will establish the result. For $t = \{t_j\} \in l^\infty, i \in \mathbb{N}$, we have

$$\begin{aligned}\|Ut\| = \left\|\sum_{j=1}^{\infty} t_j x'_{p_j}\right\| &\ge \left|\left(\sum_{j=1}^{\infty} t_j x'_{p_j}\right)(x_{p_i})\right| \\ &\ge |t_i|\,|x'_{p_i}(x_{p_i})| - \sum_{j=1,j\ne i}^{\infty} \left|t_j x'_{p_j}(x_{p_i})\right| \\ &\ge |t_i|\,\delta - \|t\|_\infty \sum_{j=1,j\ne i}^{\infty} \left|x'_{p_j}(x_{p_i})\right| \\ &\ge |t_i|\,\delta - \|t\|_\infty\, \delta/2.\end{aligned}$$

Taking the supremum over i gives $\|Ut\| \ge (\delta/2)\|t\|_\infty$ which implies U has a continuous inverse. □

Diestel and Faires have also shown that the converse of Theorem 5 holds giving a characterization of dual spaces containing a copy of l^∞. Their proof uses geometric properties of Banach spaces and the series in Example 1 as a prototype of series which are weak* subseries convergent but not norm subseries convergent.

There are other applications of the matrix Theorem 2 to Banach spaces and vector valued measures given in [AS].

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