Tatra Mt. Math. Publ. **44** (2009), 15–25
DOI: 10.2478/v10127-009-0044-3

Tatra Mountains
Mathematical Publications

SOME CONTINUOUS OPERATIONS ON PAIRS OF CLIQUISH FUNCTIONS

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ABSTRACT. The algebraic or lattice operations in the classes of cliquish or quasicontinuous functions are well known [Z. Grande: *On the maximal multiplicative family for the class of quasicontinuous functions*, Real Anal. Exchange **15** (1989–1990), 437–441, Z. Grande, L. Soltysik: *Some remarks on quasicontinuous real functions*, Problemy Mat. **10** (1990), 79–86]. This also pertains to the symmetrical quasicontinuity or symmetrical cliquishness [Z. Grande: *On the maximal additive and multiplicative families for the quasicontinuities of Piotrowski and Vallin*, Real Anal. Exchange **32** (2007), 511–518]. In this article, we examine the superpositions $F(f,g)$, where F is a continuous operation and f,g are cliquish (symmetrically cliquish) or f is continuous (f is symmetrically quasicontinuous with continuous sections) and g is quasicontinuous (symmetrically quasicontinuous).

Let (X,T_X) and (Y,T_Y) be topological spaces, let (Z,ρ) and (Z_i,ρ_i), $(i=1,2)$ be metric spaces.

Definition ([10, 11])**.** A function $f\colon X\to Z$ is said to be

• *quasicontinuous* (*cliquish*) at a point $x\in X$ if for every set $U\in T_X$ containing x and for each positive real ε there is a nonempty set $U'\in T_X$ contained in U such that $f(U')\subset K\big(f(x),\varepsilon\big)=\big\{t\in Z;\ \rho(t,f(x))<\varepsilon\big\}$ $\big(\operatorname{diam}\big(f(U')\big)=\sup\big\{\rho(f(t),f(u));\ t,u\in U'\big\}<\varepsilon\big)$.

Definition. A function $f\colon X\times Y\to Z$ is said to be:

• *quasicontinuous at* (x,y) *with respect to* x (alternatively y) if for every set $U\times V\in T_X\times T_Y$ containing (x,y) and for each positive real ε there are nonempty sets $U'\in T_X$ contained in U and $V'\in T_Y$ contained in V such that $x\in U'$ (alternatively $y\in V'$) and $f(U'\times V')\subset K\big(f(x,y),\varepsilon\big)$ ([10]–[12]);

2000 Mathematics Subject Classification: 26A15, 26A21, 54C05.
Keywords: quasicontinuity, symmetrical quasicontinuity, cliquishness, symmetrical cliquishness, strong quasicontinuity, strong cliquishness, continuous operations, maximal family for operation.

• *cliquish at* (x, y) *with respect to* x (alternatively y) if for every set $U\times V\in T_X\times T_Y$ containing (x, y) and for each positive real ε there are nonempty sets $U' \in T_X$ contained in U and $V' \in T_Y$ contained in V such that $x \in U'$ (alternatively $y \in V'$) and $\operatorname{diam}\big(f(U' \times V')\big) < \varepsilon$) ([3]);

• *symmetrically quasicontinuous* (*symmetrically cliquish*) at (x, y) if it is quasicontinuous (cliquish) at (x, y) with respect to x and with respect to y.

Evidently the cliquishness (the quasicontinuity) with respect to x (or to y) of a function $f\colon X \times Y \to Z$ implies its cliquishness (its quasicontinuity) and the respective quasicontinuities imply the respective cliquishness.

I. The superposition $x \to F\big(f(x), g(x)\big)$ on pairs of cliquish functions (f, g)

We start from the well-known examples. Let (a_n) be an enumeration of all rationals such that $a_n \neq a_m$ for $n \neq m$.

1. Let $f_1(a_n) = \frac{1}{n}$ for $n \geq 1$. Then $f_1\colon \mathbb{Q} \to (0, \infty)$ is a cliquish function (we consider the spaces $\mathbb{Q}$ and $(0, \infty)$ with the natural metric), but its inversion

$$\frac{1}{f_1(a_n)} = n \qquad \text{for} \quad n \geq 1$$

is not cliquish at any point.

2. Observe that the given $F_1\colon (0, \infty)^2 \to (0, \infty)$ defined by the formula

$$F_1(u, v) = \frac{1}{u + v}$$

is continuous, but the superposition

$$F_1\big(f_1(a_n), f_1(a_n)\big) = \frac{n}{2}, \qquad\qquad n \geq 1,$$

is not cliquish.

3. Analogously, the given $g_1\colon \mathbb{Q}^2 \to (0, \infty)$ defined by the formula

$$g_1(a_n, a_m) = \frac{1}{nm} \quad \text{for} \quad n, m \geq 1,$$

is symmetrically cliquish, but the superposition

$$F_1(g_1, g_1) = \frac{1}{2g_1}$$

is not cliquish at any point.

So, there are cliquish functions $f, g\colon \mathbb{Q} \to (0,\infty)$ (symmetrically cliquish functions $f, g\colon \mathbb{Q}^2 \to (0,1)$) and a continuous function $F\colon (0,\infty)^2 \to (0,\infty)$ such that the superposition $F(f,g)$ is not cliquish. Moreover the following theorems are true.

Theorem 1. *Let $F\colon Z_1 \times Z_2 \to Z$ be a uniformly continuous function. If the functions $f\colon X \to Z_1$ and $g\colon X \to Z_2$ are cliquish at each point $x \in X$ then the function $h(x) = F\big(f(x), g(x)\big)$ for $x \in X$ is also cliquish at each point x.*

P r o o f. Fix a real $\varepsilon > 0$, a point $x \in X$ and an open set $U \in T_X$ with $x \in U$. From the uniform continuity of F there is a real $\delta > 0$ such that for $z_1, z_2 \in Z_1$ and $z_3, z_4 \in Z_2$ if $\rho_1(z_1, z_2) < \delta$ and $\rho_2(z_3, z_4) < \delta$ then $\rho\big(F(z_1, z_3), F(z_2, z_4)\big) < \frac{\varepsilon}{2}$. Since f is cliquish at x, there is a nonempty open set $U' \in T_X$ such that

$$U' \subset U \quad \text{and} \quad \mathrm{diam}\big(\mathrm{f(U')}\big) < \delta.$$

Fix a point $v \in U'$. Since g is cliquish at v, there is a nonempty open set $U'' \in T_X$ such that

$$U'' \subset U' \quad \text{and} \quad \mathrm{diam}\big(\mathrm{g(U'')}\big) < \delta.$$

Observe that for x_1, x_2 belonging to U'' we have

$$\rho_1\big(f(x_1), f(x_2)\big) < \delta \quad \text{and} \quad \rho_2\big(g(x_1),\, g(x_2)\big) < \delta,$$

thus the inequality

$$\rho\Big(F\big(f(x_1),\, g(x_1),\, F\big(f(x_2),\, g(x_2)\big)\Big) < \frac{\varepsilon}{2}$$

is true. So, $\mathrm{diam}\big(h(U'')\big) \le \frac{\varepsilon}{2} < \varepsilon$ and the proof is finished. □

Theorem 2. *Let $F\colon Z_1 \times Z_2 \to Z$ be a uniformly continuous function. If the functions $f\colon X \times Y \to Z_1$ and $g\colon X \times Y \to Z_2$ are cliquish at each point $(x, y) \in X \times Y$ with respect to x (alternatively to y), then the function $h(x, y) = F\big(f(x, y), g(x, y)\big)$ for $(x, y) \in X \times Y$ is also cliquish at each point (x, y) with respect to x (alternatively to y).*

P r o o f. Fix a real $\varepsilon > 0$, a point $(x, y) \in X \times Y$ and open sets $U \in T_X$ and $V \in T_Y$ with $(x, y) \in U \times Y$. From the uniform continuity of F there is a real $\delta > 0$ such that for $z_1, z_2 \in Z_1$ and $z_3, z_4 \in Z_2$ if

$$\rho_1(z_1, z_2) < \delta \quad \text{and} \quad \rho_2(z_3, z_4) < \delta \qquad \text{then} \quad \rho\big(F(z_1, z_3),\, F(z_2, z_4)\big) < \frac{\varepsilon}{2}.$$

Since f is cliquish at (x, y) with respect to x, there are nonempty open sets $U' \in T_X$ and $V' \in T_Y$ such that

$$x \in U' \subset U,\, V' \subset V \quad \text{and} \quad \mathrm{diam}\big(\mathrm{f(U' \times V')}\big) < \delta.$$

Fix a point $v \in V'$. Since g is cliquish at (x, v) with respect to x, there are nonempty open sets $U'' \in T_X$ and $V'' \in T_Y$ such that

$$x \in U'' \subset U',\, V'' \subset V' \quad \text{and} \quad \operatorname{diam}\bigl(\mathrm{g(U'' \times V'')}\bigr) < \delta.$$

Observe that for (x_1, y_1) and (x_2, y_2) belonging to $U'' \times V''$ we have

$$\rho_1\bigl(f(x_1, y_1),\, f(x_2, y_2)\bigr) < \delta \quad \text{and} \quad \rho_2\bigl(g(x_1, y_1),\, g(x_2, y_2)\bigr) < \delta,$$

thus the inequality

$$\rho\Bigl(F\bigl(f(x_1, y_1),\, g(x_1, y_1)\bigr),\, F\bigl(f(x_2, y_2),\, g(x_2, y_2)\bigr)\Bigr) < \frac{\varepsilon}{2}$$

is true. So, $\operatorname{diam}\bigl(h(U'' \times V'')\bigr) \le \frac{\varepsilon}{2} < \varepsilon$ and in the case where f and g are cliquish at (x, y) with respect to x the proof is finished. In the alternative case where f and g are cliquish at (x, y) with respect to y the proof is analogous. □

Corollaries and remarks

From Theorem 1 we obtain

Corollary 1. *Let $F\colon Z_1 \times Z_1 \to Z$ be a continuous function. If the functions $f, g\colon X \times Y \to Z_1$ are cliquish at each point $(x, y) \in X \times Y$ with respect to x (alternatively to y) and if the closures $\mathrm{cl}\bigl(\mathrm{f(X \times Y)}\bigr)$ and $\mathrm{cl}\bigl(\mathrm{g(X \times Y)}\bigr)$ are compact, then the functions $h_1(x, y) = F\bigl(f(x, y), g(x, y)\bigr)$ and $h_2(x, y) = F\bigl(g(x, y), f(x, y)\bigr)$ for $(x, y) \in X \times Y$ are also cliquish at each point (x, y) with respect to x (alternatively to y).*

From Theorem 2 we have

Corollary 2. *Let $F\colon Z_1 \times Z_1 \to Z$ be a uniformly continuous function. If the functions $f, g\colon X \times Y \to Z_1$ are symmetrically cliquish at each point $(x, y) \in X \times Y$, then the functions $h_1(x, y) = F\bigl(f(x, y),\, g(x, y)\bigr)$ and $h_2(x, y) = F\bigl(g(x, y),\, f(x, y)\bigr)$ for $(x, y) \in X \times Y$ are also symmetrically cliquish at each point (x, y).*

In the case $Z = Z_1 = \mathbb{R}$ and $\rho(z_1, z_2) = \rho_1(z_1, z_2) = |z_1 - z_2|$, the functions $F_1(z_1, z_2) = \max(z_1, z_2)$ and $F_2(z_1, z_2) = \min(z_1, z_2)$ are uniformly continuous. So, from Theorem 2 we obtain

Corollary 3. *If the functions $f, g\colon X \times Y \to \mathbb{R}$ are cliquish at each point $(x, y) \in X \times Y$ with respect to x (alternatively to y), then the functions $\max(f, g)$ and $\min(f, g)$ are also cliquish at each point (x, y) with respect to x (alternatively with respect to y).*

In the case where $(Z, \|\cdot\|) = (Z_1, \|\cdot\|)$ is a normed space, for all reals r_1, r_2 the function $F(z_1, z_2) = r_1 z_1 + r_2 z_2$ is uniformly continuous. So, from Theorem 2, we obtain

Corollary 4. *If $(Z, \|\cdot\|)$ is a normed space and the functions $f, g\colon X \times Y \to Z$ are cliquish at each point $(x, y) \in X \times Y$ with respect to x (alternatively with respect to y), then for all real r_1, r_2 the function $r_1 f + r_2 g$ is also cliquish at each point $(x, y) \in X \times Y$ with respect to x (alternatively with respect to y).*

From Corollary 1 for the case $Z = Z_1 = \mathbb{R}$ and $\rho(z_1, z_2) = \rho_1(z_1, z_2) = |z_1 - z_2|$ it follows the following

Corollary 5. *If the bounded functions $f, g\colon X \times Y \to \mathbb{R}$ are cliquish at each point $(x, y) \in X \times Y$ with respect to x (alternatively with respect to y), then the product fg is also cliquish at each point (x, y) with respect to x (alternatively with respect to y).*

The product $F(x, y) = xy$ for $x, y \in \mathbb{R}$ is not uniformly continuous in $\mathbb{R}^2$ with the natural metric. Similarly, the scalar product $F(x, y) = (x|y)$ for $x, y \in \mathbb{R}^n$ is not uniformly continuous in $\mathbb{R}^{2n}$ with natural metric. So, for the proof of the cliquishness (or of the symmetrical cliquishness) of the product or the scalar product of two cliquish (or symmetrically cliquish) functions Theorem 2 is not sufficient, however, we can use the following remarks.

Remark 1. Suppose that (X, T_X) is a Baire space. If the functions $f, g\colon X \to Z_1$ are cliquish and $F\colon Z_1 \times Z_1 \to Z$ is a continuous function then the superposition $h(x) = F\big(f(x), g(x)\big)$ for $x \in X$ is cliquish.

P r o o f. Denote by $C(f)$ $(C(g))$ the set of all continuity points of f (of g). Since (X, T_X) is a Baire space, the sets $C(f)$ and $C(g)$ are residual in X (see [2], [11]). Consequently, the intersection $C(f) \cap C(g)$ is residual in X and $C(f) \cap C(g) \subset C\big(F(f, g)\big)$, so the superposition $F(f, g)$ is cliquish. $\square$

Remark 2. Suppose (X, T_X) (alternatively (Y, T_Y)) is a Baire space and functions $f, g\colon X \times Y \to Z_1$ are cliquish with respect to y (alternatively to x). If a function $F\colon Z_1 \times Z_1 \to Z$ is continuous then the superpositions $h_1(x) = F\big(f(x, y), g(x, y)\big)$ and $h_2(x) = F\big(g(x, y), f(x, y)\big)$ for $x \in X$ are cliquish with respect to y (alternatively to x).

P r o o f. Similarly as in the proof of the previous remark, we observe that the sections $\big(C(f)\big)^y$ and $\big(C(g)\big)^y$, $y \in Y$, are residual in X (see [6]). Consequently, the sections $\big(C(F(f, g))\big)^y \supset \big(C(f) \cap C(g)\big)^y$, $y \in Y$, are residual in X and the superposition h_1 is cliquish with respect to y. Similarly, we can prove the cliquishness of h_1 with respect to x and the cliquishness of h_2 with respect to x and to y. $\square$

Corollary 6. *Suppose (X, T_X) and (Y, T_Y) are Baire spaces and the functions $f, g\colon X \times Y \to Z_1$ are symmetrically cliquish. If a function $F\colon Z_1 \times Z_1 \to Z$ is continuous then the superposition $h(x) = F\big(f(x, y), g(x, y)\big)$ is symmetrically cliquish.*

II. The superposition $F(f, g)$, where one of the functions f and g is quasicontinuous and the second is continuous

Recall that there are quasicontinuous real-valued functions whose sum is not quasicontinuous (see [8]). We start from the following easy observation:

Remark 3. Let $F\colon Z_1 \times Z_1 \to Z$ be a continuous function. If a function $f\colon X \to Z_1$ is quasicontinuous at a point $x_0 \in X$ and a function $g\colon X \to Z_1$ is continuous at x_0 then the functions $h_1(x) = F\big(f(x), g(x)\big)$ and $h_2(x) = F\big(g(x), f(x)\big)$, $x \in X$, are also quasicontinuous at the point x_0.

P r o o f. Both functions $k_1(x) = \big(f(x), g(x)\big)$ and $k_2(x) = \big(g(x), f(x)\big)$, $x \in X$, are quasicontinuous at x_0, so their superpositions $F(k_1)$ and $F(k_2)$ with a continuous function F are quasicontinuous at x_0. $\square$

Theorem 3. *Let $F\colon Z_1 \times Z_1 \to Z$ be a continuous function. If the functions $f, g\colon X \times Y \to Z_1$ are quasicontinuous at each point $(x, y) \in X \times Y$ with respect to x (alternatively with respect to y) and at each point $(x, y) \in X \times Y$ at least one of them has continuous vertical section (horizontal section) then the functions $h_1(x, y) = F\big(f(x, y), g(x, y)\big)$ and $h_2(x, y) = F\big(g(x, y), f(x, y)\big)$ for $(x, y) \in X \times Y$ are also quasicontinuous at each point (x, y) with respect to x (alternatively with respect to y).*

P r o o f. Fix a real $\varepsilon > 0$, a point $(x, y) \in X \times Y$ and open sets $U \in T_X$ and $V \in T_Y$ with $(x, y) \in U \times V$. We can assume that the vertical section g_x is continuous at y. From the continuity of F at the point $\big(f(x, y), g(x, y)\big)$, there is a real $\delta > 0$ such that for $z_1, z_2 \in Z_1$ if $\rho_1\big(z_1, f(x, y)\big) < \delta$ and $\rho_1\big(z_2, g(x, y)\big) < \delta$, then

$$\rho\big(F(z_1, z_2), F(f(x, y), g(x, y))\big) < \frac{\varepsilon}{3}.$$

From the continuity of the section g_x at y, it follows that there is a set $V_1 \in T_Y$ containing y such that $V_1 \subset V$ and $\rho_1\big(g(x, w), g(x, y)\big) < \delta$ for $w \in V_1$. Since f is quasicontinuous at (x, y) with respect to x, there are nonempty open sets $U' \in T_X$ and $V' \in T_Y$ such that

$$x \in U' \subset U,\ V' \subset V_1 \quad \text{and} \quad f(U' \times V') \subset K\big(f(x, y), \delta\big).$$

Fix a point $v \in V'$. By the continuity of F at $\big(f(x, v), g(x, v)\big)$, there is a real $\delta_1 \in (0, \delta)$ such that for $z_3, z_4 \in Z_1$ if $\rho_1\big(z_3, f\big(x, v\big)\big) < \delta_1$ and $\rho_1\big(z_4, g(x, v)\big) < \delta_1$, then

$$\rho\big(F(z_3, z_4), F(f(x, v), g(x, v))\big) < \frac{\varepsilon}{3}.$$

Since g is quasicontinuous at (x, v) with respect x, there are open sets $U'' \in T_X$ and $V'' \in T_Y$ such that

$$x \in U'' \subset U',\ V'' \subset V' \quad \text{and} \quad g\big(U'' \times V''\big)\big) \subset K\big(g(x, v), \delta_1\big).$$

Observe that for (x_1, y_1) belonging to $U'' \times V''$ we have

$$\rho_1\big(f(x_1,\, y_1),\, f(x, y)\big) < \delta \quad \text{and} \quad \rho_1\big(g(x_1,\, y_1),\, g(x, v)\big) < \delta_1,$$

thus the inequalities

$$\begin{aligned}
&\rho\Big(F\big(f(x_1,\, y_1),\, g(x_1, y_1)\big),\, F\big(f(x, y),\, g(x, y)\big)\Big) \\
&\le \rho\Big(F\big(f(x_1, y_1),\, g(x_1, y_1)\big),\, F\big(f(x, v),\, g(x, v)\big)\Big) \\
&\quad + \rho\Big(F\big(f(x, v),\, g(x, v)\big),\, F\big(f(x, y),\, g(x, y)\big)\Big) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} < \varepsilon
\end{aligned}$$

are true. So, $h_1(U'' \times V'') \subset K\big(h_1(x, y), \varepsilon\big)$, and the proof for h_1 is completed. Proofs of the quasicontinuity of h_2 at each point (x, y) with respect to x and of the quasicontinuity with respect to y are analogous. □

Corollary 7. *Let $F\colon Z_1 \times Z_1 \to Z$ be a continuous function. If the functions $f, g\colon X \times Y \to Z_1$ are symmetrically quasicontinuous at each point $(x, y) \in X \times Y$ and the sections g_x and g^y are continuous, then the function $h(x, y) = F\big(f(x, y),\, g(x, y)\big)$ for $(x, y) \in X \times Y$ is also symmetrically quasicontinuous at each point (x, y).*

In the case $Z = Z_1 = \mathbb{R}$ and $\rho(z_1, z_2) = \rho_1(z_1, z_2) = |z_1 - z_2|$, the functions $F_1(z_1, z_2) = \max(z_1, z_2)$ and $F_2(z_1, z_2) = \min(z_1, z_2)$ are uniformly continuous. So, from Theorem 3, we obtain

Corollary 8. *If the functions $f, g\colon X \times Y \to \mathbb{R}$ are quasicontinuous at each point $(x, y) \in X \times Y$ with respect to x (alternatively to y) and if the function $g\colon X \times Y \to \mathbb{R}$ has continuous vertical (horizontal) sections then the functions $\max(f, g)$, $\min(f, g)$ and fg are also quasicontinuous at each point (x, y) with respect to x (alternatively with respect to y).*

In the case where $(Z, \|\cdot\|) = (Z_1, \|\cdot\|)$ is a normed space then for all reals r_1, r_2 the function $F(z_1, z_2) = r_1 z_1 + r_2 z_2$ is uniformly continuous. So, from Theorem 3, we obtain

Corollary 9. *If $(Z, \|\cdot\|)$ is a normed space, the functions $f, g\colon X \times Y \to Z$ are quasicontinuous at each point $(x, y) \in X \times Y$ with respect to x (alternatively with respect to y) and the function $f\colon X \times Y \to Z$ has continuous vertical (horizontal) sections, then for all real r_1, r_2, the function $r_1 f + r_2 g$ is also quasicontinuous at each point $(x, y) \in X \times Y$ with respect to x (alternatively with respect to y).*

III. Continuous operations and kinds of the cliquishness and quasicontinuity

Consider the real line $\mathbb{R}$ with the natural metric and denote by T_d the density topology in $\mathbb{R}$ ([1], [13]). A special kind of the cliquishness (the quasicontinuity) of the functions from $\mathbb{R}$ to $\mathbb{R}$ is a strong cliquishness (strong quasicontinuity).

Definition ([7])**.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *strongly quasicontinuous* (resp. *strongly cliquish*) at a point x if for each positive real η and for each set $A \in T_d$ containing x, there is an open interval I such that $I \cap A \neq \emptyset$ and $f(A \cap I) \subset \big(f(x) - \eta,\, f(x) + \eta\big)$ $\big(\operatorname{diam}\big(f(A \cap I)\big) < \eta\big)$.

Theorem 4. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a uniformly continuous function. If the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are strongly cliquish at each point $x \in \mathbb{R}$, then the function $h(x) = F\big(f(x), g(x)\big)$ for $x \in \mathbb{R}$ is also strongly cliquish at each point x.*

P r o o f. Fix a real $\varepsilon > 0$, a point $x \in X$ and a set $U \in T_d$ with $x \in U$. From the uniform continuity of F there is a real $\delta > 0$ such that for $z_1, z_2, z_3, z_4 \in \mathbb{R}$ if

$$\max(|z_1 - z_2|,\, |z_3 - z_4|) < \delta, \quad \text{then} \quad |F(z_1, z_3) - F(z_2, z_4)| < \frac{\varepsilon}{2}.$$

Since f is strongly cliquish at x, there is an open interval I such that

$$I \cap U \neq \emptyset \quad \text{and} \quad \operatorname{diam}\big(\mathrm{f(I \cap U)}\big) < \delta.$$

Fix a point $v \in U \cap I$. Since g is strongly cliquish at v, there is an open interval $I' \subset I$ such that

$$I' \cap U \neq \emptyset \quad \text{and} \quad \operatorname{diam}\big(\mathrm{g(U \cap I')}\big) < \delta.$$

Observe that for any points x_1, x_2 belonging to $U \cap I'$, we have

$$|f(x_2) - f(x_1| < \delta \quad \text{and} \quad |g(x_1) - g(x_2)| < \delta,$$

thus the inequality

$$\big|F\big(f(x_1),\, g(x_1)\big) - F\big(f(x_2),\, g(x_2)\big)\big| < \frac{\varepsilon}{2}$$

is true. So, $\operatorname{diam}\big(h(U' \cap I)\big) \leq \frac{\varepsilon}{2} < \varepsilon$. Thus, the function h is strongly cliquish and the proof is finished. □

Theorem 5. *Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function, let $f\colon \mathbb{R} \to \mathbb{R}$ be a strongly quasicontinuous function and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the functions $h_1(x) = F\big(f(x),\, g(x)\big)$ and $h_2(x) = F\big(g(x),\, f(x)\big)$ are strongly quasicontinuous.*

Proof. Fix a point $x \in \mathbb{R}$, a set $A \in T_d$ containing x and a positive real ε. From the continuity of F, there is a real $\delta > 0$ such that for $u, v \in \mathbb{R}$ if

$$\max\bigl(|u - f(x)|, |v - g(x)|\bigr) < \delta, \quad \text{then} \quad \bigl|F(u,v) - F\bigl(f(x), g(x)\bigr)\bigr| < \varepsilon.$$

Since g is continuous there is an open interval

$$I \ni x \quad \text{with} \quad g(I) \subset \bigl(g(x) - \delta, g(x) + \delta\bigr).$$

From the strong quasicontinuity of f, it follows that there is an open interval $I_1 \subset I$ such that

$$I_1 \cap A \neq \emptyset \quad \text{and} \quad f(I_1 \cap A) \subset \bigl(f(x) - \delta, f(x) + \delta\bigr).$$

Consequently, for $y \in I_1 \cap A$ we have

$$|f(y) - f(x)| < \delta,\ |g(y) - g(x)| < \delta \quad \text{and} \quad \bigl|F\bigl(f(y), g(y)\bigr) - F\bigl(f(x), g(x)\bigr)\bigr| < \varepsilon.$$

Thus, h_1 (and similarly h_2) is strongly quasicontinuous. □

The paper pertains to continuous operations on pairs of functions. However, answering to a question of the referee, we show that quasicontinuity of F in Theorem 3, 4 and 5 and Remark 3 is not sufficient.

Indeed, the function

$$F(u,v) = \begin{cases} 0 & \text{if} \quad v = u \text{ and } u \in \mathbb{Q}, \\ 0 & \text{if} \quad v < u, \\ 1, & \text{othervise on} \quad \mathbb{R}^2 \end{cases}$$

is quasicontinuous and the superposition $(x,y) \to F(x,x)$ is not cliquish.

IV. Maximal F-families

Analogously as in the cases of maximal families for the addition and the multiplication of functions, we can define maximal F-families. Limit our consideration to continuous operations $F\colon \mathbb{R}^2 \to \mathbb{R}$. If Φ is a family of functions from $\mathbb{R}$ to $\mathbb{R}$, then put

$$\mathrm{Max}_{\mathrm{lF}}(\Phi) = \bigl\{f \in \Phi;\ F(f,g) \in \Phi \qquad \text{for each} \quad g \in \Phi\bigr\},$$

$$\mathrm{Max}_{\mathrm{rF}}(\Phi) = \bigl\{f \in \Phi;\ F(g,f) \in \Phi \qquad \text{for each} \quad g \in \Phi\bigr\}.$$

Observe that there are families Φ for which $\mathrm{Max}_{\mathrm{l}\Phi} \neq \mathrm{Max}_{\mathrm{r}\Phi}$. For example,

1. If $F_3(u,v) = u^2$ and $\Phi = b\Delta'$ is the family of all bounded derivatives then $\mathrm{Max}_{\mathrm{lF_3}}(\mathrm{b}\Delta')$ is the family of all bounded approximately continuous functions ([1], [9]) and $\mathrm{Max}_{\mathrm{rF_3}}(\mathrm{b}\Delta') = \emptyset$.

2. Let $\mathcal{Q}$ denotes the family of all quasicontinuous functions from $\mathbb{R}$ to $\mathbb{R}$. Observe that for the operation $F_4(u,v) = u + v^2$ and the family $\mathcal{Q}$ we have also $\mathrm{Max}_{\mathrm{lF}_4}(\mathcal{Q}) \neq \mathrm{Max}_{\mathrm{rF}_4}(\mathcal{Q})$. Indeed, let

$$f(x) = 1 \quad \text{for } x \geq 0 \qquad \text{and} \qquad f(x) = -1 \quad \text{for } x < 0.$$

Since f^2 is constant and for the quasicontinuous function

$$g(x) = 2 \quad \text{for } x > 0 \qquad \text{and} \qquad g(x) = 1 \quad \text{for } x \leq 0,$$

the sum $f + g^2 \notin \mathcal{Q}$, the function $f \in \mathrm{Max}_{\mathrm{rF}_4}(\mathcal{Q}) \setminus \mathrm{Max}_{\mathrm{lF}_4}(\mathcal{Q})$.

Let $\mathcal{C}$ denote the family of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. From Remark 3 it follows that for each continuous operation $F\colon \mathbb{R}^2 \to \mathbb{R}$ we have

$$\mathcal{C} \subset \mathrm{Max}_{\mathrm{lF}}(\mathcal{Q}) \cap \mathrm{Max}_{\mathrm{rF}}(\mathcal{Q}).$$

Remark 4. Let $F(u,v) = au + bv$, where $a, b \neq 0$. Then

$$\mathrm{Max}_{\mathrm{lF}}(\mathcal{Q}) = \mathrm{Max}_{\mathrm{rF}}(\mathcal{Q}) = \mathcal{C}.$$

Proof. It is sufficient to prove $\mathrm{Max}_{\mathrm{lF}}(\mathcal{Q}) = \mathrm{Max}_{\mathrm{rF}}(\mathcal{Q}) \subset \mathcal{C}$. Fix a function $f \in \mathcal{Q} \setminus \mathcal{C}$. If there is a point $x \in \mathbb{R}$ at which f has at least one limit number $y_1 \in \mathbb{R}$ different from $f(x)$, then we put

$$g(t) = \frac{-ay_1}{b} \quad \text{for } t = x \qquad \text{and} \qquad g(t) = \frac{-af(t)}{b} \quad \text{for } t \neq x.$$

Then $g \in \mathcal{Q}$, $F\big(f(t), g(t)\big) = af(t) - af(t) = 0$ for $t \neq x$ and $F\big(f(x), g(x)\big) = af(x) - ay_1 \neq 0$. So, the superposition $t \to F\big(f(t), g(t)\big)$, $t \in \mathbb{R}$, is not quasicontinuous.

In the opposite case there are a point x and a sequence of points $x_n \neq x$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} |f(x_n)| = \infty$. Since f is quasicontinuous at the points x_n, $n \geq 1$, there are pairwise disjoint open intervals $I_n = (a_n, b_n)$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$, $\lim_{n\to\infty} |f(a_n)| = \infty$ and $\mathrm{diam}\big(f(I_n)\big) < \frac{1}{n}$ for $n \geq 1$. For each integer $n \geq 1$ we find a closed interval $J_n \subset I_n$. Let $c \in \mathbb{R}$ be a real such that $af(x) - bc \neq 0$. Put

$$g(t) = -c \qquad \text{for} \quad t \in J_n,\ n \geq 1, \qquad g(x) = -c,$$

and

$$g(t) = \frac{-af(t)}{b}, \quad \text{otherwise on } \mathbb{R}.$$

Then $g \in \mathcal{Q}$ and the superposition $t \to F\big(f(t), g(t)\big)$ is not quasicontinuous at the point x. This proves that $\mathrm{Max}_{\mathrm{lF}}(\mathcal{Q}) = \mathcal{C}$. The proof of the equality $\mathrm{Max}_{\mathrm{rF}}(\mathcal{Q}) = \mathcal{C}$ is similar. □

In [5], it is proved that for the operation $F(u,v) = uv$ we have

$$\mathrm{Max}_{\mathrm{lF}}(\mathcal{Q}) = \mathrm{Max}_{\mathrm{rF}}(\mathcal{Q}) \neq \mathcal{C}.$$

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Received November 13, 2008

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