Tatra Mt. Math. Publ. **46** (2010), 65–69
DOI: 10.2478/v10127-010-0020-y

TATRA MOUNTAINS
*Mathematical Publications*

# APPLICATION OF HUREWICZ THEOREM TO CLASSIFICATION OF $\Pi^1_1$-COMPLETE SETS

Michal Staš

ABSTRACT. We present a simple proof of a dichotomy theorem by Kechris-Louveau-Woodin [Kechris, A. S., Louveau, A.: *Descriptive set theory and the structure of sets of uniqueness,* London Math. Soc. Lecture Note Ser. **128** (1987), 104–138] telling that every $\Pi^1_1$ $\sigma$-ideal of compact subsets of a Polish space is either $G_\delta$ or else $\Pi^1_1$-complete.

The descriptive hierarchy of sets raised a question of describing $\Pi^1_1$ (coanalytic) sets which are not $\Sigma^1_1$ (analytic), i.e., the true $\Pi^1_1$ sets. Following M. J. Souslin [4], we know that a set which is both analytic and coanalytic is Borel. Since there are $\Pi^1_1$ but not Borel sets, an important kind of true $\Pi^1_1$ sets is that of $\Pi^1_1$-complete set. In particular, $\Pi^1_1$-complete set on a hyperspace.

The main goal of this note is a simple proof of the Kechris-Louveau-Woodin Dichotomy Theorem [3]. The original proof by A. S. Kechris based on the game theory using Hurewicz Theorem [1] is elementary, however, rather complicated. We will use common set theoretical terminology and notations, say those of [2].

Let $X$ be a Polish space. A subset $P$ of $X$ is called **$\boldsymbol{\Pi^1_1}$-hard** if for any Polish space $Y$ and any $\Pi^1_1$ subset $Q$ of $Y$, there is a Borel function $f\colon Y \to X$ such that $Q = f^{-1}[P]$. A $\Pi^1_1$ set which is $\Pi^1_1$-hard is said to be **$\boldsymbol{\Pi^1_1}$-complete**. Replacing $\Pi^1_1$ with $\Sigma^1_1$, we obtain the definition of $\Sigma^1_1$-hard and $\Sigma^1_1$-complete set.

Following Souslin's theorem again, we know that no $\Pi^1_1$-complete set is $\Sigma^1_1$. If $X, Y$ are Polish spaces, $f\colon X \to Y$ is a Borel function, $P \subseteq X$ is $\Pi^1_1$-complete and $Q \subseteq Y$ is $\Pi^1_1$ and such that $P = f^{-1}[Q]$, then $Q$ is also $\Pi^1_1$-complete. This is a standard procedure for showing the $\Pi^1_1$-completeness of a set. Likewise, the $\Pi^1_1$-hardness of a set.

---

2010 Mathematics Subject Classification: Primary 54B20, 54A20; Secondary 03E75, 54E35, 54H05.
Keywords: analytic set, coanalytic set, $\Pi^1_1$-complete set, hyperspace, $\sigma$-ideal of compact sets, Hurewicz theorem.
The work on this research has been partially supported by the grant 1/0032/09 of Slovak Grant Agency VEGA and by the grant 30/2009/M of Faculty of Science of Pavol Jozef Šafárik University in Košice VVGS PF.

For each topological space $X$ let $\mathcal{K}(X)$ denote the space of all compact subsets of $X$ equipped with the **Vietoris topology** generated by the sets

$$\big\{K \in \mathcal{K}(X) : K \cap V \neq \emptyset\big\} \quad \text{and} \quad \big\{K \in \mathcal{K}(X) : K \subseteq V\big\},$$

$V$ being a non-empty open set.

The Vietoris topology is induced by the **Hausdorff metric** on $\mathcal{K}(X)$ which is defined as

$$\delta(K,L) = \begin{cases} \max\big\{\max\limits_{x \in K} \operatorname{dist}(x,L), \max\limits_{y \in L} \operatorname{dist}(y,K)\big\} & \text{if } K, L \neq \emptyset, \\ 0 & \text{if } K = L = \emptyset, \\ 1 & \text{if exactly one of } K, L \text{ is } \emptyset. \end{cases}$$

Any subtree of ${}^{<\omega}\omega$ can be identified with its characteristic function as a member of ${}^{{}^{<\omega}\omega}2$. If ${}^{{}^{<\omega}\omega}2$ is given the product topology with $2 = \{0,1\}$ discrete, then it is homeomorphic to the Cantor space ${}^{\omega}2$. The set of all subtrees of ${}^{<\omega}\omega$ is a closed subset of ${}^{\omega}2$. Moreover, the set of those subtrees of ${}^{<\omega}\omega$ having all branches finite, is a $\Pi^1_1$-complete set.

A. S. Kechris [3] has proved the next result. We now present a slightly modified version of his proof.

**Lemma 1.** *If $Q$ is a countable dense subset of ${}^{\omega}2$ then $\mathcal{K}(Q)$ is $\Pi^1_1$-complete.*

Proof. Let $Q \subseteq {}^{\omega}2$ be a countable dense set. $Q$ is an $\mathrm{F}_\sigma$ set, so $\mathcal{K}(Q)$ is $\Pi^1_1$.

Let $P \subseteq {}^{\omega}2$ be a $\Pi^1_1$-complete set as mentioned above. Since ${}^{\omega}2 \setminus P$ is $\Sigma^1_1$, there exists a continuous mapping $\lambda\colon {}^{\omega}\omega \to {}^{\omega}2$ such that $\lambda({}^{\omega}\omega) = {}^{\omega}2 \setminus P$. Let $\pi\colon \omega \times \omega \to \omega$ be any bijection and $t\colon {}^{\omega}\omega \to {}^{\omega}2$ be the mapping defined by

$$t(v)\big(\pi(n,m)\big) = \begin{cases} 0 & \text{if } v(n) = m, \\ 1 & \text{other,} \end{cases}$$

for $v \in {}^{\omega}\omega$. Then the set $t({}^{\omega}\omega)$ is equal to

$$\bigcap_n \bigcup_m \Big\{\delta \in {}^{\omega}2 : \delta\big(\pi(n,m)\big) = 0\Big\}$$
$$\cap \bigcap_n \bigcap_m \bigcap_{k \neq m} \Big\{\delta \in {}^{\omega}2 : \delta\big(\pi(n,m)\big) = 0 \to \delta(\pi(n,k)) = 1\Big\}$$

and therefore, is a $\mathrm{G}_\delta$ subset of ${}^{\omega}2$. Let $G \subseteq {}^{\omega}2 \times {}^{\omega}2$ be such that

$$(x,y) \in G \Leftrightarrow \lambda\big(t^{-1}(y)\big) = x.$$

Then $G$ is a $\mathrm{G}_\delta$ set. If we denote $F = ({}^{\omega}2 \times {}^{\omega}2) \setminus G$, then

$$x \in P \Leftrightarrow (\forall\, y \in {}^{\omega}2)\, (x,y) \in F.$$

Since ${}^\omega 2 \times {}^\omega 2$ is homeomorphic to ${}^\omega 2$, there exists a continuous $h\colon {}^\omega 2 \times {}^\omega 2 \to {}^\omega 2$ such that $h^{-1}(Q) = F$. The mapping $\mu\colon {}^\omega 2 \to \mathcal{K}({}^\omega 2)$ defined by

$$\mu(x) = h\big[\{x\},{}^\omega 2\big]$$

is continuous and

$$x \in P \Leftrightarrow \mu(x) \in \mathcal{K}(Q).$$

Thus, $P = \mu^{-1}\big(\mathcal{K}(Q)\big)$ and therefore $\mathcal{K}(Q)$ is $\Pi^1_1$-complete. □

Note that if $Q$ is a set of all eventually periodic sequences in ${}^\omega 2$, then we can reduce $\mathcal{K}(Q)$ to $\mathcal{K}\big(\mathbb{Q} \cap [0,1]\big)$. So, we obtain

**Corollary 2** ([1, W. Hurewicz]). $\mathcal{K}\big(\mathbb{Q} \cap [0,1]\big)$ *is* $\Pi^1_1$*-complete set.*

A complementary result is that the set $\big\{K \in \mathcal{K}([0,1]) : K \cap \mathbb{I} \neq \emptyset\big\}$ is $\Sigma^1_1$-complete. Moreover, if $X$ is a Polish, and $G \subseteq X$ is a $\mathrm{G}_\delta$ set but not an $\mathrm{F}_\sigma$ set, then $\big\{K \in \mathcal{K}(X) : K \cap G \neq \emptyset\big\}$ is $\Sigma^1_1$-complete. Thus, if $F \subseteq X$ is an $\mathrm{F}_\sigma$ set but not a $\mathrm{G}_\delta$ set, then $\mathcal{K}(F)$ is $\Pi^1_1$-complete.

A subset $\mathcal{I} \subseteq \mathcal{K}(X)$ is called a **$\sigma$-ideal of compact sets** of $X$ if the following conditions are fulfilled

(1) if $K \in \mathcal{K}(X), K \subseteq L \in \mathcal{I}$ then $K \in \mathcal{I}$,

(2) if $K_n \in \mathcal{I}, n \in \omega$ and $\bigcup_n K_n \in \mathcal{K}(X)$ then $\bigcup_n K_n \in \mathcal{I}$.

Let $\mathcal{S}$ be a non-empty subset of $\mathcal{K}(X)$. We denote by $\mathcal{S}_\sigma$ the set

$$\mathcal{S}_\sigma = \left\{ M \in \mathcal{K}(X) :\ M = \bigcup_n M_n, M_n \in \mathcal{S}, n \in \omega \right\}.$$

Let us remark that $\mathcal{S}_\sigma$ is closed under countable unions which are in $\mathcal{K}(X)$, i.e., it satisfies condition (2) of the previous definition, but need not be $\sigma$-ideal of compact sets of $X$.

**Theorem 3.** *Let $X$ be a Polish space, and $\mathcal{S}$ be a non-empty subset of $\mathcal{K}(X)$ such that $\mathcal{S}$ contains a countable perfect subset $\mathcal{L}$ in $\mathcal{S}$.*[1] *If $\mathcal{L}$ is closed in $\mathcal{S}_\sigma$ and*

$$(\forall \mathcal{M} \in \mathcal{K}(\overline{\mathcal{L}})) \left( \bigcup \mathcal{M} \in \mathcal{S}_\sigma \wedge M \in \mathcal{M} \to M \in \mathcal{S}_\sigma \right)$$

*then $\mathcal{S}_\sigma$ is $\Pi^1_1$-hard.*

Proof. $\overline{\mathcal{L}}$ is a closed subset of $\mathcal{K}(X)$, so we consider $\overline{\mathcal{L}}$ with a subspace topology of $\mathcal{K}(X)$. Let $h\colon \mathcal{K}(\overline{\mathcal{L}}) \to \mathcal{K}(X)$ be the mapping defined by

$$h(\mathcal{M}) = \bigcup \mathcal{M} = \bigcup \{M : M \in \mathcal{M}\}.$$

Then $h$ is continuous, see, e.g., [3]. By the assumptions, we can verify that

$$\mathcal{M} \subseteq \overline{\mathcal{L}} \cap \mathcal{S} = \mathcal{L} \Leftrightarrow \bigcup \mathcal{M} \in \mathcal{S}_\sigma$$

[1] A perfect set is a non-empty closed set without isolated points.

for any $\mathcal{M} \in \mathcal{K}(\overline{\mathcal{L}})$. Thus,

$$h^{-1}(\mathcal{S}_\sigma) = \mathcal{K}(\mathcal{L}) = \{\mathcal{M} \in \mathcal{K}(\overline{\mathcal{L}}) : \mathcal{M} \subseteq \mathcal{L}\}.$$

Since $\mathcal{L}$ is an $\mathrm{F}_\sigma$, but not a $\mathrm{G}_\delta$ set, then $\mathcal{K}(\mathcal{L})$ is $\Pi^1_1$-complete. Thus $\mathcal{S}_\sigma$ is $\Pi^1_1$-hard. □

**Corollary 4.** *Let $X$ be a Polish space, and $\mathcal{S}$ be a non-empty subset of $\mathcal{K}(X)$ such that $\mathcal{S}$ contains a countable perfect subset $\mathcal{L}$ in $\mathcal{S}$. If there exists a $\sigma$-ideal $\mathcal{I} \supseteq \mathcal{S}_\sigma$ of compact sets of $X$ such that $\mathcal{L}$ is closed in $\mathcal{I}$, then $\mathcal{S}_\sigma$ is $\Pi^1_1$-hard.*

Proof. Let $\mathcal{I} \supseteq \mathcal{S}_\sigma$ be a $\sigma$-ideal of compact sets of $X$ such that $\mathcal{I} \cap \overline{\mathcal{L}} = \mathcal{L}$. Then it is easy to see that $\mathcal{L}$ is closed in $\mathcal{S}_\sigma$. Let $\mathcal{M}$ be any non-empty set from $\mathcal{K}(\overline{\mathcal{L}})$ such that $\bigcup \mathcal{M} \in \mathcal{S}_\sigma$. If $M \in \mathcal{M}$, then $M \subseteq \bigcup \mathcal{M} \in \mathcal{S}_\sigma \subseteq \mathcal{I}$. Thus, $M \in \mathcal{I} \cap \overline{\mathcal{L}} = \mathcal{L} \subseteq \mathcal{S} \subseteq \mathcal{S}_\sigma$. By the previous theorem, $\mathcal{S}_\sigma$ is $\Pi^1_1$-hard. □

If we take $X = [0,1]$ and $S = \mathcal{K}(\mathbb{Q} \cap [0,1])$, then the set of all singletons on $\mathcal{K}(\mathbb{Q} \cap [0,1])$ is a countable perfect subset of $S$. Therefore, by Corollary 4, the set $S = S_\sigma = \mathcal{K}(\mathbb{Q} \cap [0,1])$ is $\Pi^1_1$-hard, and hence $\Pi^1_1$-complete.

A. S. Kechris [3] presents a proof of Corollary 5 based on game theory, in which he uses a theorem of Hurewicz [1], telling that if $A$ is a coanalytic non-$G_\delta$ set in a Polish space $X$, then there exists a homeomorphism $f\colon {}^\omega 2 \to X$ and a countable dense set $Q \subseteq {}^\omega 2$ such that $f[Q] \subseteq A$ and $f[{}^\omega 2 \setminus Q] \subseteq X \setminus A$. An elementary proof of a generalization of Hurewicz's theorem, using the notion of Hurewicz scheme, is presented in [5]. As its application, we suggest a simple proof of the next assertion.

**Corollary 5.** *Let $\mathcal{I}$ be a $\Pi^1_1$ $\sigma$-ideal of compact sets of a Polish space $X$ and $\mathcal{S}$ be a non-empty subset of $\mathcal{I}$. If $\mathcal{S}, \mathcal{I}$ cannot be separated by a $G_\delta$ set, then $\mathcal{S}_\sigma$ is $\Pi^1_1$-hard.*

Proof. Assume that there is no $\mathrm{G}_\delta$ set separating $\mathcal{S}, \mathcal{I}$, i.e., $\mathcal{K}(X) \setminus \mathcal{I}, \mathcal{K}(X) \setminus \mathcal{S}$ cannot be separated by an $\mathrm{F}_\sigma$ set. According to the main theorem of [5], there exists a countable perfect subset $\mathcal{L} \subseteq \mathcal{S}$ such that $\overline{\mathcal{L}} \setminus \mathcal{L} \subseteq \mathcal{K}(X) \setminus \mathcal{I}$. Then, by Corollary 4, $\mathcal{S}_\sigma$ is $\Pi^1_1$-hard. □

In particular, if we take $\mathcal{S} = \mathcal{I}$, we get the Kechris-Louveau-Woodin's Dichotomy Theorem [3].

**Theorem 6** ([3, Kechris, Louveau, Woodin])**.** *Every $\Pi^1_1$ $\sigma$-ideal of compact subsets of a Polish space $X$ is either $G_\delta$ or else $\Pi^1_1$-complete.*

Thus, we know that any $\Pi^1_1$-complete set is true $\Pi^1_1$, and from the previous theorem, that every true $\Pi^1_1$ $\sigma$-ideal of compact subsets is $\Pi^1_1$-complete.

**Acknowledgements.** I would like to thank Prof. Lev Bukovský and the referee for valuable comments.

## REFERENCES

[1] HUREWICZ, W.: *Relativ perfekte Teile von Punktmengen und Mengen* (A), Fund. Math. **9** (1925), 78–109.
[2] KECHRIS, A. S.: *Classical Descriptive Set Theory*. Springer-Verlag, Berlin, 1994.
[3] KECHRIS, A. S.—LOUVEAU, A.: *Descriptive set theory and the structure of sets of uniqueness,* London Math. Soc. Lecture Note Ser. **128** (1987), 104–138.
[4] SOUSLIN, M. J.: *Sur une définition des ensembles measurables* B *sans nombres transfinis*, C. R. Acad. Sci. **1**64 (1917), 88–91.
[5] STAŠ, M.: *Hurewicz scheme,* Acta Univ. Carolin. Math. Phys. **49** (2008), 75–78.

Received November 30, 2009

*Institute of Mathematics*
*University of P. J. Šafárik*
*Jesenná 5*
*SK–040-01 Košice*
*SLOVAKIA*
*E-mail*: michal.stas@upjs.sk