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NOTES ON THE IDEAL (a)

Andrzej Nowik

ABSTRACT. For any topologies $\tau_1 \subseteq \tau_2$, we consider the ideal of sets A such that for each nonempty $U \in \tau_2$ there exists $W \in \tau_1$ such that $U \cap W \neq \emptyset$ and $A \cap U \cap W = \emptyset$. For τ_1, the standard Euclidean topology, and for τ_2, the density topology, we obtain the ideal (a) investigated by Z. Grande and E. Strońska.

1. Notations and definitions

Let τ_e denote the standard Euclidean topology in $\mathbb{R}$. For any topology τ in $\mathbb{R}$, $\mathsf{ND}(\tau)$ is the ideal of nowhere dense sets of τ. For simplicity, we put $\mathsf{ND} = \mathsf{ND}(\tau_e)$, and for every set $X \subseteq \mathbb{R}$, we denote by $\mathsf{ND}(X)$ the collection of all nowhere dense subsets of X in the Euclidean topology of X. By $\mathrm{cl}_\tau(X)$ we denote the closure of X in the topology τ. If τ is any topology in $\mathbb{R}$ then by τ_X we denote the subspace topology in X induced from τ, and by $\mathsf{ND}_{\tau_X}(X)$ the ideal of all nowhere dense subsets of X in the subspace topology τ_X.

Denote by μ the Lebesgue measure in $\mathbb{R}$. For every measurable A, by $\Phi(A)$ let us denote the set of density points of A:

$$\Phi(A) = \left\{ x \in \mathbb{R} \colon \liminf_{h \to 0^+} \frac{\mu(A \cap [x-h, x+h])}{2h} = 1 \right\}.$$

We write τ_d to denote the density topology on $\mathbb{R}$, namely: $U \in \tau_d \iff U$ is Lebesgue measurable and $U \subseteq \Phi(U)$.

Let $\mathcal{N}, \mathcal{M}$ denote the σ-ideal of null sets of reals, the σ-ideal of meager sets (i.e., sets of first category), respectively.

For any topology on $\mathbb{R}$, by $\mathcal{M}(\tau)$ we denote the σ-ideal of meager sets in the topology τ.

For any collection $\mathcal{F}$ of subsets of $\mathbb{R}$ such that there is no countable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\cup \mathcal{F}_0 = \mathbb{R}$, by $\sigma(\mathcal{F})$, we denote the σ-ideal $\{X \subseteq \mathbb{R} \colon \exists_{\mathcal{F}_0 \subseteq \mathcal{F}} |\mathcal{F}_0| \leq \aleph_0 \wedge X \subseteq \cup \mathcal{F}_0\}$.

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Let us recall the following definition of special σ-ideal of small subsets of $\mathbb{R}$:

Definition 1.1. Given a set $X \subseteq \mathbb{R}$, the porosity of X at a real $r \in \mathbb{R}$ is defined by

$$p(X,r) = \limsup_{\epsilon \to 0^+} \frac{\lambda(X,(r-\epsilon,r+\epsilon))}{\epsilon},$$

where $\lambda(X,I)$ denotes the maximal length of an open subinterval of the interval I which is disjoint from X.

A set X is *porous* ($X \in \mathcal{P}$) if and only if $\forall_{a \in X} p(X,a) > 0$.

Let $\sigma\mathcal{P}$ denote the sigma-ideal generated by porous sets.

Notice that we have $\sigma\mathcal{P} \subseteq \mathcal{N} \cap \mathcal{M}$.

2. The ideal (a)

Let us recall the following definition introduced by Z. Grande and E. Strońska:

Definition 2.1 ([GS]). We say that a set $A \subseteq \mathbb{R}$ satisfies condition (a) (this notation is taken from [GS]) if for each nonempty set $U \subseteq \mathrm{cl}(A)$ belonging to τ_d the intersection $A \cap U$ is a nowhere dense subset (in the Euclidean topology) of U.

Observation 2.2. Suppose that $A \subseteq \mathbb{R}$. The following conditions are equivalent:

(1) $A \in (a)$;

(2) $\forall_{U \in \tau_d \setminus \{\emptyset\}} A \cap U \in \mathsf{ND}(U)$;

(3) $\forall_{U \in \tau_d \setminus \{\emptyset\}} \exists_{W \in \tau_e} U \cap W \neq \emptyset \wedge U \cap W \cap A = \emptyset$.

The definition of property (a) can be formulated in the general case of the bitopological space.

Throughout the rest of this section, we will assume that (X, τ_1, τ_2) is a bitopological space (i.e., τ_1 and τ_2 are topologies on X, see [D]) such that τ_1 is coarser than τ_2: $\tau_1 \subseteq \tau_2$.

Definition 2.3. Define $A \in (a)(\tau_1, \tau_2)$ if for each nonempty set $U \subseteq \mathrm{cl}_{\tau_1}(A)$ such that $U \in \tau_2$, the intersection $A \cap U$ is a nowhere dense subset in the subspace topology τ_{1_U} of U and denote

$$\mathsf{ND}(\tau_1,\tau_2) = \{A \subseteq X \colon \forall_{U \in \tau_2 \setminus \{\emptyset\}} \exists_{W \in \tau_1} U \cap W \neq \emptyset \wedge U \cap W \cap A = \emptyset\}.$$

Notice that the first definition generalizes Definition 2.1 to any bitopological space. We have

Theorem 2.4. *Suppose that $X \subseteq \mathbb{R}$. The following conditions are equivalent:*

(1) $A \in (a)(\tau_1, \tau_2)$;

(2) $\forall_{U \in \tau_2 \setminus \{\emptyset\}} A \cap U \in \mathsf{ND}_{\tau_{1_U}}(U)$;

(3) $A \in \mathsf{ND}(\tau_1, \tau_2)$.

Proof.

(2) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (3). Suppose that $A \in (a)(\tau_1, \tau_2)$ and let $U \in \tau_2 \setminus \{\emptyset\}$. If $U \subseteq \mathrm{cl}_{\tau_1} A$, then $U \cap A \in \mathsf{ND}_{\tau_{1_U}}(U)$, so there exists $W \in \tau_1 \setminus \{\emptyset\}$ such that $W \cap U \neq \emptyset$ and $W \cap U \cap A = \emptyset$. On the other hand, assume that $U \setminus \mathrm{cl}_{\tau_1} A \neq \emptyset$. Then define $W = X \setminus \mathrm{cl}_{\tau_1} A$. It is easy to see that $W \in \tau_1$, $W \cap U \neq \emptyset$ and $W \cap U \cap A = \emptyset$.

(3) $\Rightarrow$ (2). Suppose that $A \in \mathsf{ND}(\tau_1, \tau_2)$ and let $U \in \tau_2 \setminus \{\emptyset\}$. Let $W_1 \in \tau_1$ be such that $W_1 \cap U \neq \emptyset$. Then $U \cap W_1 \in \tau_2 \setminus \{\emptyset\}$, so there exists $W \in \tau_1$ such that $U \cap W_1 \cap W \neq \emptyset$ and $U \cap W_1 \cap W \cap A = \emptyset$. This shows that $A \cap U \in \mathsf{ND}_{\tau_{1_U}}(U)$. $\square$

Notice the following easy observations and conclusions:

(1) $\mathsf{ND}(\tau_e, \tau_d) = (a)$;

(2) Assume that
$$\forall_{x \in X} X \setminus \{x\} \in \tau_1 \wedge \{x\} \notin \tau_2 \tag{1}$$
and $|X| \geq 2$. Then $\mathsf{ND}(\tau_1, \tau_2)$ is an ideal of sets which contains singletons;

(3) $\mathsf{ND}(\tau_1, \tau_2) \subseteq \mathsf{ND}(\tau_1) \cap \mathsf{ND}(\tau_2)$.

Let us notice that in [D] there was defined another version of nowhere dense sets in bitopological space, namely, let us recall the following definition.

Definition 2.5 ([D]). A subset $A \subseteq X$ is $(2,1) - \mathsf{ND}(\tau_1, \tau_2)$ if and only if $\mathrm{int}_{\tau_2}\big(\mathrm{cl}_{\tau_1}(A)\big) = \emptyset$.

However, this property is different from the notion $\mathsf{ND}(\tau_1, \tau_2)$ since, as we can easy observe, $(2,1) - \mathsf{ND}(\tau_e, \tau_d)$ is the ideal generated by closed measure zero sets.

We have

Theorem 2.6. *For any $A \subseteq X$, the following conditions are equivalent:*

(1) $A \in \mathsf{ND}(\tau_1, \tau_2)$;

(2) $\forall_{\substack{\tau_1 \subseteq \tau \subseteq \tau_2 \\ \tau\, topology}} A \in \mathsf{ND}(\tau)$;

Proof.

(1) $\Rightarrow$ (2). Suppose that τ is a topology on $\mathbb{R}$ such that $\tau_1 \subseteq \tau \subseteq \tau_2$. Take $U \in \tau \setminus \{\emptyset\}$. Then $U \in \tau_2 \setminus \{\emptyset\}$, so there exists $W \in \tau_1$ such that $U \cap W \neq \emptyset$ and $U \cap W \cap A = \emptyset$. Since $W \in \tau$, this shows that $A \in \mathsf{ND}(\tau)$.

(2) $\Rightarrow$ (1). Suppose that $U \in \tau_2 \setminus \{\emptyset\}$. Let τ be a topology generated by $\tau_1 \cup \{U\}$, i.e., $\tau = \{U_1 \cup (U \cap U_2) \colon U_1, U_2 \in \tau_1\}$. Since $A \in \mathsf{ND}(\tau)$, we conclude that there

exists $V \in \tau \setminus \{\emptyset\}$ such $V \subseteq U$ and $V \cap A = \emptyset$. Let $U_1, U_2 \in \tau_1$ be such that $V = U_1 \cup (U \cap U_2)$. Then $V = U \cap V = U \cap (U_1 \cup U_2)$. Define $W = U_1 \cup U_2$, then $W \in \tau_1$, $U \cap W \neq \emptyset$ and $U \cap W \cap A = \emptyset$, which shows that $A \in \mathsf{ND}(\tau_1, \tau_2)$. $\square$

Corollary 2.7. *For every $X \subseteq \mathbb{R}$, $X \in (a)$ if and only if for every topology τ on $\mathbb{R}$ finer than the Euclidean topology and coarser than the density topology, we have $X \in \mathsf{ND}(\tau)$.*

Theorem 2.8. *Suppose that A is a closed set in the topology τ_1. Then, $A \setminus \mathrm{int}_{\tau_2}(A) \in \mathsf{ND}(\tau_1, \tau_2)$.*

P r o o f. Let $U \in \tau_2 \setminus \{\emptyset\}$. Suppose that $U \setminus A \neq \emptyset$. Define $W = \mathbb{R} \setminus A$. Then, $U \cap W \neq \emptyset$ and $U \cap W \cap \big(A \setminus \mathrm{int}_{\tau_2}(A)\big) = \emptyset$.

On the other hand, suppose that $U \subseteq A$. Then $U \subseteq \mathrm{int}_{\tau_2}(A)$. Put $W = \mathbb{R}$, then $U \cap W \neq \emptyset$ and $U \cap W \cap \big(A \setminus \mathrm{int}_{\tau_2}(A)\big) = \emptyset$.

This shows that $A \setminus \mathrm{int}_{\tau_2}(A) \in \mathsf{ND}(\tau_1, \tau_2)$. $\square$

As a corollary we obtain the following result, which was not explicitly stated in [GS], but probably implicitly contained in this article.

Corollary 2.9 ([GS])**.** *For every closed (in the Euclidean topology) set $E \subseteq \mathbb{R}$ we have $E \setminus \Phi(E) \in (a)$.*

P r o o f. This follows from the fact that $\mathsf{ND}(\tau_e, \tau_d) = (a)$ and that for every closed set $E \subseteq \mathbb{R}$ (in the Euclidean topology), we have $\mathrm{int}_{\tau_d}(E) = \Phi(E)$. $\square$

3. Porous sets

The aim of this chapter is to prove the following inclusion.

Theorem 3.1. $\mathcal{P} \subseteq (a)$.

P r o o f. Suppose that $A \in \mathcal{P}$, and let $U \in \tau_d \setminus \{\emptyset\}$. If $U \cap A = \emptyset$, then define $W = \mathbb{R}$; then $U \cap W \neq \emptyset$ and $U \cap W \cap A = \emptyset$. On the other hand, suppose that $U \cap A \neq \emptyset$. Let $x_0 \in U \cap A$ be arbitrary. Since $A \in \mathcal{P}$, we have

$$\exists_{\eta>0} \forall_{\epsilon>0} \exists_{0<\delta<\epsilon} \lambda\big(A, (x_0 - \delta, x_0 + \delta)\big) > \delta \cdot \eta. \tag{2}$$

Since x_0 is a density point of U let $\epsilon_0 > 0$ be such that

$$\forall_{0<\epsilon<\epsilon_0} \frac{\mu([x_0 - \epsilon, x_0 + \epsilon] \cap U)}{2\epsilon} > \left(1 - \frac{\eta}{2}\right).$$

Let us choose (by (2)) $\delta \in (0, \epsilon_0)$ such that

$$\lambda\big(A, (x_0 - \delta, x_0 + \delta)\big) > \delta \cdot \eta. \tag{3}$$

Then, we have

$$\frac{\mu([x_0-\delta,x_0+\delta]\cap U)}{2\delta} > 1-\frac{\eta}{2}. \tag{4}$$

By (3), let $J \subseteq (x_0-\delta, x_0+\delta)$ be an open interval such that $A \cap J = \emptyset$ and $|J| > \delta \cdot \eta$. Suppose, on the contrary, that $U \cap J = \emptyset$. Then,

$$(x_0-\delta,x_0+\delta)\cap U \subseteq (x_0-\delta,x_0+\delta)\setminus J,$$

so,

$$\begin{aligned}\mu\bigl(U\cap[x_0-\delta,x_0+\delta]\bigr) &= \mu\bigl(U\cap(x_0-\delta,x_0+\delta)\bigr)\\ &\le 2\delta-|J|\\ &< 2\delta-\eta\delta\\ &= \delta(2-\eta).\end{aligned}$$

Thus,

$$\frac{\mu(U\cap[x_0-\delta,x_0+\delta])}{2\delta} < 1-\frac{\eta}{2},$$

which is a contradiction to (4).

Therefore, $U \cap J \neq \emptyset$. Define $W = J$, then, $U \cap W \neq \emptyset$, $W \in \tau_e$ and $W \cap U \cap A = \emptyset$.

This shows that $A \in (a)$. □

Let us notice that the inclusion from Theorem 3.1 is not reversible, since there exists a closed set of measure zero which is not porous and the set belongs (as it was stated in [GS]) to the ideal (a).

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University of Gdańsk
Institute of Mathematics
Wita Stwosza 57
PL–80-952 Gdańsk
POLAND
E-mail: matan@julia.univ.gda.pl