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THE SET OF DISCONTINUITIES OF DENSITY-TYPE-APPROXIMATELY CONTINUOUS FUNCTIONS

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ABSTRACT. A characterization of the set of discontinuities for different density-type-approximately continuous functions is given.

1. Introduction

It is well-known that the following two statements for a set $A \subset \mathbb{R}$ are equivalent (see [G]):

(i) A is an F_σ set,

(ii) there is a function $f\colon \mathbb{R} \to \mathbb{R}$ such that A is the set of discontinuities of f.

An analogous result has been established for approximately continuous functions (see [B, p. 48]):

Theorem 1.1. *The following two statements for a set $A \subset [a,b]$ are equivalent:*

(i) *A is dense in $[a,b]$ and of type G_δ.*

(ii) *There is an approximately continuous function f such that A is the set of continuities of f.*

The aim of this paper is to verify whether the same result can be established for some density-type topologies. Our main result is as follows:

Theorem 1.2. *The following two statements for a set $A \subset \mathbb{R}$ (or $A \subset \mathbb{R}^2$) are equivalent:*

(i) *A is an F_σ first category set,*

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(ii) *There is a continuous function* $f\colon (\mathbb{R},\tau) \to (\mathbb{R},\mathcal{O})$ *(or* $f\colon (\mathbb{R}^2,\tau) \to (\mathbb{R},\mathcal{O})$*) such that* A *is the set of discontinuities of* f*, where* τ *denotes one of the following topologies: the density topology* [W1]*, simple density topology* [WA]*, complete density topology* [WW]*,* ψ*-density topology* [TW-B]*,* $\langle s\rangle$*-density topology generated by a sequence* $\langle s\rangle$ [FH] *(respectively density, strong density* [W1]*, product of density topologies or ordinary* ψ*-density topology on the plane* [F]*) and* $\mathcal{O}$ *stands for the Euclidean topology.*

The main idea of the proof comes from [P-WS], where a theorem for the density topology case was proved (as in [B]).

What is interesting the situation in the category case (I-density topology) is different. Some additional assumptions of a set A are needed to receive the existence of I-approximately continuous function $f\colon (\mathbb{R},\mathcal{T}_I) \to (\mathbb{R},\mathcal{O})$ for which A is equal to the set of all points of ordinary discontinuity (see [W2]).

2. Definitions and basic facts about density type topologies

Each of the topologies mentioned above is defined as a family of those Lebesgue measurable sets A such that each point of A is a point of some kind of density of A:

$$\tau = \big\{A \in \mathcal{L} : A \subset \Phi(A)\big\},$$

where $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable sets on the line, and $\Phi(A)$ denotes the set of some kind of density points of A for $A \in \mathcal{L}$.

We obtain different topologies by using different type of densities.

Definition 2.1 ([W1])**.** We say that a point $x \in \mathbb{R}$ is a density point of $A \in \mathcal{L}$ if and only if

$$\lim_{h\to 0^+} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1,$$

where λ stands for the Lebesgue measure on the line. We denote the density topology by $\mathcal{T}$.

Definition 2.2 ([W1], [TW-B])**.** Let $\mathcal{C}$ denote the class of continuous increasing functions $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, such that $\lim_{t\to 0^+} \psi(t) = 0$.

Let $\psi \in \mathcal{C}$. We say that $x \in \mathbb{R}$ is a ψ-density point of $A \in \mathcal{L}$ if and only if

$$\lim_{h\to 0^+} \frac{\lambda(A' \cap [x-h, x+h])}{2h\psi(2h)} = 0,$$

where A' denotes a complement of the set A.

We denote the set of ψ-density point of A by $\Phi_\psi(A)$, so ψ-density topology $\mathcal{T}_\psi$ is defined by

$$\mathcal{T}_\psi = \big\{A \in \mathcal{L} : A \subset \Phi_\psi(A)\big\}.$$

Proposition 2.3 ([W1], [TW-B]). *For each* $\psi \in \mathcal{C} : \mathcal{O} \subsetneq \mathcal{T}_\psi \subsetneq \mathcal{T}$.

Definition 2.4 ([WA]). We say that $x \in \mathbb{R}$ is a simple density point of the set $A \in \mathcal{L}$ if $\big\{\chi_{(n\cdot(A-x))\cap[-1,1]}\big\}_{n\in\mathbb{N}}$ converges to $\chi_{[-1,1]}$ almost everywhere, where $A - x = \{t - x : t \in A\}$, $n \cdot A = \{nt : t \in A\}$ and χ_A denotes a characteristic function of the set A.

The set of all simple density point of A will be denoted by $\Phi_s(A)$, so the simple density topology $\mathcal{T}_s$ is given by

$$\mathcal{T}_s = \big\{A \in \mathcal{L} : A \subset \Phi_s(A)\big\}.$$

Definition 2.5 ([WW]). We say that $x \in \mathbb{R}$ is a complete density point of a set $A \in \mathcal{L}$ if

$$\sum_{n=1}^{\infty} \lambda\Big(\big(n(A' - x)\big) \cap [-1,1]\Big) < +\infty.$$

The set of all complete density points of A will be denoted by $\Phi_c(A)$, so the complete density topology $\mathcal{T}_c$ is defined by

$$\mathcal{T}_c = \big\{A \in \mathcal{L} : A \subset \Phi_c(A)\big\}.$$

Proposition 2.6 ([WW], [WA]). $\mathcal{O} \subsetneq \mathcal{T}_c \subsetneq \mathcal{T}_s \subsetneq \mathcal{T}$.

Definition 2.7 ([FH]). Let S be the family of all unbounded and nondecreasing sequences of positive reals. Every sequence $\{s_n\} \in S$ is denoted by $\langle s \rangle$.

Let $\langle s \rangle \in S$. We say that $x \in \mathbb{R}$ is an $\langle s \rangle$-density point of $A \in \mathcal{L}$ if

$$\lim_{n\to\infty} \frac{\lambda(A \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$

The set of all $\langle s \rangle$-density points of A will be denoted by $\Phi_{\langle s \rangle}(A)$, so $\langle s \rangle$-density topology $\mathcal{T}_{\langle s \rangle}$ is defined by

$$\mathcal{T}_{\langle s \rangle} = \big\{A \in \mathcal{L} : A \subset \Phi_{\langle s \rangle}(A)\big\}.$$

Proposition 2.8 ([FH]). *For every sequence* $\langle s \rangle \in S$*, the following inclusion* $\mathcal{T} \subset \mathcal{T}_{\langle s \rangle}$ *holds and* $\mathcal{T} = \mathcal{T}_{\langle s \rangle}$ *if and only if* $\liminf \frac{s_n}{s_{n+1}} > 0$.

Let $\mathcal{L}_2$ denote the σ-algebra of Lebesgue measurable sets on the plane, and for $h > 0$ and $p = (x_0, y_0) \in \mathbb{R}^2$ put $Q(p,h) = [x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$.

Definition 2.9 ([W1]). We say that $p = (x_0, y_0) \in \mathbb{R}^2$ is an ordinary density point of $A \in \mathcal{L}_2$ if

$$\lim_{h \to 0^+} \frac{\lambda_2\big(A \cap Q(p,h)\big)}{4h^2} = 1,$$

where λ_2 stands for the Lebesgue measure on the plane.

The set of all ordinary density points of A will be denoted by $\Phi_2(A)$ and the ordinary density topology $\mathcal{T}_2$ is defined by

$$\mathcal{T}_2 = \big\{A \in \mathcal{L}_2 : A \subset \Phi_2(A)\big\}.$$

Definition 2.10 ([W1]). We say that a point $p = (x_0, y_0) \in \mathbb{R}^2$ is a strong density point of the set $A \in \mathcal{L}_2$ if

$$\lim_{\substack{h \to 0^+ \\ k \to 0^+}} \frac{\lambda_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k])}{4hk} = 1.$$

The set of all strong density points of A will be denoted by $\Phi^s(A)$, so strong density topology $\mathcal{T}_2^s$ is defined by

$$\mathcal{T}_2^s = \big\{A \in \mathcal{L}_2 : A \subset \Phi^s(A)\big\}.$$

Proposition 2.11. *$\mathcal{O}_2 \subset \mathcal{T} \times \mathcal{T} \subset \mathcal{T}_2^s \subset \mathcal{T}_2$, where $\mathcal{O}_2$ stands for the Euclidean topology on the plane.*

Definition 2.12 ([F]). Let $\psi \in \mathcal{C}$. We say that $p = (x_0, y_0)$ is an ordinary ψ-density point of $A \in \mathcal{L}_2$ if

$$\lim_{h \to 0^+} \frac{\lambda_2(A' \cap Q(p,h))}{4h^2 \psi(4h^2)} = 0.$$

The set of all ordinary ψ-density points of A will be denoted by $\Phi_\psi^0(A)$, so ordinary ψ density topology $\mathcal{T}_\psi^0$ is defined by

$$\mathcal{T}_\psi^0 = \big\{A \in \mathcal{L}_2 : A \subset \Phi_\psi^0(A)\big\}.$$

Proposition 2.13 ([F]). *For each $\psi \in C : \mathcal{O}_2 \subsetneq \mathcal{T}_\psi^0 \subsetneq \mathcal{T}_2$.*

Definition 2.14. For every topology τ which was mentioned above, we say that a function is τ-approximately continuous if it is continuous as a function $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \mathcal{O})$ or $f : (\mathbb{R}^2, \tau) \to (\mathbb{R}, \mathcal{O})$.

Proposition 2.15 ([W1]). *Every $\mathcal{T}$-approximately continuous and every $\mathcal{T}_2$-approximately continuous function is a Baire one function.*

Corollary 2.16. *Every $\mathcal{T} \times \mathcal{T}$, $\mathcal{T}_2^s$, $\mathcal{T}_\psi$, $\mathcal{T}_\psi^0$ (for any $\psi \in \mathcal{C}$) $\mathcal{T}_s$ or $\mathcal{T}_c$-approximately continuous function is Baire one function.*

Proposition 2.17 ([E, Th. 1.4.7]). *If f_n is τ-approximately continuous for any $n \in \mathbb{N}$ and a sequence $\{f_n\}_{n \in \mathbb{N}}$ uniformly converges to f, then f is τ-approximately continuous.*

Proposition 2.18 ([L]). *Let $\langle s\rangle \in S$. If f is $\mathcal{T}_{\langle s\rangle}$-approximately continuous then f is a Baire one function.*

3. Proof of the main theorem

(ii) $\Longrightarrow$ (i) follows immediately from Proposition 2.18 and Corollary 2.16 for each topology considered here since the set of discontinuities of a Baire one function is a first category F_σ set [Ł].

(i) $\Longrightarrow$ *(ii)*

$\mathcal{T}_\psi$**-case.** Let $\psi \in \mathcal{C}$. Let A be an F_σ first category set. There exists a countable set C and a sequence of pairwise disjoint nowhere dense perfect sets P_i such that $A = \bigcup_{i\in\mathbb{N}} P_i \cup C$ (see [S, vol. II, p. 537]).

First, we want to build a $\mathcal{T}_\psi$-approximately continuous function which is discontinuous at the point 0.

Since $\psi \in \mathcal{C}$, there exists $n_0 \in \mathbb{N}$ such that $\psi\big(\frac{1}{2^{n_0}}\big) \le 1$. Let

$$a_n = \frac{1}{2^n} - \frac{1}{4^n}\psi\big(\frac{1}{2^n}\big), \quad b_n = \frac{1}{2^n} \text{ for } n \ge n_0$$

and

$$B = \bigcup_{n=n_0}^{\infty} [a_n, b_n].$$

Let $h < \frac{1}{2^{n_0}}$. Then there exists $n \ge n_0$, $n \in \mathbb{N}$ such that $\frac{1}{2^{n+1}} \le h < \frac{1}{2^n}$, so

$$\begin{aligned}\lambda\big(B \cap [0,h]\big) \le\ & \lambda\left(B \cap \left[0, \frac{1}{2^n}\right]\right) = \sum_{i=n}^{\infty} \frac{1}{4^i}\,\psi\left(\frac{1}{2^i}\right)\\ & \le \psi\left(\frac{1}{2^n}\right) \cdot \sum_{i=n}^{\infty} \frac{1}{4^i} \ < \psi\left(\frac{1}{2^n}\right) \cdot \frac{1}{4^{n-1}}.\end{aligned}$$

Therefore,

$$\frac{\lambda(B \cap [0,h])}{2h\psi(2h)} < \frac{\psi\big(\frac{1}{2^n}\big) \cdot \frac{1}{4^{n-1}}}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\big(2 \cdot \frac{1}{2^{n+1}}\big)} = \frac{1}{2^{n-2}}.$$

Since for $h \to 0^+$ we have n tending to $+\infty$, so $0 \in \Phi_\psi(B')$ and $B' \in \mathcal{T}_\psi$, because each point from $B' \setminus \{0\}$ is an inner one in the Euclidean topology.

Let

$$h_0(x) = \begin{cases} 0 & \text{for} \quad x \in B', \\ 1 & \text{for} \quad x = \frac{b_n + a_n}{2}, n \in \mathbb{N}, \\ \text{linear and continuous} & \text{for} \quad x \in \big[a_n, \frac{b_n+a_n}{2}\big] \text{ and } x \in \big[\frac{b_n+a_n}{2}, b_n\big]. \end{cases}$$

Then h_0 has the desired properties.

We define $h(x) = 0$ if $C = \emptyset$. Otherwise, let $\{r_n\}_{n\in\mathbb{N}}$ be an enumeration of a considered countable set C. We define

$$h_n(x) = \frac{1}{2^n} h_0(x - r_n) \quad \text{and} \quad \mathrm{h}(x) = \sum_{n=1}^{\infty} h_n(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

Then the series is uniformly convergent on $\mathbb{R}$, so by Proposition 2.17, the function h is $\mathcal{T}_\psi$-approximately continuous. For a given n_0, the function $\sum_{n\neq n_0} h_n$ is continuous at the point r_{n_0}. Thus,

$$h = h_{n_0} + \sum_{n\neq n_0} h_n$$

is discontinuous there. Therefore h is discontinuous on C, and continuous on C'.

Let $\mathcal{P}$ be a nowhere dense perfect set.

Let $\{I_n\}_{n\in\mathbb{N}}$ be the family of connected bounded components of the complement of $\mathcal{P}$. Let

$$\sum_{n\in\mathbb{N}} z_n < +\infty$$

such that $z_n > 0$ for $n \in \mathbb{N}$. We denote by J_n an interval centered at the midpoint of I_n and such that

$$\lambda(J_n) = \min\left(z_n \cdot \lambda(I_n)\psi\left(\frac{1}{2}\lambda(I_n)\right), \frac{1}{n}\lambda(I_n)\right), \qquad n \in \mathbb{N}.$$

Let $p \in \mathcal{P}$. We want to show that

$$\lim_{h\to 0^+} \frac{\lambda(\bigcup_{n\in\mathbb{N}} J_n \cap [p, p+h])}{2h\psi(2h)} = 0.$$

We can assume that p is not a left end-point of any I_n (since in the opposite case there exists $h > 0$ such that $\bigcup_{n\in\mathbb{N}} J_n \cap [p, p+h] = \emptyset$).

We fix $n_0 \in \mathbb{N}$. Then there exists $h > 0$ such that

$$[p, p+h] \cap \bigcup_{n=1}^{n_0} J_n = \emptyset.$$

Let $\{n_k\}_{k\in\mathbb{N}}$ be a subsequence of $\{n\}_{n\in\mathbb{N}}$ consisting of all n_k's such that

$$(p, p+h] \cap J_{n_k} \neq \emptyset.$$

Then $n_k > n_0$ for any $k \in \mathbb{N}$.

It is easily seen that $h > \frac{\lambda(I_{n_k}) - \lambda(J_{n_k})}{2}$, so

$$\lambda(I_{n_k}) < 2h + \lambda(J_{n_k}) \leq 2h + \frac{1}{n_k}\lambda(I_{n_k}),$$

thus

$$(1 - \frac{1}{n_k})\lambda(I_{n_k}) \leq 2h, \quad k \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}\frac{\lambda\big(\bigcup_{n\in\mathbb{N}} J_n \cap [p,p+h]\big)}{2h\psi(2h)} &= \frac{\lambda\big(\bigcup_{k\in\mathbb{N}} J_{n_k} \cap [p,p+h]\big)}{2h\psi(2h)} \\ &\le \sum_{k\in\mathbb{N}} \frac{\lambda(J_{n_k})}{2h\psi(2h)} \\ &\le \sum_{k\in\mathbb{N}} \frac{z_n\cdot\lambda(I_{n_k})\cdot\psi\big(\frac{1}{2}\lambda(I_{n_k})\big)}{\big(1-\frac{1}{n_k}\big)\lambda(I_{n_k})\cdot\psi\big((1-\frac{1}{n_k})\lambda(I_{n_k})\big)} \\ &\le \sum_{k\in\mathbb{N}} \frac{z_{n_k}\psi\big(\frac{1}{2}\lambda(I_{n_k})\big)}{\frac{1}{2}\psi\big(\frac{1}{2}\lambda(I_{n_k})\big)} \\ &= 2\cdot\sum_{k\in\mathbb{N}} z_{n_k} \le 2\cdot\sum_{n=n_0}^{\infty} z_n.\end{aligned}$$

Since for $n_0 \to +\infty$ the rest 0 series $\sum_{n=n_0}^{\infty} z_n$ tends to 0, we have

$$\lim_{h\to 0^+} \frac{\lambda\big(\bigcup_{n\in\mathbb{N}} J_n \cap [p,p+h]\big)}{2h\psi(2h)} = 0.$$

The argument similar to that above shows that also

$$\lim_{h\to 0^+} \frac{\lambda\big(\bigcup_{n\in\mathbb{N}} J_n \cap [p-h,p]\big)}{2h\psi(2h)} = 0.$$

Consequently,

$$p \in \Phi_\psi\bigg(\Big(\bigcup_{n\in\mathbb{N}} J_n\Big)'\bigg).$$

We define g_P to be 0 on the complement of $\bigcup_{n\in\mathbb{N}} J_n$, $g_P(J_n) = [0,1]$ for $n \in \mathbb{N}$ and g_P continuous on I_n, $n \in \mathbb{N}$. Then g_P is $\mathcal{T}_\psi$-approximately continuous everywhere and discontinuous exactly at all points of $\mathcal{P}$.

Now, for each set $\mathcal{P}_i$, $i \in \mathbb{N}$, we define a function $g_i(x) = \frac{1}{3^i} g_{P_i}(x)$ and let $g(x) = \sum_{i\in\mathbb{N}} g_i(x)$. The function g is $\mathcal{T}_\psi$-approximately continuous everywhere and continuous at all points from the complement of $\bigcup_{i\in\mathbb{N}} \mathcal{P}_i$.

To end the construction, let $f = g + h$. Of course, f is $\mathcal{T}_\psi$-approximately continuous and the set A is exactly the set of discontinuities of f.

$\mathcal{T}_c$-case. Let $\widetilde{\psi}\colon \mathbb{R}^+ \to \mathbb{R}^+$ be given by

$$\widetilde{\psi}(x) = \begin{cases} \frac{1}{n^2} & \text{for } x \in \big[\frac{2}{n}, \frac{2}{n-1}\big), \quad n > 1, \\ 1 & \text{for } x \in [1,+\infty). \end{cases}$$

An easy computation shows that if for an arbitrarily chosen set A, $A \subset \mathbb{R}$, if

$$\lim_{n\to 0^+} \frac{\lambda(A' \cap [-h,h])}{2h\widetilde{\psi}(2h)} = 0,$$

then

$$\lim_{n\to\infty}\frac{\lambda\big(A'\cap\big[\frac{1}{n},\frac{1}{n}\big]\big)}{\frac{2}{n}\widetilde{\psi}\big(\frac{2}{n}\big)}=0\quad\text{and}\quad\sum_{n=1}^{\infty}\lambda\big(n\cdot A'\cap[-1,1]\big)<+\infty,$$

so

$$0\in\phi_c(A).$$

For any non-decreasing function $\widetilde{\psi}$ transforming $\mathbb{R}^+$ into $\mathbb{R}^+$ and such that $\lim_{x\to 0^+}\widetilde{\psi}(x)=0$, we can define a $\widetilde{\psi}$-density topology as for a function from $\mathcal{C}$ and for each function with such properties, we can find a function from the family $\mathcal{C}$ such that the topologies regulated by these two functions coincide [AW].

Therefore, there exists a function $\psi\in\mathcal{C}$ such that $\mathcal{T}_\psi\subset\mathcal{T}_c$, hence the function from $\mathcal{T}_\psi$-case works for $\mathcal{T}_c$ case.

$\mathcal{T}_s$-case. According to Proposition 2.6, the function from $\mathcal{T}_c$-case is also $\mathcal{T}_s$-approximately continuous on $\mathbb{R}$.

$T_{\langle s\rangle}$-case. Obviously, it follows from the theorem for approximately continuous functions [B], since the approximately continuous function is also $\mathcal{T}_{\langle s\rangle}$-approximately continuous on $\mathbb{R}$.

$\mathcal{T}^0_\psi$-case. Let $\psi\in\mathcal{C}$. Firstly we build a $\mathcal{T}^0_\psi$-approximately continuous function which is discontinuous at the point $(0,0)$.

Since $\psi\in\mathcal{C}$, then there exists $n_0\in\mathcal{N}$ such that $\psi\big(\frac{1}{2^{n_0}}\big)\le 1$. Put

$$B=\bigcup_{n=n_0}^{\infty}\left(\left[\frac{1}{2^n}-\frac{1}{4^n}\psi\left(\frac{1}{4^n}\right),\frac{1}{2^n}\right]\times\left[-\frac{1}{2\cdot 4^n}\psi\left(\frac{1}{4^n}\right),\frac{1}{2\cdot 4^n}\psi\left(\frac{1}{4^n}\right)\right]\right).$$

Let $h\in\mathbb{R}$, $h<\frac{1}{2^n}$. Then there exists $n\in\mathbb{N}$, $n\ge n_0$ such that $\frac{1}{2^{n+1}}\le h<\frac{1}{2^n}$. Monotonicity of ψ yields

$$\begin{aligned}\lambda_2\Big(B\cap Q\big((0,0),h\big)\Big)&\le\lambda_2\left(B\cap Q\left((0,0),\frac{1}{2^n}\right)\right)=\left(\sum_{i=n}^{\infty}\left(\frac{1}{4^i}\psi\left(\frac{1}{4^i}\right)\right)\right)^2\\&\le\left(\psi\left(\frac{1}{4^n}\right)\right)^2\cdot\left(\sum_{i=n}^{\infty}\frac{1}{4^i}\right)^2<\left(\frac{1}{4^{n-1}}\right)^2\cdot\psi\left(\frac{1}{4^n}\right),\end{aligned}$$

hence

$$\frac{\lambda_2(B\cap Q((0,0),h))}{4h^2\psi(4h^2)}<\frac{\big(\frac{1}{4^{n-1}}\big)^2\cdot\psi\big(\frac{1}{4^n}\big)}{4\cdot\big(\frac{1}{2^{n+1}}\big)^2\cdot\psi\Big(4\cdot\big(\frac{1}{2^{n+1}}\big)^2\Big)}=\frac{4}{2^n}\cdot\frac{\psi\big(\frac{1}{4^n}\big)}{\psi\big(\frac{1}{4^n}\big)}=\frac{4}{2^n}.$$

Since for $h\to 0^+$, we have n tending to $+\infty$, so $(0,0)\in\Phi^0_\psi(B')$ and $B'\in\mathcal{T}^0_\psi$.

Let

$$h_0(x,y)=\begin{cases}0 \quad \text{for} \quad (x,y)\in B',\\ 1 \quad \text{for} \quad (x,y)=\Big(\frac{1}{2^n}-\frac{1}{2\cdot 4^n}\psi\big(\frac{1}{4^n}\big),0\Big),\ n\in\mathbb{N},\\ \text{continuous on } \Big[\frac{1}{2^n}-\frac{1}{4^n}\psi\big(\frac{1}{4^n}\big),\frac{1}{2^n}\Big]\times\Big[-\frac{1}{2\cdot 4^n}\psi\big(\frac{1}{4^n}\big),\frac{1}{2\cdot 4^n}\psi\big(\frac{1}{4^n}\big)\Big]\\ \text{and such that the image of this interval is equal to } [0,1], \text{ for } n\in\mathbb{N}.\end{cases}$$

Then h_0 has the desired properties.

Let $\mathcal{P}$ be a nowhere dense perfect set $\mathcal{P}\subset[0,1]\times[0,1]$. Let $\{A_l\}_{l\in\mathbb{N}}$ be the family of connected components of $\big([0,1]\times[0,1]\big)\setminus\mathcal{P}$.

Consider the squares of the following form:

$$S_{(k,m,p)}=\left[\frac{k}{2^p},\frac{k+1}{2^p}\right]\times\left[\frac{m}{2^p},\frac{m+1}{2^p}\right],\qquad p\in\mathbb{N},\quad k,m\in\{0,\dots,2^p-1\}.$$

Fix $p=1$. From the squares $S_{(k,m,1)}$, $k,m\in\{0,1\}$ we choose those ones which are contained in $\bigcup_{l\in\mathbb{N}}A_l$ and $\operatorname{dist}\big(S_{(k,m,1)},\mathcal{P}\big)>\frac{1}{4}$.

For $p=2$, we choose from the squares $S_{(k,m,2)}$, $k,m\in\{0,\dots,3\}$ those ones which are contained in $\bigcup_{l\in\mathbb{N}}A_l\setminus\bigcup_{n=1}^{n_1}I_n$ and $\operatorname{dist}\big(S_{(k,m,2)},\mathcal{P}\big)>\frac{1}{8}$ and we arrange them in a sequence $\{I_n\}_{n=n_1+1}^{n_2}$.

For $p\in\mathbb{N}$ we choose from the squares $S_{(k,m,p)}$, $k,m\in\{0,\dots,2^p-1\}$ those which are contained in $\bigcup_{l\in\mathbb{N}}A_l\setminus\bigcup_{n=1}^{n_p-1}I_n$ and $\operatorname{dist}\big(S_{(k,m,p)},\mathcal{P}\big)>\frac{1}{2^{p+1}}$ and we arrange them in a sequence $\{I_n\}_{n=n_{p-1}+1}^{n_p}$, and so on. It is easily seen that

$$\bigcup_{l\in\mathbb{N}}A_l=\bigcup_{n\in\mathbb{N}}I_n.$$

Now, in each square $I_n=S_{(k,m,p)}$, $n\in\mathbb{N}$, we put a closed square J_n centered at the midpoint of I_n with sides parallel to the axis and such that

$$\lambda_2(J_1)=\lambda_2(I_1)$$

and

$$\lambda_2(J_n)=\min\left(\frac{1}{n^2}\lambda_2(I_n),\frac{1}{2}\lambda_2(J_{n-1}),\frac{1}{n}\lambda_2(I_n)\psi\left(\frac{\lambda_2(I_n)}{4}\right)\right)\quad\text{for}\quad n>1.$$

Then

$$\lambda_2\left(\bigcup_{n=n_0}^{\infty}J_n\right)\le 2\lambda_2(J_{n_0})\qquad\text{for}\quad n_0>1.$$

Let $z\in\mathcal{P}$, $r>0$, and let $n(r)=\min\big\{n\in\mathbb{N}:Q(z,r)\cap J_n\neq\emptyset\big\}$. Then there exists exactly one number $p\in\mathcal{N}$ such that $n(r)\in\{n_{p-1}+1,\dots,n_p\}$.

We denote this number by $p(r)$. Thus, the square $Q(z,r)$ has common points with at least one of the squares $J_{n_{p(r)-1}+1},\dots J_{n_{p(r)}}$. Therefore

$$r>\frac{1}{2}\sqrt{\lambda_2\Big(I_{n(r)}\Big)}-\frac{1}{2}\,\frac{1}{n(r)}\cdot\sqrt{\lambda_2\Big(I_{n(r)}\Big)}=\frac{1}{2}\sqrt{\lambda_2\Big(I_{n(r)}\Big)}\left(1-\frac{1}{n(r)}\right).$$

Hence for $r\to 0$, $n(r)$ tends to $+\infty$ and $p(r)$ tends to $+\infty$.

The task is to show that z is a $\mathcal{T}_\psi^0$ density point of the complement of $\bigcup_{n\in\mathbb{N}} J_n$. It follows that for $r>0$:

$$\lambda_2\Bigg(\bigcup_{n\in\mathbb{N}} J_n\cap Q(z,r)\Bigg)\leq\lambda_2\Bigg(\bigcup_{n=n(r)}^{\infty} J_n\cap Q(z,r)\Bigg)\leq\lambda_2\Bigg(\bigcup_{n=n(r)}^{\infty} J_n\Bigg),$$

so,

$$\begin{aligned}\frac{\lambda_2\big(\bigcup_{n\in\mathbb{N}} J_n\cap Q(z,r)\big)}{4r^2\psi(4r^2)}&\leq\frac{2\cdot\lambda_2\left(J_{n(r)}\right)}{\lambda_2\left(I_{n(r)}\right)\left(1-\frac{1}{n(r)}\right)^2\psi\left(\lambda_2\left(I_{n(r)}\right)\left(1-\frac{1}{n(r)}\right)^2\right)}\\&\leq\frac{2\frac{1}{n(r)}\lambda_2\left(I_{n(r)}\right)\psi\left(\frac14\lambda_2\left(I_{n(r)}\right)\right)}{\lambda_2\left(I_{n(r)}\right)(1-\frac12)^2\psi\left((1-\frac12)^2\lambda_2\left(I_{n(r)}\right)\right)}\\&=\frac{8}{n(r)}.\end{aligned}$$

Hence

$$\lim_{r\to0}\frac{\lambda_2\left(\bigcup_{n\in\mathbb{N}} J_n\cap Q(z,r)\right)}{4r^2\psi(4r^2)}=0.$$

Next, we build a function which is $\mathcal{T}_\psi^0$-approximately continuous and which is discontinuous on $\mathcal{P}$. We define $g_{\mathcal{P}}$ to be 0 on the complement of $\bigcup_{n\in\mathbb{N}} J_n$, $g_{\mathcal{P}}(J_n)=[0,1]$, $g_{\mathcal{P}}$ continuous on I_n, $n\in\mathbb{N}$.

We have shown that $\mathcal{P}\subset\Phi_\psi^0\big(\big(\bigcup_{n\in\mathbb{N}} J_n\big)'\big)$. Of course, $\big(\bigcup_{n\in\mathbb{N}} J_n\big)'\setminus\mathcal{P}$ is contained in the interior in the Euclidean topology of the set $\big(\bigcup_{n\in\mathbb{N}} J_n\big)'$, so $\big(\bigcup_{n\in\mathbb{N}} J_n\big)'\in\mathcal{T}_\psi^0$. Since arbitrarily close to the each point of $\mathcal{P}$, we can find the square J_n, $n\in\mathbb{N}$, the function $g_{\mathcal{P}}$ has the desired properties.

Let A be an F_σ first category set. As in $\mathcal{T}_\psi$-case, we find a countable set C and a sequence of nowhere dense perfect sets $\mathcal{P}_i$ such that $A=\bigcup_{i\in\mathbb{N}}\mathcal{P}_i\cup C$ and a construction of the function h which is $\mathcal{T}_\psi^0$-approximately continuous and discontinuous only on C can be also done almost in the same manner.

Let $g_i=\frac{1}{3^i}g_{\mathcal{P}_i}$, $i\in\mathbb{N}$, and $g=\sum_{i\in\mathbb{N}} g_i$. (If $\mathcal{P}_i$ is an unbounded perfect set for some $i\in\mathbb{N}$, we consider sets $\mathcal{P}_{i,n,m}=\mathcal{P}_i\cap\big([n,n+1]\times[m,m+1]\big)$ for $n,m\in Z$.) The function g is $\mathcal{T}_\psi^0$-approximately continuous function (Proposition 2.17) and it is discontinuous at any point from $\bigcup_{i\in\mathbb{N}}\mathcal{P}_i$. Indeed, if $x\in\bigcup_{i\in\mathbb{N}}\mathcal{P}_i$, then denoting by i_0 the smallest index such that $x\in\mathcal{P}_{i_0}$, we get $\operatorname{osc}(g_{i_0},x)=\frac{1}{3^{i_0}}$, $\operatorname{osc}\big(\sum_{i<i_0} g_i,x\big)=0$ and $\operatorname{osc}\big(\sum_{i>i_0} g_i,x\big)\leq\sum_{i>i_0}\frac{1}{3^i}=\frac{1}{2\cdot3^{i_0}}$, therefore $\operatorname{osc}(g,x)>0$. Of course, when $x\notin\bigcup_{i\in\mathbb{N}}\mathcal{P}_i$, then g is continuous at x.

The function $f=g+h$, where h is defined for the set $\mathcal{C}$ as in $\mathcal{T}_\psi$-case, is the desired one.

$\mathcal{T}\times_{\mathcal{T}}$-case. We can repeat the construction of the function from the previous case if we can build a $\mathcal{T}\times\mathcal{T}$-approximately continuous function which is discontinuous at the point $(0,0)$ and for a nowhere dense perfect set $\mathcal{P}$ we can define a family of squares $\{J_n\}_{n\in\mathbb{N}}$ such that if $\{I_n\}_{n\in\mathbb{N}}$ is the sequences of squares defined in $\mathcal{T}_\psi^0$ case, then $J_n \subset I_n$, J_n is centered at the centre of I_n, for $n \in \mathbb{N}$ and each point of $\mathcal{P}$ is an inner point of the complement of $\bigcup_{n\in\mathbb{N}} J_n$ in $\mathcal{T}\times\mathcal{T}$.

We begin by taking an interval set $A = \bigcup_{n\in\mathbb{N}}(a_n, b_n)$, such that

$$a_{n+1} \le b_{n+1} \le a_n,\ n \in \mathbb{N},\ a_n \searrow 0 \quad \text{and} \quad 0 \in \Phi(A').$$

Then let

$$B = \bigcup_{n\in\mathbb{N}} (a_n, b_n) \times \left(-\frac{b_n - a_n}{2}, \frac{b_n - a_n}{2}\right).$$

Hence $(0,0) \in \operatorname{Int}(\mathbb{R}^2 \setminus B)$ since $(\mathbb{R}\setminus A)\times\mathbb{R} \in \mathcal{T}\times\mathcal{T}$ and $(\mathbb{R}\setminus A)\times\mathbb{R} \subset \mathbb{R}^2\setminus B$.

It follows that the function

$$h_0(x,y) = \begin{cases} 0 \quad \text{for } (x,y) \in B', \\ 1 \quad \text{for } (x,y) = \left(\frac{a_n+b_n}{2}, 0\right), n \in \mathbb{N}, \\ \text{continuous on } [a_n, b_n] \times \left[-\frac{b_n-a_n}{2}, \frac{b_n-a_n}{2}\right] \text{ and such that} \\ \text{the image of this interval is equal to } [0,1], \text{ for } n \in \mathbb{N} \end{cases}$$

has the desired properties.

Now, let P be a nowhere dense perfect set $P \subset [0,1]\times[0,1]$ and $\{I_n\}_{n\in\mathbb{N}}$ be the sequence of squares defined in $\mathcal{T}_\psi^0$-case. Then, in each square $I_n = S_{(k,m,p)}$, $n \in \mathbb{N}$ we put a square J_n centered at the midpoint of I_n with sides parallel to the axis and such that

$$\lambda_2(J_1) = \lambda_2(I_1) \quad \text{and} \quad \lambda_2(J_n) = \min\left(\frac{1}{n^2}\cdot\lambda_2(I_n), \frac{1}{4}\lambda_2(J_{n-1})\right) \qquad \text{for} \quad n > 1.$$

Additionally, we denote

$$I_n = (a_n, b_n) \times (c_n, d_n), \qquad n \in \mathbb{N}.$$

Consider $z = (x_0, y_0) \in \mathcal{P}$. Let $A_1 = \bigcup_{n\in N_1}(a_n, b_n)$, where

$$N_1 = \Big\{n \in \mathbb{N} : I_n \subset \big\{(x,y) : (y - x - y_0 + x_0)(y + x - y_0 - x_0) < 0\big\}\Big\}$$

and $A_2 = \bigcup_{n\in N_2}(c_n, d_n)$, where $N_2 = \mathbb{N}\setminus N_1$. Then $\bigcup_{n\in\mathbb{N}} J_n \subset (A_1\times\mathbb{R}) \cup (\mathbb{R}\times A_2)$. To show that $z \in \operatorname{Int}_{\mathcal{T}\times\mathcal{T}}\big(\mathbb{R}^2\setminus\bigcup_{n\in\mathbb{N}} J_n\big)$, it suffices to prove that x_o is a density point of $\mathbb{R}\setminus A_1$ and y_0 is a density point of $\mathbb{R}\setminus A_2$.

Let $r > 0$ and let $n(r) = \min\big\{n \in N_1 : (x_0, x_0 + r) \cap (a_n, b_n)\big\}$. Then there exists exactly one number $p \in \mathbb{N}$ such that $n(r) \in \{n_{p-1}+1, \dots, n_p\}$. We denote this number by $p(r)$. Thus, the interval $(x_0, x_0 + r)$ has common points with at least one of the intervals $(a_{n_{p(r)-1}+1}, b_{n_{p(r)-1}+1}), \dots, (a_{n_{p(r)}}, b_{n_{p(r)}})$. Therefore

$$r > \frac{1}{2}\sqrt{\lambda_2\Big(I_{n(r)}\Big)} - \frac{1}{2}\cdot\frac{1}{n(r)}\sqrt{\lambda_2\Big(I_{n(r)}\Big)} = \frac{1}{2}\cdot\frac{1}{2^{p(r)}}\left(1 - \frac{1}{n(r)}\right),$$

hence for $r \to 0$, we have $n(r)$ tending to $+\infty$ and $p(r)$ tending to $+\infty$.

We observe that

$$\lambda\big((x_0, x_0 + r) \cap A_1\big) = \lambda\left((x_0, x_0 + r) \cap \bigcup_{n \in N_1 \cap [n(r), +\infty)} (a_n, b_n)\right)$$
$$\leq \sum_{n \in N_1 \cap [n(r), +\infty)} \lambda(a_n, b_n) \leq 2 \cdot \big(b_{n(r)} - a_{n(r)}\big)$$
$$\leq 2 \cdot \frac{1}{n(r)} \cdot \frac{1}{2^{p(r)}}.$$

Consequently,

$$\frac{\lambda\big((x_0, x_0 + r) \cap A_1\big)}{r} \leq \frac{2 \cdot \frac{1}{n(r)} \cdot \frac{1}{2^{p(r)}}}{\frac{1}{2 \cdot 2^{p(r)}}} \left(1 - \frac{1}{n(r)}\right) = \frac{4}{n(r) - 1},$$

hence

$$\lim_{r \to 0^+} \frac{\lambda\big((x_0, x_0 + r) \cap A_1\big)}{r} = 0 \quad \text{and similarly,} \quad \lim_{r \to 0^+} \frac{\lambda\big((x_0 - r, x_0) \cap A_1\big)}{r} = 0,$$

which yields that x_0 is a density point of $\mathbb{R} \setminus A_1$.

Similar arguments apply to the set A_2 and the point y_0.

The construction of the function with desired properties runs as in $\mathcal{T}_\psi^0$ case.

$\mathcal{T}_2$-case. We get the result immediately by the previous case and Proposition 2.11. □

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