THE SET OF DISCONTINUITIES OF
DENSITY-TYPE-APPROXIMATELY CONTINUOUS
FUNCTIONS

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ABSTRACT. A characterization of the set of discontinuities for different density-
type-approximately continuous functions is given.

1. Introduction

It is well-known that the following two statements for a set $A \subset \mathbb{R}$ are equivalent (see [G]):

(i) $A$ is an $F_\sigma$ set,
(ii) there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $A$ is the set of discontinuities of $f$.

An analogous result has been established for approximately continuous functions (see [B, p. 48]):

1.1 The following two statements for a set $A \subset [a,b]$ are equivalent:

(i) $A$ is dense in $[a,b]$ and of type $G_\delta$.
(ii) There is an approximately continuous function $f$ such that $A$ is the set of
continuities of $f$.

The aim of this paper is to verify whether the same result can be established for some density-type topologies. Our main result is as follows:

1.2 The following two statements for a set $A \subset \mathbb{R}$ (or $A \subset \mathbb{R}^2$) are equivalent:

(i) $A$ is an $F_\sigma$ first category set,

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There is a continuous function \( f : (\mathbb{R}, \tau) \to (\mathbb{R}, \mathcal{O}) \) (or \( f : (\mathbb{R}^2, \tau) \to (\mathbb{R}, \mathcal{O}) \)) such that \( A \) is the set of discontinuities of \( f \), where \( \tau \) denotes one of the following topologies: the density topology \([W1]\), simple density topology \([WA]\), complete density topology \([WW]\), \( \psi \)-density topology \([TW-B]\), \( \langle s \rangle \)-density topology generated by a sequence \( \langle s \rangle \) \([FH]\) (respectively density, strong density \([W1]\), product of density topologies or ordinary \( \psi \)-density topology on the plane \([F]\)) and \( \mathcal{O} \) stands for the Euclidean topology.

The main idea of the proof comes from \([P-WS]\), where a theorem for the density topology case was proved (as in \([B]\)).

What is interesting the situation in the category case (\( I \)-density topology) is different. Some additional assumptions of a set \( A \) are needed to receive the existence of \( I \)-approximately continuous function \( f : (\mathbb{R}, \mathcal{T}_I) \to (\mathbb{R}, \mathcal{O}) \) for which \( A \) is equal to the set of all points of ordinary discontinuity (see \([W2]\)).

2. Definitions and basic facts about density type topologies

Each of the topologies mentioned above is defined as a family of those Lebesgue measurable sets \( A \) such that each point of \( A \) is a point of some kind of density of \( A \):

\[
\tau = \{ A \in \mathcal{L} : A \subset \Phi(A) \},
\]

where \( \mathcal{L} \) denotes the \( \sigma \)-algebra of Lebesgue measurable sets on the line, and \( \Phi(A) \) denotes the set of some kind of density points of \( A \) for \( A \in \mathcal{L} \).

We obtain different topologies by using different type of densities.

**Definition 2.1 ([W1]).** We say that a point \( x \in \mathbb{R} \) is a density point of \( A \in \mathcal{L} \) if and only if

\[
\lim_{h \to 0^+} \frac{\lambda(A \cap [x - h, x + h])}{2h} = 1,
\]

where \( \lambda \) stands for the Lebesgue measure on the line. We denote the density topology by \( \mathcal{T} \).

**Definition 2.2 ([W1], [TW-B]).** Let \( \mathcal{C} \) denote the class of continuous increasing functions \( \psi : \mathbb{R}^+ \to \mathbb{R}^+ \), where \( \mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \} \), such that \( \lim_{t \to 0^+} \psi(t) = 0 \).

Let \( \psi \in \mathcal{C} \). We say that \( x \in \mathbb{R} \) is a \( \psi \)-density point of \( A \in \mathcal{L} \) if and only if

\[
\lim_{h \to 0^+} \frac{\lambda(A' \cap [x - h, x + h])}{2h\psi(2h)} = 0,
\]

where \( A' \) denotes a complement of the set \( A \).
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We denote the set of $\psi$-density point of $A$ by $\Phi_\psi(A)$, so $\psi$-density topology $T_\psi$ is defined by

$$T_\psi = \{ A \in \mathcal{L} : A \subset \Phi_\psi(A) \}.$$ 

**Proposition 2.3 ([W1], [TW-B]).** For each $\psi \in \mathcal{C}$, $O \subset T_\psi \subset T$.

**Definition 2.4 ([WA]).** We say that $x \in \mathbb{R}$ is a simple density point of the set $A \in \mathcal{L}$ if $\{ \chi_{n(A-x)} \cap [-1,1] \}_{n \in \mathbb{N}}$ converges to $\chi_{[-1,1]}$ almost everywhere, where $A - x = \{ t - x : t \in A \}$, $n \cdot A = \{ nt : t \in A \}$ and $\chi_A$ denotes a characteristic function of the set $A$.

The set of all simple density point of $A$ will be denoted by $\Phi_s(A)$, so the simple density topology $T_s$ is given by

$$T_s = \{ A \in \mathcal{L} : A \subset \Phi_s(A) \}.$$ 

**Definition 2.5 ([WW]).** We say that $x \in \mathbb{R}$ is a complete density point of a set $A \in \mathcal{L}$ if

$$\sum_{n=1}^{\infty} \lambda\left( (nA' - x) \cap [-1,1] \right) < +\infty.$$

The set of all complete density points of $A$ will be denoted by $\Phi_c(A)$, so the complete density topology $T_c$ is defined by

$$T_c = \{ A \in \mathcal{L} : A \subset \Phi_c(A) \}.$$ 

**Proposition 2.6 ([WW], [WA]).** $O \subset T_c \subset T_s \subset T$.

**Definition 2.7 ([FH]).** Let $S$ be the family of all unbounded and nondecreasing sequences of positive reals. Every sequence $\{ s_n \} \in S$ is denoted by $\langle s \rangle$.

Let $\langle s \rangle \in S$. We say that $x \in \mathbb{R}$ is a $\langle s \rangle$-density point of $A \in \mathcal{L}$ if

$$\lim_{n \to \infty} \frac{\lambda(A \cap [x - \frac{1}{s_n}, x + \frac{1}{s_n}])}{\frac{2}{s_n}} = 1.$$ 

The set of all $\langle s \rangle$-density points of $A$ will be denoted by $\Phi_{\langle s \rangle}(A)$, so $\langle s \rangle$-density topology $T_{\langle s \rangle}$ is defined by

$$T_{\langle s \rangle} = \{ A \in \mathcal{L} : A \subset \Phi_{\langle s \rangle}(A) \}.$$ 

**Proposition 2.8 ([FH]).** For every sequence $\langle s \rangle \in S$, the following inclusion $T \subset T_{\langle s \rangle}$ holds and $T = T_{\langle s \rangle}$ if and only if $\lim \inf \frac{s_n}{s_{n+1}} > 0$.

Let $\mathcal{L}_2$ denote the $\sigma$-algebra of Lebesgue measurable sets on the plane, and for $h > 0$ and $p = (x_0, y_0) \in \mathbb{R}^2$ put $Q(p, h) = [x_0 - h, x_0 + h] \times [y_0 - h, y_0 + h]$. 

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**Definition 2.9 ([W1])**. We say that \( p = (x_0, y_0) \in \mathbb{R}^2 \) is an ordinary density point of \( A \in L_2 \) if
\[
\lim_{h \to 0^+} \frac{\lambda_2(A \cap Q(p, h))}{4h^2} = 1,
\]
where \( \lambda_2 \) stands for the Lebesgue measure on the plane.

The set of all ordinary density points of \( A \) will be denoted by \( \Phi_2(A) \) and the ordinary density topology \( T_2 \) is defined by
\[
T_2 = \{ A \in L_2 : A \subset \Phi_2(A) \}.
\]

**Definition 2.10 ([W1])**. We say that a point \( p = (x_0, y_0) \in \mathbb{R}^2 \) is a strong density point of the set \( A \in L_2 \) if
\[
\lim_{h \to 0^+} \frac{\lambda_2(A \cap ([x_0 - h, x_0 + h] \times [y_0 - k, y_0 + k]))}{4hk} = 1.
\]

The set of all strong density points of \( A \) will be denoted by \( \Phi^s(A) \), so strong density topology \( T_2^s \) is defined by
\[
T_2^s = \{ A \in L_2 : A \subset \Phi^s(A) \}.
\]

**Proposition 2.11**. \( O_2 \subset T \times T \subset T_2^s \subset T_2 \), where \( O_2 \) stands for the Euclidean topology on the plane.

**Definition 2.12 ([F])**. Let \( \psi \in C \). We say that \( p = (x_0, y_0) \) is an ordinary \( \psi \)-density point of \( A \in L_2 \) if
\[
\lim_{h \to 0^+} \frac{\lambda_2(A \cap Q(p, h))}{4h^2 \psi(4h^2)} = 0.
\]

The set of all ordinary \( \psi \)-density points of \( A \) will be denoted by \( \Phi_\psi^0(A) \), so ordinary \( \psi \) density topology \( T_\psi^0 \) is defined by
\[
T_\psi^0 = \{ A \in L_2 : A \subset \Phi_\psi^0(A) \}.
\]

**Proposition 2.13 ([F])**. For each \( \psi \in C : O_2 \not\subseteq T_\psi^0 \not\subseteq T_2 \).

**Definition 2.14**. For every topology \( \tau \) which was mentioned above, we say that a function is \( \tau \)-approximately continuous if it is continuous as a function \( f : (\mathbb{R}, \tau) \to (\mathbb{R}, O) \) or \( f : (\mathbb{R}^2, \tau) \to (\mathbb{R}, O) \).

**Proposition 2.15 ([W1])**. Every \( T \)-approximately continuous and every \( T_2 \)-approximately continuous function is a Baire one function.

**Corollary 2.16**. Every \( T \times T \), \( T_2^s \), \( T_\psi \), \( T_\psi^0 \) (for any \( \psi \in C \)) \( T_s \) or \( T_c \)-approximately continuous function is Baire one function.

**Proposition 2.17 ([F] Th. 1.4.7)**. If \( f_n \) is \( \tau \)-approximately continuous for any \( n \in \mathbb{N} \) and a sequence \( \{f_n\}_{n \in \mathbb{N}} \) uniformly converges to \( f \), then \( f \) is \( \tau \)-approximately continuous.
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**Proposition 2.18** ([L]). Let \( \langle s \rangle \in S \). If \( f \) is \( T_{\langle s \rangle} \)-approximately continuous then \( f \) is a Baire one function.

3. Proof of the main theorem

(ii) \( \implies \) (i) follows immediately from Proposition 2.18 and Corollary 2.16 for each topology considered here since the set of discontinuities of a Baire one function is a first category \( F_\sigma \) set [

(i) \( \implies \) (ii)

\( T_\psi \)-case. Let \( \psi \in C \). Let \( A \) be an \( F_\sigma \) first category set. There exists a countable set \( C \) and a sequence of pairwise disjoint nowhere dense perfect sets \( P_i \) such that \( A = \bigcup_{i \in \mathbb{N}} P_i \cup C \) (see [S] vol. II, p. 537).

First, we want to build a \( T_\psi \)-approximately continuous function which is discontinuous at the point 0.

Since \( \psi \in C \), there exists \( n_0 \in \mathbb{N} \) such that \( \psi\left(\frac{1}{2^{n_0}}\right) \leq 1 \). Let

\[
a_n = \frac{1}{2^n} - \frac{1}{4^n} \psi\left(\frac{1}{2^n}\right), \quad b_n = \frac{1}{2^n} \quad \text{for } n \geq n_0
\]

and

\[
B = \bigcup_{n=n_0}^{\infty} [a_n, b_n].
\]

Let \( h < \frac{1}{2^n} \). Then there exists \( n \geq n_0, n \in \mathbb{N} \) such that \( \frac{1}{2^{n+1}} \leq h < \frac{1}{2^n} \), so

\[
\lambda(B \cap [0, h]) \leq \lambda\left(B \cap \left[0, \frac{1}{2^n}\right]\right) = \sum_{i=n}^{\infty} \frac{1}{4^i} \psi\left(\frac{1}{2^i}\right)
\]

\[
\leq \psi\left(\frac{1}{2^n}\right) \cdot \sum_{i=n}^{\infty} \frac{1}{4^i} \leq \psi\left(\frac{1}{2^n}\right) \cdot \frac{1}{4^{n-1}}.
\]

Therefore,

\[
\frac{\lambda(B \cap [0, h])}{2h \psi(2h)} < \frac{\psi\left(\frac{1}{2^n}\right) \cdot \frac{1}{4^{n-1}}}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\left(\frac{1}{2^{n+1}}\right)} = \frac{1}{2^{n-2}}.
\]

Since for \( h \to 0^+ \) we have \( n \) tending to \( +\infty \), so \( 0 \in \Phi_{\psi}(B') \) and \( B' \in T_\psi \), because each point from \( B' \setminus \{0\} \) is an inner one in the Euclidean topology.

Let

\[
h_0(x) = \begin{cases} 
0 & \text{for } x \in B', \\
1 & \text{for } x = \frac{b_n + a_n}{2}, n \in \mathbb{N}, \\
\text{linear and continuous} & \text{for } x \in \left[a_n, \frac{b_n + a_n}{2}\right] \text{ and } x \in \left[\frac{b_n + a_n}{2}, b_n\right].
\end{cases}
\]

Then \( h_0 \) has the desired properties.
We define \( h(x) = 0 \) if \( C = \emptyset \). Otherwise, let \( \{r_n\}_{n \in \mathbb{N}} \) be an enumeration of a considered countable set \( C \). We define
\[
h_n(x) = \frac{1}{2^n} h_0(x - r_n) \quad \text{and} \quad h(x) = \sum_{n=1}^{\infty} h_n(x) \quad \text{for} \quad x \in \mathbb{R}.
\]
Then the series is uniformly convergent on \( \mathbb{R} \), so by Proposition 2.17, the function \( h \) is \( T_\psi \)-approximately continuous. For a given \( n_0 \), the function \( \sum_{n \neq n_0} h_n \) is continuous at the point \( r_{n_0} \). Thus,
\[
h = h_{n_0} + \sum_{n \neq n_0} h_n
\]
is discontinuous there. Therefore \( h \) is discontinuous on \( C \), and continuous on \( C' \).

Let \( P \) be a nowhere dense perfect set.

Let \( \{I_n\}_{n \in \mathbb{N}} \) be the family of connected bounded components of the complement of \( P \). Let
\[
\sum_{n \in \mathbb{N}} z_n < +\infty
\]
such that \( z_n > 0 \) for \( n \in \mathbb{N} \). We denote by \( J_n \) an interval centered at the midpoint of \( I_n \) and such that
\[
\lambda(J_n) = \min \left( z_n \cdot \lambda(I_n) \psi \left( \frac{1}{2} \lambda(I_n) \right), \frac{1}{n} \lambda(I_n) \right), \quad n \in \mathbb{N}.
\]

Let \( p \in P \). We want to show that
\[
\lim_{h \to 0^+} \frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p, p + h])}{2h \psi(2h)} = 0.
\]

We can assume that \( p \) is not a left end-point of any \( I_n \) (since in the opposite case there exists \( h > 0 \) such that \( \bigcup_{n \in \mathbb{N}} J_n \cap [p, p + h] = \emptyset \)).

We fix \( n_0 \in \mathbb{N} \). Then there exists \( h > 0 \) such that
\[
[p, p + h] \cap \bigcup_{n=1}^{n_0} J_n = \emptyset.
\]

Let \( \{n_k\}_{k \in \mathbb{N}} \) be a subsequence of \( \{n\}_{n \in \mathbb{N}} \) consisting of all \( n_k \)'s such that
\[
(p, p + h] \cap J_{n_k} \neq \emptyset.
\]

Then \( n_k > n_0 \) for any \( k \in \mathbb{N} \).

It is easily seen that \( h > \frac{\lambda(I_{n_k}) - \lambda(J_{n_k})}{2} \), so
\[
\lambda(I_{n_k}) < 2h + \lambda(J_{n_k}) \leq 2h + \frac{1}{n_k} \lambda(I_{n_k}),
\]
thus
\[
(1 - \frac{1}{n_k}) \lambda(I_{n_k}) \leq 2h, \quad k \in \mathbb{N}.
\]
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Therefore,
\[
\frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p, p+h])}{2h\psi(2h)} = \frac{\lambda(\bigcup_{k \in \mathbb{N}} J_{nk} \cap [p, p+h])}{2h\psi(2h)}
\leq \sum_{k \in \mathbb{N}} \frac{\lambda(J_{nk})}{2h\psi(2h)}
\leq \sum_{k \in \mathbb{N}} \frac{z_n \cdot \lambda(I_{nk}) \cdot \psi(\frac{1}{2}\lambda(I_{nk}))}{(1 - \frac{1}{n}) \lambda(I_{nk}) \cdot \psi((1 - \frac{1}{n})\lambda(I_{nk}))}
\leq \sum_{k \in \mathbb{N}} \frac{z_{nk} \psi(\frac{1}{2}\lambda(I_{nk}))}{\frac{1}{2} \psi\left(\frac{1}{2}\lambda(I_{nk})\right)}
\leq 2 \cdot \sum_{k \in \mathbb{N}} z_{nk} \leq 2 \cdot \sum_{n = n_0}^{\infty} z_n.
\]

Since for \( n_0 \to +\infty \) the rest 0 series \( \sum_{n = n_0}^{\infty} z_n \) tends to 0, we have
\[
\lim_{h \to 0^+} \frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p, p+h])}{2h\psi(2h)} = 0.
\]

The argument similar to that above shows that also
\[
\lim_{h \to 0^+} \frac{\lambda(\bigcup_{n \in \mathbb{N}} J_n \cap [p-h, p])}{2h\psi(2h)} = 0.
\]

Consequently,
\[
p \in \Phi_\psi\left(\left(\bigcup_{n \in \mathbb{N}} J_n\right)\right).
\]

We define \( g_P \) to be 0 on the complement of \( \bigcup_{n \in \mathbb{N}} J_n \), \( g_P(J_n) = [0, 1] \) for \( n \in \mathbb{N} \) and \( g_P \) continuous on \( I_n \), \( n \in \mathbb{N} \). Then \( g_P \) is \( \mathcal{T}_\psi \)-approximately continuous everywhere and discontinuous exactly at all points of \( \mathcal{P} \).

Now, for each set \( \mathcal{P}_i \), \( i \in \mathbb{N} \), we define a function \( g_i(x) = \frac{1}{3}g_P_i(x) \) and let \( g(x) = \sum_{i \in \mathbb{N}} g_i(x) \). The function \( g \) is \( \mathcal{T}_\psi \)-approximately continuous everywhere and continuous at all points from the complement of \( \bigcup_{i \in \mathbb{N}} \mathcal{P}_i \).

To end the construction, let \( f = g + h \). Of course, \( f \) is \( \mathcal{T}_\psi \)-approximately continuous and the set \( A \) is exactly the set of discontinuities of \( f \).

**\( \mathcal{T}_c \)-case.** Let \( \tilde{\psi} : \mathbb{R}^+ \to \mathbb{R}^+ \) be given by
\[
\frac{1}{n^2} \quad \text{for} \ x \in \left[\frac{2}{n}, \frac{2}{n-1}\right), \quad n > 1,
1 \quad \text{for} \ x \in [1, +\infty).
\]

An easy computation shows that if for an arbitrarily chosen set \( A \), \( A \subset \mathbb{R} \), if
\[
\lim_{n \to 0^+} \frac{\lambda(A' \cap [-h, h])}{2h\psi(2h)} = 0,
\]

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then
\[
\lim_{n \to \infty} \frac{\lambda(A' \cap \left[\frac{1}{n}, \frac{1}{n+1}\right])}{\frac{2}{n} \psi(\frac{2}{n})} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda(n \cdot A' \cap [-1, 1]) < +\infty,
\]
so
\[0 \in \phi_c(A).
\]

For any non-decreasing function \(\tilde{\psi}\) transforming \(\mathbb{R}^+\) into \(\mathbb{R}^+\) and such that \(\lim_{x \to 0^+} \tilde{\psi}(x) = 0\), we can define a \(\tilde{\psi}\)-density topology as for a function from \(\mathcal{C}\) and for each function with such properties, we can find a function from the family \(\mathcal{C}\) such that the topologies regulated by these two functions coincide \([AW]\).

Therefore, there exists a function \(\psi \in \mathcal{C}\) such that \(\mathcal{T}_\psi \subset \mathcal{T}_c\), hence the function from \(\mathcal{T}_\psi\)-case works for \(\mathcal{T}_c\) case.

\(\mathcal{T}_c\)-case. According to Proposition 2.6, the function from \(\mathcal{T}_c\)-case is also \(\mathcal{T}_c\)-approximately continuous on \(\mathbb{R}\).

\(\mathcal{T}_{(s)}\)-case. Obviously, it follows from the theorem for approximately continuous functions \([B]\), since the approximately continuous function is also \(\mathcal{T}_{(s)}\)-approximately continuous on \(\mathbb{R}\).

\(\mathcal{T}_\psi^0\)-case. Let \(\psi \in \mathcal{C}\). Firstly we build a \(\mathcal{T}_\psi^0\)-approximately continuous function which is discontinuous at the point \((0, 0)\).

Since \(\psi \in \mathcal{C}\), then there exists \(n_0 \in \mathcal{N}\) such that \(\psi(\frac{1}{2^{n_0}}) \leq 1\).

Put
\[
B = \bigcup_{n=n_0}^{\infty} \left( \left[ \frac{1}{2^n} - \frac{1}{4^n} \psi\left(\frac{1}{4^n}\right) \cdot \frac{1}{2^n} \right] \times \left[ -\frac{1}{2} \cdot \frac{1}{4^n} \psi\left(\frac{1}{4^n}\right) \cdot \frac{1}{2} \cdot \frac{1}{4^n} \psi\left(\frac{1}{4^n}\right) \right] \right).
\]

Let \(h \in \mathbb{R}, h < \frac{1}{2^n}\). Then there exists \(n \in \mathbb{N}, n \geq n_0\) such that \(\frac{1}{2^n} \leq h < \frac{1}{2^{n+1}}\).

Monotonicity of \(\psi\) yields
\[
\lambda_2\left(B \cap Q((0, 0), h)\right) \leq \lambda_2\left(B \cap Q((0, 0), \frac{1}{2^n})\right) = \left(\sum_{i=1}^{\infty} \left(\frac{1}{4^i} \psi\left(\frac{1}{4^i}\right)\right)^2\right)^2
\]
\[
\leq \left(\psi\left(\frac{1}{4^n}\right)^2 \cdot \left(\sum_{i=1}^{\infty} \frac{1}{4^i}\right)^2\right) < \left(\frac{1}{4^{n-1}}\right)^2 \cdot \psi\left(\frac{1}{4^n}\right),
\]

hence
\[
\frac{\lambda_2(B \cap Q((0, 0), h))}{4h^2 \psi(4h^2)} < \frac{\left(\frac{1}{4^{n-1}}\right)^2 \cdot \psi\left(\frac{1}{4^n}\right)}{4 \cdot \left(\frac{1}{2^{n+1}}\right)^2 \cdot \psi\left(\frac{1}{2^{n+1}}\right)^2} = \frac{4}{2^n} \cdot \psi\left(\frac{1}{4^n}\right) = \frac{4}{2^n}.
\]

Since for \(h \to 0^+\), we have \(n\) tending to \(+\infty\), so \((0, 0) \in \Phi_\psi^0(B')\) and \(B' \in \mathcal{T}_\psi^0\).

Let
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\[ h_0(x, y) = \begin{cases} 
0 & \text{for } (x, y) \in B', \\
1 & \text{for } (x, y) = \left( \frac{1}{n^2} - \frac{1}{2^{1/n}} \psi \left( \frac{1}{n^2} \right), 0 \right), \ n \in \mathbb{N}, \\
\text{continuous on} & \left[ \frac{1}{n^2} - \frac{1}{2^{1/n}} \psi \left( \frac{1}{n^2} \right), \frac{1}{2^{1/n}} \right] \times \left[ -\frac{2^{1/n}}{4^{1/n}} \psi \left( \frac{1}{n^2} \right), \frac{1}{2^{1/n}} \psi \left( \frac{1}{n^2} \right) \right], \\
\text{and such that the image of this interval is equal to } [0, 1], & \text{for } n \in \mathbb{N}. 
\end{cases} \]

Then \( h_0 \) has the desired properties.

Let \( \mathcal{P} \) be a nowhere dense perfect set \( \mathcal{P} \subset [0, 1] \times [0, 1] \). Let \( \{A_l\}_{l \in \mathbb{N}} \) be the family of connected components of \( ([0, 1] \times [0, 1]) \setminus \mathcal{P} \).

Consider the squares of the following form:

\[ S(k,m,p) = \left[ \frac{k}{2^p}, \frac{k+1}{2^p} \right] \times \left[ \frac{m}{2^p}, \frac{m+1}{2^p} \right], \quad p \in \mathbb{N}, \ k, m \in \{0, \ldots, 2^p - 1\}. \]

Fix \( p = 1 \). From the squares \( S(k,m,1), k, m \in \{0, 1\} \) we choose those ones which are contained in \( \bigcup_{l \in \mathbb{N}} A_l \) and \( \text{dist}(S(k,m,1), \mathcal{P}) > \frac{1}{4} \).

For \( p = 2 \), we choose from the squares \( S(k,m,2), k, m \in \{0, \ldots, 3\} \) those ones which are contained in \( \bigcup_{l \in \mathbb{N}} A_l \setminus \bigcup_{n=1}^{2^{p-1}} I_n \) and \( \text{dist}(S(k,m,2), \mathcal{P}) > \frac{1}{8} \) and we arrange them in a sequence \( \{I_n\}_{n=n_1+1}^{n_2} \).

For \( p \in \mathbb{N} \) we choose from the squares \( S(k,m,p), k, m \in \{0, \ldots, 2^p - 1\} \) those which are contained in \( \bigcup_{l \in \mathbb{N}} A_l \setminus \bigcup_{n=1}^{2^{p-1}} I_n \) and \( \text{dist}(S(k,m,p), \mathcal{P}) > \frac{1}{2^{p-2}} \) and we arrange them in a sequence \( \{I_n\}_{n=n_p-1+1}^{n_p} \), and so on. It is easily seen that

\[ \bigcup_{l \in \mathbb{N}} A_l = \bigcup_{n \in \mathbb{N}} I_n. \]

Now, in each square \( I_n = S(k,m,p), n \in \mathbb{N} \), we put a closed square \( J_n \) centered at the midpoint of \( I_n \) with sides parallel to the axis and such that

\[ \lambda_2(J_1) = \lambda_2(I_1) \]

and

\[ \lambda_2(J_n) = \min \left( \frac{1}{n^2} \lambda_2(I_n), \frac{1}{n} \lambda_2(I_{n-1}), \frac{1}{n} \lambda_2(I_k) \psi \left( \frac{\lambda_2(I_n)}{4} \right) \right) \quad \text{for } n > 1. \]

Then

\[ \lambda_2 \left( \bigcup_{n=n_0}^{\infty} J_n \right) \leq 2 \lambda_2 (J_{n_0}) \quad \text{for } n_0 > 1. \]

Let \( z \in \mathcal{P}, r > 0 \), and let \( n(r) = \min \{n \in \mathbb{N} : Q(z,r) \cap J_n \neq \emptyset \} \). Then there exists exactly one number \( p \in \mathbb{N} \) such that \( n(r) \in \{n_{p-1} + 1, \ldots, n_p\} \).

We denote this number by \( p(r) \). Thus, the square \( Q(z,r) \) has common points with at least one of the squares \( J_{n_{p(r)}-1+1}, \ldots, J_{n_{p(r)}} \). Therefore

\[ r > \frac{1}{2} \sqrt{\lambda_2(I_{n(r)}) - \frac{1}{2} \frac{1}{n(r)}} \cdot \sqrt{\lambda_2(I_{n(r)})} = \frac{1}{2} \sqrt{\lambda_2(I_{n(r)})(1 - \frac{1}{n(r)})}. \]

Hence for \( r \to 0 \), \( n(r) \) tends to \( +\infty \) and \( p(r) \) tends to \( +\infty \).
The task is to show that $z$ is a $T_0^0$ density point of the complement of $\bigcup_{n\in\mathbb{N}} J_n$.

So, for $r > 0$:

$$\lambda_2\left(\bigcup_{n\in\mathbb{N}} J_n \cap Q(z, r)\right) \leq \lambda_2\left(\bigcup_{n=n(r)}^{\infty} J_n \cap Q(z, r)\right) \leq \lambda_2\left(\bigcup_{n=n(r)}^{\infty} J_n\right),$$

so,

$$\frac{\lambda_2\left(\bigcup_{n\in\mathbb{N}} J_n \cap Q(z, r)\right)}{4r^2\psi(4r^2)} \leq \frac{2 \cdot \lambda_2\left(J_{n(r)}\right)}{\lambda_2\left(I_{n(r)}\right)\left(1 - \frac{1}{n(r)}\right)^2\psi(\lambda_2\left(I_{n(r)}\right)\left(1 - \frac{1}{n(r)}\right)^2)}$$

$$\leq \frac{2}{n(r)}\lambda_2\left(I_{n(r)}\right)\psi\left(\frac{\lambda_2\left(I_{n(r)}\right)}{n(r)}\right)$$

$$= \frac{8}{n(r)}.$$

Hence

$$\lim_{r \to 0} \frac{\lambda_2\left(\bigcup_{n\in\mathbb{N}} J_n \cap Q(z, r)\right)}{4r^2\psi(4r^2)} = 0.$$

Next, we build a function which is $T_0^0$-approximately continuous and which is discontinuous on $\mathcal{P}$. We define $g_P$ to be 0 on the complement of $\bigcup_{n\in\mathbb{N}} J_n$, $g_P(J_n) = [0, 1]$, $g_P$ continuous on $I_n$, $n \in \mathbb{N}$.

We have shown that $\mathcal{P} \subset \Phi_0^0\left((\bigcup_{n\in\mathbb{N}} J_n)^\prime\right)$. Of course, $\left(\bigcup_{n\in\mathbb{N}} J_n\right)^\prime \setminus \mathcal{P}$ is contained in the interior in the Euclidean topology of the set $\left(\bigcup_{n\in\mathbb{N}} J_n\right)^\prime$, so $\left(\bigcup_{n\in\mathbb{N}} J_n\right)^\prime \in T_0^0$. Since arbitrarily close to the each point of $\mathcal{P}$, we can find the square $J_n$, $n \in \mathbb{N}$, the function $g_P$ has the desired properties.

Let $A$ be an $F_\sigma$ first category set. As in $T_0^0$-case, we find a countable set $C$ and a sequence of nowhere dense perfect sets $\mathcal{P}_i$ such that $A = \bigcup_{i\in\mathbb{N}} \mathcal{P}_i \cup C$ and a construction of the function $h$ which is $T_0^0$-approximately continuous and discontinuous only on $C$ can be also done almost in the same manner.

Let $g_i = \frac{1}{2^i} g_{\mathcal{P}_i}$, $i \in \mathbb{N}$, and $g = \sum_{i\in\mathbb{N}} g_i$. (If $\mathcal{P}_i$ is an unbounded perfect set for some $i \in \mathbb{N}$, we consider sets $\mathcal{P}_{i,n,m} = \mathcal{P}_i \cap ([n,n+1] \times [m,m+1])$ for $n, m \in \mathbb{Z}$.) The function $g$ is $T_0^0$-approximately continuous function (Proposition 2.17) and it is discontinuous at any point from $\bigcup_{i\in\mathbb{N}} \mathcal{P}_i$. Indeed, if $x \in \bigcup_{i\in\mathbb{N}} \mathcal{P}_i$, then denoting by $i_0$ the smallest index such that $x \in \mathcal{P}_{i_0}$, we get $\text{osc}(g_{i_0}, x) = \frac{1}{3^{i_0}}$, $\text{osc}\left(\sum_{i<i_0} g_i, x\right) = 0$ and $\text{osc}\left(\sum_{i>i_0} g_i, x\right) \leq \sum_{i>i_0} \frac{1}{3^i} = \frac{1}{2-3^{i_0}}$, therefore $\text{osc}(g, x) > 0$. Of course, when $x \notin \bigcup_{i\in\mathbb{N}} \mathcal{P}_i$, then $g$ is continuous at $x$.

The function $f = g + h$, where $h$ is defined for the set $C$ as in $T_0^0$-case, is the desired one.

$\mathcal{T} \times \mathcal{T}$-case. We can repeat the construction of the function from the previous case if we can build a $\mathcal{T} \times \mathcal{T}$-approximately continuous function which is discontinuous at the point $(0,0)$ and for a nowhere dense perfect set $\mathcal{P}$ we can define...
a family of squares \( \{J_n\}_{n \in \mathbb{N}} \) such that if \( \{I_n\}_{n \in \mathbb{N}} \) is the sequences of squares defined in \( T_\psi \) case, then \( J_n \subset I_n \), \( J_n \) is centered at the centre of \( I_n \), for \( n \in \mathbb{N} \) and each point of \( \mathcal{P} \) is an inner point of the complement of \( \bigcup_{n \in \mathbb{N}} J_n \) in \( \mathcal{T} \times \mathcal{T} \).

We begin by taking an interval set \( A = \bigcup_{n \in \mathbb{N}} (a_n, b_n) \), such that

\[
a_{n+1} \leq b_{n+1} \leq a_n, \quad n \in \mathbb{N}, \quad a_n \searrow 0 \quad \text{and} \quad 0 \in \Phi(A').
\]

Then let

\[
B = \bigcup_{n \in \mathbb{N}} (a_n, b_n) \times \left( -\frac{b_n - a_n}{2}, \frac{b_n - a_n}{2} \right).
\]

Hence \( (0, 0) \in \text{Int}(\mathbb{R}^2 \setminus B) \) since \( (\mathbb{R} \setminus A) \times \mathbb{R} \in \mathcal{T} \times \mathcal{T} \) and \( (\mathbb{R} \setminus A) \times \mathbb{R} \subset \mathbb{R}^2 \setminus B \).

It follows that the function

\[
h_0(x, y) = \begin{cases} 
0 & \text{for } (x, y) \in B', \\
1 & \text{for } (x, y) = \left( \frac{a_n + b_n}{2}, 0 \right), \quad n \in \mathbb{N}, \\
\text{continuous on } [a_n, b_n] \times \left[ -\frac{b_n - a_n}{2}, \frac{b_n - a_n}{2} \right] \text{ and such that} \\
\text{the image of this interval is equal to } [0, 1], \quad \text{for } n \in \mathbb{N}
\end{cases}
\]

has the desired properties.

Now, let \( P \) be a nowhere dense perfect set \( P \subset [0, 1] \times [0, 1] \) and \( \{I_n\}_{n \in \mathbb{N}} \) be the sequence of squares defined in \( T_\psi \)-case. Then, in each square \( I_n = S_{(k,m,p)} \), \( n \in \mathbb{N} \) we put a square \( J_n \) centered at the midpoint of \( I_n \) with sides parallel to the axis and such that

\[
\lambda_2(J_1) = \lambda_2(I_1) \quad \text{and} \quad \lambda_2(J_n) = \min \left( \frac{1}{n^2} \cdot \lambda_2(I_n), \frac{1}{4} \lambda_2(J_{n-1}) \right) \quad \text{for } n > 1.
\]

Additionally, we denote

\[
I_n = (a_n, b_n) \times (c_n, d_n), \quad n \in \mathbb{N}.
\]

Consider \( z = (x_0, y_0) \in \mathcal{P} \). Let \( A_1 = \bigcup_{n \in N_1} (a_n, b_n) \), where

\[
N_1 = \left\{ n \in \mathbb{N} : I_n \subset \left\{ (x, y) : (y - x - y_0 + x_0)(y + x - y_0 - x_0) < 0 \right\} \right\}
\]

and \( A_2 = \bigcup_{n \in N_2} (c_n, d_n) \), where \( N_2 = \mathbb{N} \setminus N_1 \). Then \( \bigcup_{n \in \mathbb{N}} J_n \subset (A_1 \times \mathbb{R}) \cup (\mathbb{R} \times A_2) \). To show that \( z \in \text{Int}_{\mathcal{T} \times \mathcal{T}} (\mathbb{R}^2 \setminus \bigcup_{n \in \mathbb{N}} J_n) \), it suffices to prove that \( x_0 \) is a density point of \( \mathbb{R} \setminus A_1 \) and \( y_0 \) is a density point of \( \mathbb{R} \setminus A_2 \).

Let \( r > 0 \) and let \( n(r) = \min \{ n \in N_1 : (x_0, x_0 + r) \cap (a_n, b_n) \} \}. Then there exists exactly one number \( p \in \mathbb{N} \) such that \( n(r) \in \{ n_{p-1} + 1, \ldots, n_p \} \). We denote this number by \( p(r) \). Thus, the interval \( (x_0, x_0 + r) \) has common points with at least one of the intervals \((a_{n_{p(r)} - 1 + 1}, b_{n_{p(r)} - 1 + 1}), (a_{n_{p(r)}}, b_{n_{p(r)}}))\). Therefore

\[
r > \frac{1}{2} \sqrt{\lambda_2(I_{n(r)})} - \frac{1}{2} \cdot \frac{1}{n(r)} \sqrt{\lambda_2(I_{n(r)})} = \frac{1}{2} \cdot \frac{1}{2p(r)} \left( 1 - \frac{1}{n(r)} \right),
\]

hence for \( r \to 0 \), we have \( n(r) \) tending to \( +\infty \) and \( p(r) \) tending to \( +\infty \).
We observe that
\[
\lambda((x_0, x_0 + r) \cap A_1) = \lambda\left((x_0, x_0 + r) \cap \bigcup_{n \in N_1 \cap [n(r), +\infty)} (a_n, b_n)\right)
\leq \sum_{n \in N_1 \cap [n(r), +\infty)} \lambda(a_n, b_n) \leq 2 \cdot (b_{n(r)} - a_{n(r)})
\leq 2 \cdot \frac{1}{n(r)} \cdot \frac{1}{2^{p(r)}}.
\]
Consequently,
\[
\frac{\lambda((x_0, x_0 + r) \cap A_1)}{r} \leq 2 \cdot \frac{1}{n(r)} \cdot \frac{1}{2^{p(r)}} \left(1 - \frac{1}{n(r)}\right) = \frac{4}{n(r) - 1},
\]
hence
\[
\lim_{r \to 0^+} \frac{\lambda((x_0, x_0 + r) \cap A_1)}{r} = 0 \quad \text{and similarly,} \quad \lim_{r \to 0^+} \frac{\lambda((x_0 - r, x_0) \cap A_1)}{r} = 0,
\]
which yields that \(x_0\) is a density point of \(\mathbb{R} \setminus A_1\).

Similar arguments apply to the set \(A_2\) and the point \(y_0\).

The construction of the function with desired properties runs as in \(T_0^\psi\) case.

\textbf{\(T_2\)-case.} We get the result immediately by the previous case and Proposition 2.11. \(\square\)

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