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ON SOME CHARACTERIZATIONS OF BAIRE CLASS ONE FUNCTIONS AND BAIRE CLASS ONE LIKE FUNCTIONS

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ABSTRACT. The basis of our considerations is a characterization of Baire class one functions presented by [Pen-Yee Lee — Wee-Kee Tang — Doingsheng Zhao: *An equivalent definition of functions of the first Baire class*, in: Proc. Amer. Math. Soc., Vol. 129, 2000, pp. 2273–2275]. In the first part of this paper, we will prove an analogous characterization for functions belonging to the class B_1^{**}. In the last part, we will consider various classes of functions connected with the characterization mentioned above, which permits to give a new characterization of Baire class one functions.

Introduction

Despite the fact that the class of Baire one functions is well-known and has been investigated for more than hundred years, new results permitting to discover new properties of these functions have appeared recently. New characterizations of the Baire one functions seem to be especially interesting (e.g., [1], [6], [3]). Our investigations will be based on characterization presented in [3]. In the first part of this paper, we will consider real functions defined on compact metric spaces. The main result of this part is contained in Theorem 5, which gives a characterization of functions belonging to the family B_1^{**}. In [7], the author presented some possible directions of generalizations of Baire class one functions. In the last part of this paper, we will continue these considerations. The main results of this part of the paper are connected with a characterization of Baire class one functions and a diagram, which finishes the considerations.

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Preliminaries

We will use standard definitions and notations (see e.g., [2]). In particular, by $\mathbb{N}$ ($\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$) we denote the set of positive integers (rational numbers, real numbers and positive real numbers, respectively).

Let (X, d_X) be a metric space. We will write it simply X so that no confusion can arise. For an arbitrary set $A \subset X$, $\overline{A}$ will denote the closure of A. We will use the symbol $B(x_0, r)$ to denote an open ball with the centre at x_0 and the radius r. We say that $A \subset X$ is an od-set, if A is an open dense set.

If $f\colon X \to Y$ is an arbitrary function and $A \subset X$, then by $f \upharpoonright A$ we denote the restriction of f to the set A. Let us denote by C_f (D_f) the set of all continuity (discontinuity) points of f.

Let (X, d_X), (Y, d_Y) be metric spaces and $f\colon X \to Y$ be a function.

If $x \in C_f$ and $\varepsilon > 0$, by $\sigma_\varepsilon^f(x)$, we will denote a positive real number such that for each $y \in X$

$$\text{if} \quad d_X(x, y) < \sigma_\varepsilon^f(x), \quad \text{then} \quad d_Y(f(x), f(y)) < \varepsilon. \tag{1}$$

If $A \subset X$, $\varepsilon > 0$ and $f \upharpoonright A$ is uniformly continuous, by $\sigma_\varepsilon^f(A)$, we will denote a positive real number such that for each $x, y \in A$

$$\text{if} \quad d_X(x, y) < \sigma_\varepsilon^f(A), \quad \text{then} \quad d_Y(f(x), f(y)) < \varepsilon. \tag{2}$$

A function $f\colon X \to Y$ is said to be of the first Baire class, if $f^{-1}(G)$ is $\mathcal{F}_\sigma$ for every open set $G \subset Y$ (cf. [3], [7]). The family of all Baire class one functions $f\colon X \to Y$ will be denoted by B_1.

A function $f\colon X \to Y$ is said to be B_1^{**} function if $D_f = \emptyset$ or $f \upharpoonright D_f$ is a continuous function (cf. [5]).

1. Baire class one functions

In the beginning, we will remind the characterisation of B_1 functions proved by Peng-Yee Lee, Wee-Kee Tang and Doingsheng Zhao:

Theorem 1 ([3]). *Let (X, d_X), (Y, d_Y) be complete separable metric spaces and $f\colon X \to Y$. The following statements are equivalent*

(a) $f \in B_1$,

(b) *for any $\varepsilon > 0$ there exists a function $\delta_\varepsilon^f\colon X \to \mathbb{R}_+$ such that for every $x, y \in X$*

$$\textit{if} \quad d_X(x, y) < \min\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(y)\}, \ \textit{then} \quad d_Y(f(x), f(y)) < \varepsilon.$$

Note that for a fixed function f and a fixed number $\varepsilon > 0$, one can find more than one function δ_ε^f satisfying condition (b). An example of such a situation will be described in Proposition 2. First, we will introduce the notation which will be useful throughout this part of the paper:

For a fixed function $f \in B_1$ and a real number $\varepsilon > 0$, let us denote by $B_\varepsilon(f)$ the family of all functions δ_ε^f satisfying condition (b)

$$\text{if } d_X(x,y) < \min\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(y)\}, \text{ then } d_Y(f(x), f(y)) < \varepsilon.$$

Proposition 2. *Let (X, d_X) be a complete separable (nonsingleton) metric space. Let $c \in \mathbb{R}$, $r \in X$, $g\colon X \setminus \{r\} \to \mathbb{R}$ be a continuous function and $f\colon X \to \mathbb{R}$ be a function given by formula*

$$f(x) = \begin{cases} g(x) & \text{if } x \neq r, \\ c & \text{if } x = r. \end{cases}$$

For a fixed number $\varepsilon > 0$ and for arbitrary numbers $p, q \in \mathbb{R}$ such that $p > 0$, $q > 1$, let us put

$$\delta_{\varepsilon,p,q}^f(x) = \begin{cases} p & \text{if } x = r, \\ \min\left\{\frac{d_X(r,x)}{q}, \sigma_\varepsilon^f(x)\right\} & \text{if } 0 < d_X(x,r) < p, \\ \sigma_\varepsilon^f(x) & \text{if } p < d_X(x,r). \end{cases}$$

Then we have $\{\delta_{\varepsilon,p,q}^f : p > 0, q > 1\} \subset B_\varepsilon(f)$.

The notion of the family $B_\varepsilon(f)$ (for a fixed function $f \in B_1$) is a starting point for the characterization of the class B_1^{**} (Theorem 5) similar to the characterization presented in Theorem 1. First, we will extend results connected with the family B_1^{**}.

Lemma 3. *A function $f : X \to \mathbb{R}$ belongs to the class B_1^{**} if and only if either $D_f = \emptyset$ or $f \restriction \overline{D_f}$ is a continuous function.*

Proof. [1] Necessity. Suppose, contrary to our claim, that there exists $x_0 \in \overline{D_f}$ such that the function $f \restriction \overline{D_f}$ is discontinuous at x_0. We have

$$x_0 \in D_f. \tag{3}$$

Since the function $f \restriction \overline{D_f}$ is discontinuous at x_0, there exists a number $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n \in \overline{D_f}$ such that

$$d_X(x_n, x_0) < \frac{1}{n} \quad \text{and} \quad d_Y(f(x_n), f(x_0)) \geq \varepsilon_0. \tag{4}$$

Put

$$A = Y \setminus \overline{B\left(f(x_0), \frac{\varepsilon_0}{2}\right)}.$$

[1]We will adapt the proof from [5] connected with the functions defined on metric spaces.

Then from (4), one can infer that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset \overline{D_f}$, such that

$$x_n \xrightarrow[n\to\infty]{d_X} x_0 \quad \text{and} \quad f(x_n) \in A \text{ for } n \in \mathbb{N}. \tag{5}$$

Since $f\restriction D_f$ is continuous, the set of terms of the sequence $(x_n)_{n\in\mathbb{N}}$ belonging to D_f is finite, so, without loss of generality, we may assume that $(x_n)_{n\in\mathbb{N}} \subset \overline{D_f} \setminus D_f \subset C_f$.

For any open set $U_n \ni x_n$ $(n = 1, 2, \dots)$, we have $U_n \cap D_f \neq \emptyset$. The set A is open, so for each $n \in \mathbb{N}$ there exists an open set $V_n \ni f(x_n)$ such that $V_n \subset A$. Hence, from continuity of f at x_n (for $n = 1, 2, \dots$), we have that there exists a point $z_n \in \{x \in X : d_X(x, x_n) < \frac{1}{n}\} \cap D_f$ such that $f(z_n) \in V_n \subset A$. Then, $(z_n)_{n\in\mathbb{N}} \subset D_f$ and $z_n \xrightarrow[n\to\infty]{d_X} x_0$. According to continuity of $f \restriction D_f$, from (3), we have $f(z_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$, so $f(x_0) \in \overline{A}$, which contradicts the fact that $B\left(f(x_0), \frac{\varepsilon_0}{2}\right) \cap \overline{A} = \emptyset$. This means that $f \restriction \overline{D_f}$ is continuous.

Sufficiency is obvious. □

Lemma 4 ([5]). *If $f \in B_1^{**}$, then D_f is a nowhere dense set.*

Let (X, d_X) be a metric space and $f\colon X \to \mathbb{R}$ be a function for which there exists an od-set $G \subset C_f$. For fixed numbers $\varepsilon > 0$ and $\eta > 0$ we will denote by $\delta_{f,G}^{\varepsilon,\eta}\colon X \to \mathbb{R}$ a function given by the formula

$$\delta_{f,G}^{\varepsilon,\eta}(x) = \begin{cases} \min\left\{\frac{1}{2} d_X(x, X \setminus G), \sigma_\varepsilon^f(x)\right\} & \text{if } x \in G, \\ \eta & \text{if } x \notin G. \end{cases}$$

Until the end of this part, we will assume compactness of the domain of considered real functions.

Theorem 5. *A function $f\colon X \to \mathbb{R}$ belongs to the class B_1^{**} if and only if there exists an od-set $G \subset C_f$ such that for each $\varepsilon > 0$ there exists $\eta > 0$ such that $\delta_{f,G}^{\varepsilon,\eta} \in B_\varepsilon(f)$.*

Proof. Necessity. Let $f\colon X \to \mathbb{R}$ be a B_1^{**} function. Let us establish $\varepsilon > 0$. If f is a continuous function, then we put $G = X$. Now, we assume that $D_f \neq \emptyset$ and we put $G = X \setminus \overline{D_f}$. According to Lemma 4 one can deduce that G is a dense set. Moreover, we have $G \subset C_f$, so, for each $x \in G$, there exists a number $\sigma_\varepsilon^f(x) > 0$ such that

$$\text{if } d_X(x, y) < \sigma_\varepsilon^f(x), \text{ then } |f(x) - f(y)| < \varepsilon. \tag{6}$$

Since f is a B_1^{**} function, Lemma 3 permits to infer that $f \restriction (X \setminus G) = f \restriction \overline{D_f}$ is an uniformly continuous function. So there exists $\sigma_\varepsilon^f(X \setminus G) > 0$ such that for each $x, y \in X \setminus G$

$$\text{if } d_X(x, y) < \sigma_\varepsilon^f(X \setminus G), \text{ then } |f(x) - f(y)| < \varepsilon. \tag{7}$$

Now, it is sufficient to show that for $\eta = \sigma^f_\varepsilon(X \setminus G)$ the function $\delta^{\varepsilon,\eta}_{f,G}$ belongs to the family $B_\varepsilon(f)$.

Let $x, y \in X$ be such that $d_X(x, y) < \min\{\delta^{\varepsilon,\eta}_{f,G}(x), \delta^{\varepsilon,\eta}_{f,G}(y)\}$. One can consider the following cases:

(1) $x, y \in G$. Then $d_X(x, y) < \min\{\sigma^f_\varepsilon(x), \sigma^f_\varepsilon(y)\}$ and from (6) we infer that $|f(x) - f(y)| < \varepsilon$.

(2) $x, y \in X \setminus G$. Then, $d_X(x, y) < \eta$, and from (7), we infer that $|f(x) - f(y)| < \varepsilon$.

(3) $x \in X \setminus G$ and $y \in G$. Then
$$d_X(x, y) > 0 \quad \text{and} \quad d_X(x, y) < \delta^{\varepsilon,\eta}_{f,G}(y) \le \frac{1}{2} d_X(y, X \setminus G) \le \frac{1}{2} d_X(x, y),$$
which is impossible.

Sufficiency. Let $f\colon X \to \mathbb{R}$ be a function and $G \subset C_f$ be an od-set such that for each $\varepsilon > 0$ we have $\delta^{\varepsilon,\eta}_{f,G} \in B_\varepsilon(f)$ for some $\eta > 0$. We will prove that
$$f \restriction D_f$$
is continuous. Since $G \subset C_f$, we have $D_f \subset X \setminus G$, so it is sufficient to show that
$$f \restriction (X \setminus G)$$
is continuous. Fix $\varepsilon > 0$. Let $x, y \in X \setminus G$ be points such that $d_X(x, y) < \eta = \min\{\delta^{\varepsilon,\eta}_{f,G}(x), \delta^{\varepsilon,\eta}_{f,G}(y)\}$. Since $\delta^{\varepsilon,\eta}_{f,G} \in B_\varepsilon(f)$, it follows that $|f(x) - f(y)| < \varepsilon$. According to the arbitrariness of the choice of $x, y \in X \setminus G$ we may infer that
$$f \restriction (X \setminus G)$$
is an uniformly continuous function, which finishes the proof. □

Note that the sum of B_1^{**} functions does not need to belong to B_1^{**} class. For the proof of this statement, it is sufficient to consider the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ given by the formulas
$$f(x) = \begin{cases} x & \text{if } x \in \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \\ 0 & \text{if } x \notin \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases}$$
Theorem 5 allows us to give a sufficient condition for the situation when the sum of B_1^{**} functions is also a B_1^{**} function.

Theorem 6. *If for functions $f, g\colon X \to \mathbb{R}$ there exists an od-set $G \subset C_f \cap C_g$ such that $\delta^{\varepsilon,\eta_f}_{f,G} \in B_\varepsilon(f)$ and $\delta^{\varepsilon,\eta_g}_{g,G} \in B_\varepsilon(g)$ for some $\eta_f, \eta_g > 0$, then f and g belong to the common additive semi-group of B_1^{**} functions.*

P r o o f. Let G be an od-set fulfilling the assumptions of our theorem. We denote by $\mathcal{H}$ the family of functions $h\colon X \to \mathbb{R}$ such that $h \restriction G$ and $h \restriction (X \setminus G)$ are continuous. It is obvious that $\mathcal{H}$ is an additive semi-group. Let us observe that $h\in\mathcal{H}$ if and only if

$$\Big(G \subset C_h \text{ and there exists } \eta_h > 0 \text{ such that } \delta_{h,G}^{\varepsilon,\eta_h} \in B_\varepsilon(h)\Big). \tag{8}$$

Indeed, for $h \in \mathcal{H}$ it is obvious that $G \subset C_h$. The function $h \upharpoonright (X \setminus G)$ is continuous and $(X \setminus G)$ is a closed set, so $h \upharpoonright (X \setminus G)$ is a uniformly continuous function. This means that there exists a number $\sigma_\varepsilon^h(X \setminus G) > 0$ such that for each $x, y \in X \setminus G$

$$\text{if} \quad d_X(x,y) < \sigma_\varepsilon^h(X \setminus G), \quad \text{then} \quad |h(x) - h(y)| < \varepsilon. \tag{9}$$

Put

$$\eta_h = \sigma_\varepsilon^h(X \setminus G)$$

and let $x, y \in X$ be such that $d_X(x,y) < \min\{\delta_{h,G}^{\varepsilon,\eta_h}(x), \delta_{h,G}^{\varepsilon,\eta_h}(y)\}$. We will show that

$$\delta_{h,G}^{\varepsilon,\eta_h} \in B_\varepsilon(h). \tag{10}$$

If $x \in G$ or $y \in G$, then, from the definition of the number $\sigma_\varepsilon^h(x)$ or $\sigma_\varepsilon^h(y)$, we obtain that $|h(x)-h(y)| < \varepsilon$. If $x, y \in X \setminus G$, then the inequality $|h(x)-h(y)| < \varepsilon$ follows from the definition of η_h and from (9). Hence, we have (10).

Conversely, if $G \subset C_h$ and there exists $\eta_h > 0$ such that $\delta_{h,G}^{\varepsilon,\eta_h} \in B_\varepsilon(h)$, then obviously $h \upharpoonright G$ is continuous and $h \upharpoonright (X \setminus G)$ is uniformly continuous (where $\sigma_\varepsilon^h(X \setminus G) = \eta_h$). This finishes the proof of (8).

By applying Theorem 5 we have that $\mathcal{H}$ is an additive semi-group of B_1^{**} functions. Moreover, functions f and g belong to $\mathcal{H}$. □

2. Baire class one like functions

Let $\mathcal{S}$ and $\mathcal{T}$ be families of positive real valued functions defined on $\mathbb{R}$. By $\mathfrak{B}_{\mathcal{S},\mathcal{T}}$ we will denote the family of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for each function $\varepsilon \in \mathcal{T}$, there exists a function $\delta_\varepsilon^f \in \mathcal{S}$ such that for each $x, y \in \mathbb{R}$ we have

$$\text{if} \quad |x - y| < \min\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(y)\}, \quad \text{then} \quad |f(x) - f(y)| < \min\{\varepsilon(f(x)), \varepsilon(f(y))\}.$$

Some similar considerations and notation can be found in [4] and [7].

We will now emphasize (in the form of lemma) a very useful statement (the simple proof will be omitted).

Lemma 7. *Let $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, $\mathcal{W}$ be arbitrary families of positive real valued functions defined on $\mathbb{R}$. Then*

(1) *if $\mathcal{S} \subset \mathcal{U}$, then $\mathfrak{B}_{\mathcal{S},\mathcal{T}} \subset \mathfrak{B}_{\mathcal{U},\mathcal{T}}$,*

(2) *if $\mathcal{T} \subset \mathcal{W}$, then $\mathfrak{B}_{\mathcal{S},\mathcal{T}} \supset \mathfrak{B}_{\mathcal{S},\mathcal{W}}$.*

In order to construct a basic interpreting dependence between some classes of functions, we will use the following symbols:

$\mathbb{R}^{\mathbb{R}}$ **:** will denote the family of all real functions defined on $\mathbb{R}$,

B_1 **:** will denote the family of all real Baire class one functions defined on $\mathbb{R}$,

C **:** will denote the family of all continuous real functions defined on $\mathbb{R}$,

C_u **:** will denote the family of all uniformly continuous real functions defined on $\mathbb{R}$,

bC_u **:** will denote the family of all bounded uniformly continuous real functions defined on $\mathbb{R}$,

lsc **:** will denote the family of all lower semicontinuous real functions defined on $\mathbb{R}$,

usc **:** will denote the family of all upper semicontinuous real functions defined on $\mathbb{R}$,

$Const$ **:** will denote the family of all constant real functions defined on $\mathbb{R}$.

$\mathcal{X}^+$**:** If $\mathcal{X}$ is a family of functions, then we will denote by $\mathcal{X}^+$ the subfamily of the family $\mathcal{X}$ consisted of positive real valued functions.

Lemma 7 permits to establish the following diagram:

$$\begin{array}{ccccccc}
\mathfrak{B}_{Const^+,\,\mathbb{R}^{\mathbb{R}+}} & \subset & \mathfrak{B}_{Const^+,\,lsc^+} & \subset & \mathfrak{B}_{Const^+,\,C^+} & \subset & \mathfrak{B}_{Const^+,\,Const^+} \\
\cap & & \cap & & \cap & & \cap \\
\mathfrak{B}_{C^+,\,\mathbb{R}^{\mathbb{R}+}} & \subset & \mathfrak{B}_{C^+,\,lsc^+} & \subset & \mathfrak{B}_{C^+,\,C^+} & \subset & \mathfrak{B}_{C^+,\,Const^+} \\
\cap & & \cap & & \cap & & \cap \\
\mathfrak{B}_{lsc^+,\,\mathbb{R}^{\mathbb{R}+}} & \subset & \mathfrak{B}_{lsc^+,\,lsc^+} & \subset & \mathfrak{B}_{lsc^+,\,C^+} & \subset & \mathfrak{B}_{lsc^+,\,Const^+} \\
\cap & & \cap & & \cap & & \cap \\
\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\,\mathbb{R}^{\mathbb{R}+}} & \subset & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\,lsc^+} & \subset & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\,C^+} & \subset & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\,Const^+}\,.
\end{array}$$

The above considerations lead us to questions connected with the relationship between families of continuous functions, B_1 functions and classes of function exposed in the above diagram. We will give answers to some of these questions. Our main result will be a diagram placed at the end of this paper. In order to construct it, we prove the following theorems.

Remark 8. In order to simplify the notation, we will identify the functions $\varepsilon \in Const^+$ and $\delta^f_\varepsilon \in Const^+$ with the positive numbers $\bar{\varepsilon}$ and $\bar{\delta}^f_{\bar{\varepsilon}}$.

Directly from the denotation fixed in Remark 8 and definitions of the considered classes of functions, we obtain the new characterization of uniformly continuous functions and B_1 functions.

Proposition 9. $\mathfrak{B}_{Const^+,\,Const^+} = C_u \subsetneq C$.

Proposition 10. $\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\,Const^+} = B_1$.

The next theorem permits us to establish some inclusion, which will be useful in further considerations (e.g., the proof of Theorem 18).

Theorem 11. $\mathfrak{B}_{lsc^+, Const^+} \subset C$.

P r o o f. Suppose, contrary to our claim, that there exists a discontinuous function $f \in \mathfrak{B}_{lsc^+, Const^+}$. Let $x_0 \in \mathbb{R}$ be a discontinuity point of f. Then there exists a number $\bar{\varepsilon}_0 > 0$ such that for each $\eta > 0$ one can find a point $x_\eta \in \mathbb{R}$ such that

$$|x_\eta - x_0| < \eta \quad \text{and} \quad |f(x_\eta) - f(x_0)| \geq \bar{\varepsilon}_0. \tag{11}$$

Since $f \in \mathfrak{B}_{lsc^+, Const^+}$, there exists a function $\delta^f_{\bar{\varepsilon}_0} \in lsc^+$ such that for each $x, y \in \mathbb{R}$

$$\text{if} \quad |x - y| < \min\{\delta^f_{\bar{\varepsilon}_0}(x), \delta^f_{\bar{\varepsilon}_0}(y)\}, \quad \text{then} \quad |f(x) - f(y)| < \bar{\varepsilon}_0. \tag{12}$$

On account of lower semicontinuity of $\delta^f_{\bar{\varepsilon}_0} : \mathbb{R} \to \mathbb{R}^+$ at x_0, we have that there exists $\alpha_0 > 0$ such that for each $x \in \mathbb{R}$

$$\text{if} \quad |x - x_0| < \alpha_0, \quad \text{then} \quad \frac{\delta^f_{\bar{\varepsilon}_0}(x_0)}{2} < \delta^f_{\bar{\varepsilon}_0}(x). \tag{13}$$

Putting $\eta = \min\{\alpha_0, \frac{\delta^f_{\bar{\varepsilon}_0}(x_0)}{2}\}$, from (11) we obtain that there exists $x_\eta \in \mathbb{R}$ such that $|x_\eta - x_0| < \eta \leq \alpha_0$. According to (13), we have $\frac{\delta^f_{\bar{\varepsilon}_0}(x_0)}{2} < \delta^f_{\bar{\varepsilon}_0}(x_{\eta_0})$. Hence,

$$|x_\eta - x_0| < \eta \leq \frac{\delta^f_{\bar{\varepsilon}_0}(x_0)}{2} < \min\{\delta^f_{\bar{\varepsilon}_0}(x_0), \delta^f_{\bar{\varepsilon}_0}(x_\eta)\}.$$

Thus, from (12), we obtain $|f(x_0) - f(x_\eta)| < \bar{\varepsilon}_0$, contrary to (11). $\square$

On account of Theorem 11, the definition of B_1 function and Lemma 7 we easily get the following corollaries:

Corollary 12. $\mathfrak{B}_{lsc^+, Const^+} \subsetneq \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+}, Const^+}$.

Corollary 13. $\mathfrak{B}_{C^+, Const^+} \subset C$.

Proposition 14. $\mathfrak{B}_{Const^+, Const^+} \subsetneq \mathfrak{B}_{C^+, Const^+}$.

P r o o f. According to Proposition 9 and Lemma 7 (1), we have

$$C_u = \mathfrak{B}_{Const^+, Const^+} \subset \mathfrak{B}_{C^+, Const^+}.$$

We will show that $\mathfrak{B}_{Const^+, Const^+} \not\supset \mathfrak{B}_{C^+, Const^+}$. Let $f(x) = e^x$, $x \in \mathbb{R}$. Certainly, f is not uniformly continuous. It remains to prove that $f \in \mathfrak{B}_{C^+, Const^+}$. Let $\bar{\varepsilon} > 0$. Put $\delta^f_{\bar{\varepsilon}}(x) = \ln\left(\frac{\bar{\varepsilon}}{2} e^{-x} + 1\right)$, $x \in \mathbb{R}$, obviously $\delta^f_{\bar{\varepsilon}} \in C^+$. Let $x, y \in \mathbb{R}$ be such that

$$|x - y| < \min\left\{\delta^f_{\bar{\varepsilon}}(x), \delta^f_{\bar{\varepsilon}}(y)\right\}. \tag{14}$$

Without loss of generality, we can assume that $x > y$. From (14), we have $x < y+\delta_{\bar{\varepsilon}}^f(y)$. Hence, according to the monotonicity of f, we have: $|f(x)-f(y)| < e^{y+\delta_{\bar{\varepsilon}}^f(y)} - e^y = e^y\left(\frac{\bar{\varepsilon}}{2}e^{-y}+1-1\right) < \bar{\varepsilon}$. □

Now, let us note the following simple statement, which will be very useful.

Proposition 15. *Let $\mathcal{S}$ and $\mathcal{T}$ be arbitrary families of positive real valued functions defined on $\mathbb{R}$. Then Const $\subset \mathfrak{B}_{\mathcal{S},\mathcal{T}}$.*

Proposition 16. *Const* $\subsetneq \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\mathbb{R}^{\mathbb{R}+}}$.

Proof. From Proposition 15 we have $Const \subset \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\mathbb{R}^{\mathbb{R}+}}$.

We will now show, that $Const \not\supset \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\mathbb{R}^{\mathbb{R}+}}$. Let a, b, r be arbitrary real numbers such that $a \neq b$. Let us consider the function given by formula

$$f(x) = \begin{cases} a & \text{if } x = r, \\ b & \text{if } x \neq r. \end{cases}$$

Certainly, $f \notin Const$. Let $\varepsilon\colon \mathbb{R} \to \mathbb{R}_+$. In order to finish the proof that $f \in \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\mathbb{R}^{\mathbb{R}+}}$ suffices to put

$$\delta_\varepsilon^f(x) = \begin{cases} \frac{1}{2} & \text{if } x = r, \\ \frac{|r-x|}{2} & \text{if } x \in \left(r-\frac{1}{2}, r+\frac{1}{2}\right) \setminus \{r\}, \\ 1 & \text{if } x \in \mathbb{R} \setminus \left(r-\frac{1}{2}, r+\frac{1}{2}\right). \end{cases}$$

□

As it was mentioned in the above theorems and propositions, some families of Baire class one like functions are equal to some well-known classes, e.g., uniformly continuous functions. It is interesting that the family $\mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}}$ is equal to the class of constant functions. We will show it below, but first, let us note a useful lemma.

Lemma 17. *If $f \in C \setminus Const$, then there exists $x_0 \in \mathbb{R}$ such that the function f is not constant in any interval of the form $[x_0, x_0+\eta]$ for any $\eta > 0$.*

Theorem 18. $Const = \mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}}$.

Proof. On account of Proposition 15 we have $Const \subset \mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}}$.

Now, we will show that $Const \supset \mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}}$. According to Lemma 7 (2) and Theorem 11, we obtain $\mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}} \subset \mathfrak{B}_{lsc^+,Const^+} \subset C$.

Let us suppose, contrary to our claim, that $Const \not\supset \mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}}$. Then, there exists $f \in \mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}} \setminus Const$. From Lemma 17, we have that there exists a point $x_0 \in \mathbb{R}$ such that f is not constant in any interval of the form $[x_0, x_0+\eta]$, where $\eta > 0$. Since $f \in \mathfrak{B}_{lsc^+,\mathbb{R}^{\mathbb{R}+}}$, for a function

$$\varepsilon(y) = \begin{cases} \frac{|f(x_0)-y|}{2} & \text{if } y \neq f(x_0), \\ 1 & \text{if } y = f(x_0), \end{cases} \tag{15}$$

there exists a function $\delta_\varepsilon^f \in lsc^+$ such that for every $x_1, x_2 \in \mathbb{R}$ if

$$|x_1 - x_2| < \min\{\delta_\varepsilon^f(x_1), \delta_\varepsilon^f(x_2)\},$$

then

$$|f(x_1) - f(x_2)| < \min\{\varepsilon(f(x_1)), \varepsilon(f(x_2))\}. \tag{16}$$

According to the lower semicontinuity of $\delta_\varepsilon^f\colon \mathbb{R} \to \mathbb{R}_+$ at x_0, we obtain that there exists $\alpha_0 > 0$ such that for every $x \in \mathbb{R}$

$$\text{if} \quad |x - x_0| < \alpha_0, \quad \text{then} \quad \frac{\delta_\varepsilon^f(x_0)}{2} < \delta_\varepsilon^f(x). \tag{17}$$

The function f is not constant in the interval $\left[x_0, x_0 + \min\left\{\frac{\delta_\varepsilon^f(x_0)}{2}, \frac{\alpha_0}{2}\right\}\right]$, so there exists

$$x_0^* \in \left[x_0, x_0 + \min\left\{\frac{\delta_\varepsilon^f(x_0)}{2}, \frac{\alpha_0}{2}\right\}\right] \text{ such that } f(x_0) \neq f(x_0^*). \tag{18}$$

It is easy to see that

$$|x_0 - x_0^*| \leq \frac{\alpha_0}{2} < \alpha_0,$$

so from (17) we obtain that

$$\frac{\delta_\varepsilon^f(x_0)}{2} < \delta_\varepsilon^f(x_0^*).$$

Hence,

$$|x_0 - x_0^*| \leq \frac{\delta_\varepsilon^f(x_0)}{2} < \min\{\delta_\varepsilon^f(x_0), \delta_\varepsilon^f(x_0^*)\}.$$

Using (16), we have:

$$\begin{aligned} |f(x_0) - f(x_0^*)| &< \min\{\varepsilon(f(x_0)), \varepsilon(f(x_0^*))\} \\ &= \min\left\{1, \frac{|f(x_0) - f(x_0^*)|}{2}\right\} \leq \frac{|f(x_0) - f(x_0^*)|}{2}, \end{aligned}$$

so

$$|f(x_0) - f(x_0^*)| = 0,$$

which contradicts (18). □

Remark 19. Let $y_0 \in \mathbb{R}$. Note that a function $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ given by formula

$$\varepsilon(y) = \begin{cases} \frac{|y_0 - y|}{2} & \text{if } y \neq y_0, \\ 1 & \text{if } y = y_0, \end{cases}$$

is upper semicontinuous. Hence, directly from Theorem 18, we obtain the following theorem

Theorem 20. *Const* $= \mathfrak{B}_{lsc^+, usc^+}$.

From Theorem 18, Theorem 20, Lemma 7 and Proposition 15 we easily get the following corollaries:

Corollary 21. *Const* $= \mathfrak{B}_{Const^+,\,usc^+} = \mathfrak{B}_{C^+,\,usc^+}$.

Corollary 22. $\mathfrak{B}_{lsc^+,\,\mathbb{R}^{\mathbb{R}+}} \subsetneq bC_u$.

Corollary 23. $\mathfrak{B}_{lsc^+,\,\mathbb{R}^{\mathbb{R}+}} \subsetneq \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\,\mathbb{R}^{\mathbb{R}+}}$.

Corollary 24. *Const* $= \mathfrak{B}_{Const^+,\,\mathbb{R}^{\mathbb{R}+}} = \mathfrak{B}_{C^+,\,\mathbb{R}^{\mathbb{R}+}}$.

Corollary 25. $\mathfrak{B}_{Const^+,\,\mathbb{R}^{\mathbb{R}+}} \subsetneq bC_u$ *and* $\mathfrak{B}_{C^+,\,\mathbb{R}^{\mathbb{R}+}} \subsetneq bC_u$.

Let us now examine the class bC_u more precisely.

Theorem 26. $bC_u \subset \mathfrak{B}_{Const^+,\,lsc^+}$.

P r o o f. Let $f \in bC_u$. There exist numbers $m, M \in \mathbb{R}$ such that $f(\mathbb{R}) \subset [m, M]$. Establish $\varepsilon \in lsc^+$. Since $\varepsilon \restriction [m, M]$ is a lower semicontinuous function and $[m, M]$ is compact, there exists $\eta_0 > 0$ such that $\eta_0 \le \varepsilon(t)$ for each $t \in [m, M]$. The function f is uniformly continuous, so there exists $\bar{\delta}^f_\varepsilon > 0$ such that for each $x, y \in \mathbb{R}$ if $|x-y| < \bar{\delta}$, then $|f(x)-f(y)| < \eta_0$. Since we have $f(x), f(y) \in [m, M]$ for each $x, y \in \mathbb{R}$, it follows that $\eta_0 \le \varepsilon(f(x))$ and $\eta_0 \le \varepsilon(f(y))$. Consequently, for each $x, y \in \mathbb{R}$ if $|x-y| < \bar{\delta}^f_\varepsilon$, then $|f(x)-f(y)| < \eta_0 \le \min\{\varepsilon(f(x)), \varepsilon(f(y))\}$, which gives $f \in \mathfrak{B}_{Const^+,\,lsc^+}$. □

Corollary 27. $bC_u \subset \mathfrak{B}_{C^+,\,lsc^+}$ *and* $bC_u \subset \mathfrak{B}_{lsc^+,\,lsc^+}$.

Now, we are able to complete the considerations connected with the family $\mathfrak{B}_{Const^+,\,C^+}$.

Theorem 28. $\mathfrak{B}_{Const^+,C^+} \subsetneq \mathfrak{B}_{Const^+,\,Const^+}$.

P r o o f. According to Lemma 7 (2), we have $\mathfrak{B}_{Const^+,\,C^+} \subset \mathfrak{B}_{Const^+,\,Const^+}$.

We will show that

$$\mathfrak{B}_{Const^+,\,C^+} \not\supset \mathfrak{B}_{Const^+,\,Const^+}.$$

Consider the function $f(x) = x$. Obviously, $f \in C_u$, so from Proposition 9, we obtain $f \in \mathfrak{B}_{Const^+,\,Const^+}$.

Now we prove that $f \notin \mathfrak{B}_{Const^+,\,C^+}$. Suppose, contrary to our claim, that $f \in \mathfrak{B}_{Const^+,\,C^+}$. For the function $\varepsilon(x) = e^{-x}$ there exists $\bar{\delta}^f_\varepsilon > 0$ such that for each $x, y \in \mathbb{R}$

$$\text{if} \quad |x-y| < \bar{\delta}^f_\varepsilon, \quad \text{then} \quad |f(x)-f(y)| < \min\{\varepsilon(f(x)), \varepsilon(f(y))\}.$$

This means that

$$\text{if} \quad |x-y| < \bar{\delta}^f_\varepsilon, \quad \text{then} \quad |x-y| < \min\{e^{-x}, e^{-y}\}. \tag{19}$$

Let $y > 0$ be such that $e^{-y} < \bar{\delta}^f_\varepsilon$ and $x \in \mathbb{R}$ be such that $y-\bar{\delta}^f_\varepsilon < x < y-e^{-y}$. We have $x-y < -e^{-y}$ and $y-x < \bar{\delta}^f_\varepsilon$. Hence, we have $\min\{e^{-x}, e^{-y}\} = e^{-y} < |x-y|$ and $|x-y| = y-x < \bar{\delta}^f_\varepsilon$, which contradicts (19) and finishes the proof. □

Theorem 29. $\mathfrak{B}_{Const^+,C^+} \subsetneq \mathfrak{B}_{C^+,C^+}$.

P r o o f. From Lemma 7 (1), we have $\mathfrak{B}_{Const^+,C^+} \subset \mathfrak{B}_{C^+,C^+}$.

Consider the function $f(x) = x$. In the proof of Theorem 28, we have shown that $f \notin \mathfrak{B}_{Const^+,C^+}$. Now, we prove that $f \in \mathfrak{B}_{C^+,C^+}$. Let $\varepsilon \in C^+$. Put $\delta_\varepsilon^f = \frac{\varepsilon}{2} \in C^+$. Then for each $x, y \in \mathbb{R}$ such that $|x-y| < \min\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(y)\}$ we have

$$|f(x) - f(y)| = |x-y| < \min\{\delta_\varepsilon^f(x), \delta_\varepsilon^f(y)\} < \min\{\varepsilon(x), \varepsilon(y)\},$$

which gives that $f \in \mathfrak{B}_{C^+,C^+}$. □

According to [7], we have $\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},lsc^+} = B_1$. This leads us to a question whether this equality remains true if we replace the lsc^+ class some wider family of functions. The answer is "yes"; in order to prove this statement, let us define the class of weakly lower semicontinuous functions.

Definition 30. A function $f\colon \mathbb{R} \to \mathbb{R}^+$ is said to be weakly lower semicontinuous if there exists a decreasing sequence of positive real numbers $(\alpha_n)_{n\in\mathbb{N}}$ such that $\alpha_n \underset{n\to\infty}{\longrightarrow} 0$ and for each $x_0 \in \mathbb{R}$ and each $n \in \mathbb{N}$ such that $\alpha_n < f(x_0)$, there exists an open set U such that $x_0 \in U$ and

$$f(x) > \alpha_n \text{ for } x \in U.$$

The family of all weakly lower semicontinuous functions will be denoted by *wlsc*.

Now, we will prove some basic properties of weakly lower semicontinuous functions.

Proposition 31. $lsc^+ \subsetneq wlsc$.

P r o o f. Let $f \in lsc^+$. We obtain that for each $x_0 \in \mathbb{R}$ and for each $a < f(x_0)$ there exists $\eta_a > 0$ such that

$$\text{if} \quad |x - x_0| < \eta_a, \quad \text{then} \quad a < f(x). \tag{20}$$

Let us denote by $(\alpha_n)_{n\in\mathbb{N}}$ an arbitrary decreasing sequence of positive real numbers convergent to zero. For $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\alpha_n < f(x_0)$ put $U = (x_0 - \eta_{\alpha_n}, x_0 + \eta_{\alpha_n})$, then from (20) we have

$$f(x) > \alpha_n, \text{ if } x \in U,$$

which means that $f \in wlsc$. According to the arbitrariness of the choice of f we infer that $lsc^+ \subset wlsc$.

We will now show that $lsc^+ \not\supset wlsc$. Let us consider the function given by formula

$$g(x) = \begin{cases} \frac{1}{n} & \text{if } x \in (-n, -(n-1)], n \in \mathbb{N}, \\ \frac{1}{3}\sin\frac{1}{x} + 1 & \text{if } x \in (0, +\infty). \end{cases}$$

Put $\alpha_n = \frac{1}{2n}$, $n \in \mathbb{N}$. It is clear that $(\alpha_n)_{n\in\mathbb{N}}$ is a decreasing sequence of positive real numbers convergent to zero.

Let $x_0 \in \mathbb{R}$ be an arbitrary point. One can consider the following cases:

(1) If $x_0 \in (0, +\infty)$, then $g(x_0) = \frac{1}{3}\sin\frac{1}{x_0} + 1 \geq \frac{1}{3}\cdot(-1) + 1 = \frac{2}{3} > \frac{1}{2} = \alpha_1$. Put $U = (0, +\infty)$. Since $(\alpha_n)_{n\in\mathbb{N}}$ is decreasing and for each $x \in (0, +\infty)$ we have $g(x) = \frac{1}{3}\sin\frac{1}{x} + 1 \geq \frac{1}{3}\cdot(-1) + 1 = \frac{2}{3} > \frac{1}{2} = \alpha_1$, then

$$g(x) > \alpha_n, \quad \text{for } x \in U,\ n \in \mathbb{N}.$$

(2) If $x_0 = 0$, then $g(x_0) = 1 > \frac{1}{2} = \alpha_1$. Put $U = (-1, +\infty)$. Since $g(x) = 1$ for $x \in (-1, 0]$, from point (1) we have

$$g(x) > \alpha_n, \quad \text{for } x \in U,\ n \in \mathbb{N}.$$

(3) If $x_0 \in (-\infty, 0)$, then there exists $n_0 \in \mathbb{N}$ such that $x_0 \in (-n_0, -(n_0 - 1)]$, so, we have $g(x_0) = \frac{1}{n_0} > \frac{1}{2n_0} = \alpha_{n_0}$. Put $U = (-n_0, 0)$. Since g is nondecreasing in $(-\infty, 0)$, we obtain

$$g(x) > \alpha_n, \quad \text{for } x \in U,\ n \geq n_0.$$

This proves that $g \in wlsc$.

Now, we will show that $g \notin lsc^+$. Let us consider the sequence given by formula $x_k = \frac{1}{\frac{3}{2}\pi + 2k\pi}$.

We have

$$\lim_{k\to\infty} x_k = 0$$

and

$$g(x_k) = \frac{1}{3}\sin\frac{1}{x_k} + 1 = \frac{1}{3}\sin\left(\frac{3}{2}\pi + 2k\pi\right) + 1 = \frac{1}{3}\cdot(-1) + 1 = \frac{2}{3}.$$

Hence,

$$\liminf_{x\to 0} g(x) \leq \lim_{k\to\infty} g(x_k) = \frac{2}{3} < g(0) = 1,$$

which means, that g is not lower semicontinuous at 0. □

Corollary 32. *$Const^+ \subset wlsc$.*

Directly from the definition of weakly lower semicontinuous function, we get

Lemma 33. *Let $g \in wlsc$ and $(\alpha_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers convergent to zero, chosen according to the definition of weakly lower semicontinuity. Then, for each $n \in \mathbb{N}$, the set $g^{-1}((\alpha_n, +\infty))$ is open.*

Lemma 34 ([7]). *If $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$, where E_n, $n \in \mathbb{N}$ are $\mathcal{F}_\sigma$, then there exist disjoint $\mathcal{F}_\sigma$ sets F_n, $n \in \mathbb{N}$ such that $F_n \subset E_n$ and $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$.*

The above considerations and statements permit to obtain a new characterization of the first Baire class.

Theorem 35. $\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, wlsc^+} = B_1$.

P r o o f. [2] From Corollary 32, Lemma 7 (2), and Proposition 10 we have

$$\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, wlsc^+} \subset \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, Const^+} = B_1.$$

We will show now that $\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, wlsc^+} \supset \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, Const^+}$. Let $f \in \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, Const^+} = B_1$ and $\varepsilon \in wlsc$. Then there exists a decreasing sequence of positive real numbers $(\alpha_n)_{n\in\mathbb{N}}$ chosen according to the definition of weak lower semicontinuity. Put $U_n = \varepsilon^{-1}((\alpha_n, +\infty))$, $n \in \mathbb{N}$. From Lemma 33, we obtain that U_n $(n \in \mathbb{N})$ are open sets. Put $E_n = f^{-1}(U_n)$, $n \in \mathbb{N}$. Certainly, E_n $(n \in \mathbb{N})$ are $\mathcal{F}_\sigma$ sets.

Note that from the condition $\lim\limits_{n\to\infty} \alpha_n = 0$, we obtain

$$\mathbb{R} = \varepsilon^{-1}(\mathbb{R}_+) = \varepsilon^{-1}\left(\bigcup_{n=1}^{\infty}(\alpha_n, +\infty)\right) = \bigcup_{n=1}^{\infty} U_n.$$

Hence,

$$\mathbb{R} = f^{-1}\left(\bigcup_{n=1}^{\infty} U_n\right) = \bigcup_{n=1}^{\infty} E_n.$$

From Lemma 34 we may infer that there exist pairwise disjoint $\mathcal{F}_\sigma$ sets F_n $(n \in \mathbb{N})$ such that $F_n \subset E_n$ and $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$. Then there exist closed sets $\tilde{F}_{n,j}$, $j \in \mathbb{N}$ such that $F_n = \bigcup_{j=1}^{\infty} \tilde{F}_{n,j}$. Put $F_{n,i} = \bigcup_{j=1}^{i} \tilde{F}_{n,j}$ for $i \in \mathbb{N}$. For each $n \in \mathbb{N}$, we have $F_{n,i} \subset F_{n,i+1}$, $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} F_{n,i} = \bigcup_{i=1}^{\infty}\bigcup_{j=1}^{i} \tilde{F}_{n,j} = \bigcup_{j=1}^{\infty} \tilde{F}_{n,j} = F_n$.

Let $x \in \mathbb{R} = \bigcup_{n=1}^{\infty} F_n$. There exists a unique positive integer n_x such that $x \in F_{n_x}$. Put $i_x = \min\{i \in \mathbb{N} : x \in F_{n_x,i}\}$. Consider the family of closed sets

$$\mathfrak{F} = \{F_{m,j} : m \neq n_x \wedge m + j \leq n_x + i_x\}.$$

Since $x \in F_{n_x}$ and $F_{n_x} \cap F_m = \emptyset$ for $m \neq n_x$, it follows that $x \notin \bigcup \mathfrak{F}$, so there exists a number $\bar{\delta}_x > 0$ such that

$$\bigcup \mathfrak{F} \cap (x - \bar{\delta}_x, x + \bar{\delta}_x) = \emptyset,$$

hence,

$$F_{m,j} \cap (x - \bar{\delta}_x, x + \bar{\delta}_x) = \emptyset \quad \text{for} \quad m \neq n_x,\ m + j \leq n_x + i_x. \tag{21}$$

Let $x \in F_{n_x},\ y \in F_{n_y}$, where $n_x \neq n_y$.
If $n_x + i_x \leq n_y + i_y$, then from (21) we have $F_{n_x,i_x} \cap (y - \bar{\delta}_y, y + \bar{\delta}_y) = \emptyset$. According to $x \in F_{n_x,i_x}$, we conclude that $|x - y| \geq \bar{\delta}_y \geq \min\{\bar{\delta}_y, \bar{\delta}_x\}$. Similarly, if $n_x + i_x \geq n_y + i_y$, then $|x - y| \geq \bar{\delta}_x \geq \min\{\bar{\delta}_x, \bar{\delta}_y\}$.

Let us define a function $\delta_0 \in \mathbb{R}^{\mathbb{R}+}$ by formula $\delta_0(x) = \bar{\delta}_x$, $x \in \mathbb{R}$. For each $x, y \in \mathbb{R}$ we have

$$\text{if } x \in F_{n_x}, y \in F_{n_y}, \text{ and } n_x \neq n_y, \text{ then } |x - y| \geq \min\{\delta_0(x), \delta_0(y)\}. \tag{22}$$

[2]The proof is adapted from [7].

Since $f \in \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, Const^+}$, for each $n \in \mathbb{N}$, there exists a function $\delta_n \in \mathbb{R}^{\mathbb{R}+}$ such that for each $x, y \in \mathbb{R}$

$$\text{if} \quad |x-y| < \min\{\delta_n(x), \delta_n(y)\}, \quad \text{then} \quad |f(x)-f(y)| < \alpha_n. \tag{23}$$

Put $\delta(x) = \min\{\delta_0(x), \delta_{n_x}(x)\}$, $x \in \mathbb{R}$.

If we consider $x \in F_{n_x},\ y \in F_{n_y}$ such that $|x-y| < \min\{\delta(x), \delta(y)\}$, then we have $n_x = n_y$. Put $n_0 = n_x = n_y$.

From the definition of the function δ, we have

$$|x-y| < \min\{\delta(x), \delta(y)\} \leq \min\{\delta_{n_0}(x), \delta_{n_0}(y)\}.$$

Using (23), we infer that

$$|f(x)-f(y)| < \alpha_{n_0}, \quad \text{for } x, y \in F_{n_0}. \tag{24}$$

Note that

$$F_{n_0} \subset E_{n_0} = f^{-1}(U_{n_0}) = f^{-1}\left(\varepsilon^{-1}\left((\alpha_{n_0}, +\infty)\right)\right).$$

Hence,

$$f(F_{n_0}) \subset f\left(f^{-1}\left(\varepsilon^{-1}\left((\alpha_{n_0}, +\infty)\right)\right)\right) \subset \varepsilon^{-1}\left((\alpha_{n_0}, +\infty)\right).$$

This means that $\varepsilon(f(F_{n_0})) \subset (\alpha_{n_0}, +\infty)$. So, we have

$$\varepsilon(f(z)) > \alpha_{n_0}, \quad \text{for } z \in F_{n_0}.$$

Finally, from (24), we get

$$|f(x)-f(y)| < \alpha_{n_0} < \min\{\varepsilon(f(x)), \varepsilon(f(y))\}, \quad \text{if } x, y \in F_{n_0}.$$

This proves that $f \in \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, lsc^+}$. □

We obtain now an important equality contained in [7] as a corollary of Theorem 35

Corollary 36 ([7])**.** $\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, lsc^+} = B_1$.

Moreover, we have

Corollary 37. $\mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, C^+} = B_1$.

Corollary 38. $\mathfrak{B}_{lsc^+,\, lsc^+} \subsetneq \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, lsc^+}$.

Corollary 39. $\mathfrak{B}_{lsc^+,\, C^+} \subsetneq \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, C^+}$.

From the above theorems, we have

$$\begin{array}{ccccccccccccc}
\mathfrak{B}_{Const^+,\, usc^+} & = & \mathfrak{B}_{Const^+,\, \mathbb{R}^{\mathbb{R}+}} & \subsetneq & bC_u & \subset & \mathfrak{B}_{Const^+,\, lsc^+} & \subset & \mathfrak{B}_{Const^+,\, C^+} & \subsetneq & \mathfrak{B}_{Const^+,\, Const^+} & \subsetneq & C \\
\| & & \| & & & & \cap & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \\
\mathfrak{B}_{C^+,\, usc^+} & = & \mathfrak{B}_{C^+,\, \mathbb{R}^{\mathbb{R}+}} & \subsetneq & bC_u & \subset & \mathfrak{B}_{C^+,\, lsc^+} & \subset & \mathfrak{B}_{C^+,\, C^+} & \subset & \mathfrak{B}_{C^+,\, Const^+} & \subset & C \\
\| & & \| & & & & \cap & & \cap & & \cap & & \\
\mathfrak{B}_{lsc^+,\, usc^+} & = & \mathfrak{B}_{lsc^+,\, \mathbb{R}^{\mathbb{R}+}} & \subsetneq & bC_u & \subset & \mathfrak{B}_{lsc^+,\, lsc^+} & \subset & \mathfrak{B}_{lsc^+,\, C^+} & \subset & \mathfrak{B}_{lsc^+,\, Const^+} & \subset & C \\
\| & & \rotatebox{90}{$\subsetneq$} & & & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \rotatebox{90}{$\subsetneq$} & & \\
Const & \subsetneq & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, \mathbb{R}^{\mathbb{R}+}} & & \subset & & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, lsc^+} & = & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, C^+} & = & \mathfrak{B}_{\mathbb{R}^{\mathbb{R}+},\, Const^+} & = & B_1.
\end{array}$$

REFERENCES

[1] DARJI, U. B.—EVANS, M. J.—O'MALLEY, R. J.: *A first return characterization for Baire one functions*, Real Anal. Exchange **19** (1993/1994), 510–515.
[2] ENGELKING, R.: *General Topology*. PWN, Warszawa, 1976.
[3] PEN-YEE LEE—WEE-KEE TANG—DOINGSHENG ZHAO: *An equivalent definition of functions of the first Baire class*, in: Proc. Amer. Math. Soc., Vol. 129, 2000, pp. 2273–2275.
[4] LINDNER, M.: *On Baire Class One Like Functions Defined over Topological Spaces*. PhD Thesis, Łódź, 2007.
[5] PAWLAK, R.: *On some class of functions intermediate between the class B_1^{**} and the family of continuous functions*, Tatra Mt. Math. Publ. **19** (2000), 135–144.
[6] SARKHEL, D. N.: *Baire one functions*, Bull. Inst. Math. Acad. Sinica **31** (2003), 143–149.
[7] DOINGSHENG ZHAO: *Functions whose composition with Baire class one functions are Baire class one*, Soochow J. Math. **33** (2007), no. 4 543–551.

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