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SOME REMARKS ON ρ-UPPER CONTINUOUS FUNCTIONS

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ABSTRACT. The notion of a ρ-upper continuous function is a generalization of the notion of an approximately continuous function. It was introduced by S. Kowalczyk and K. Nowakowska. In [Kowalczyk, S., Nowakowska, K.: *A note on ρ-upper continuous functions*, Tatra. Mt. Math. Publ. **44** (2009), 153–158]. the authors proved that each ρ-upper continuous function is measurable and has Denjoy property. In this note we prove that there exists a measurable function having Denjoy property which is not ρ-upper continuous function for any $\rho \in [0,1)$ and there exists a function which is ρ-upper continuous for each $\rho \in [0,1)$ and is not approximately continuous. In the paper [Kowalczyk, S.—Nowakowska, K.: *A note on ρ-upper continuous functions*, Tatra. Mt. Math. Publ. **44** (2009), 153–158] there is also proved that for each $\rho \in \left(0, \frac{1}{2}\right)$ there exists a ρ-upper continuous function which is not in the first class of Baire. Here we show that there exists a function which is ρ-upper continuous for each $\rho \in [0,1)$ but is not Baire 1 function.

Let $\mathbb{R}$ denote the set of all real numbers, $\mathbb{N}$ — the set of positive integers, m — the Lebesgue measure on the real line.

Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$.

Definition 1. The numbers

$$\underline{d}^+(E,x) = \liminf_{t \to 0^+} \frac{m(E \cap [x, x+t])}{t}$$

and

$$\bar{d}^+(E,x) = \limsup_{t \to 0^+} \frac{m(E \cap [x, x+t])}{t}$$

are called the right lower density of E at x and right upper density of E at x, respectively.

The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E,x) = \bar{d}^+(E,x) \quad \text{and} \quad \underline{d}^-(E,x) = \bar{d}^-(E,x),$$

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then we call these numbers the right density and left density of E at x, respectively, and denote them by $d^+(E,x)$, $d^-(E,x)$.

Definition 2. The number

$$\bar{d}(E,x) = \limsup_{\substack{t\to 0^+,\, h\to 0^+ \\ t+h\neq 0}} \frac{m(E\cap[x-h,x+t])}{t+h}$$

is called the upper density of E at x.

The lower density of E at x, denoted by $\underline{d}(E,x)$, is defined analogously.

Definition 3. If $\bar{d}(E,x) = \underline{d}(E,x) = \rho$, the density of E at x is said to exist and the number $d(E,x) = \rho$ is called the density of E at x.

Let E be a measurable subset of the real line, $x \in \mathbb{R}$ and $\rho \in [0,1)$.

Definition 4 ([KN])**.** We say that x is a point of ρ-type upper density of E if $\bar{d}(E,x) > \rho$.

Definition 5 ([KN])**.** A function $f\colon \mathbb{R}\to\mathbb{R}$ is called ρ-upper continuous at x provided that there exists a measurable set $E\subset\mathbb{R}$ such that x is a point of ρ-type upper density of E, $x\in E$, and $f_{|E}$ is continuous at x.

If f is ρ-upper continuous at every point, we say that f is ρ-upper continuous. The class of all ρ-upper continuous functions defined on $\mathbb{R}$ will be denote by $\mathcal{UC}_\rho$.

Definition 6. We say that a function $f\colon\mathbb{R}\to\mathbb{R}$ has Denjoy property at x_0 if for each $\varepsilon>0$ and $\delta>0$ the set $\big\{x\in(x_0-\delta,x_0+\delta) : |f(x)-f(x_0)|<\varepsilon\big\}$ contains a measurable subset of positive measure.

We say that f has Denjoy property if it has Denjoy property at each point $x\in\mathbb{R}$.

Let $\mathcal{M}$ denote the family of all measurable functions $f\colon\mathbb{R}\to\mathbb{R}$ and let $\mathcal{D}$ denote the family of all functions defined on $\mathbb{R}$ having Denjoy property.

In [KN], the authors proved that $\mathcal{UC}_\rho\subset\mathcal{M}\cap\mathcal{D}$ for each $\rho\in(0,1)$ (Theorem 2.1 and Remark 2.1). It is easy to see that this inclusion holds also for $\rho=0$. Consequently,

$$\bigcup_{\rho\in(0,1)}\mathcal{UC}_\rho\subset\mathcal{UC}_0\subset\mathcal{M}\cap\mathcal{D}.$$

We will prove that the latter inclusion is proper. For this purpose, we will use a function defined by J. Borsík in [B].

Theorem 7. *There exists a measurable function $f\colon\mathbb{R}\to[0,1]$ such that f has Denjoy property and $f\notin\mathcal{UC}_\rho$ for each $\rho\in[0,1)$.*

Proof. Let f be a function defined in [B, Proposition 5], for $r = 0$. Then f is continuous at each point different from zero and for each $\varepsilon > 0$ there is an open set U such that $\underline{d}(U, 0) > 0$ and $|f(x) - f(0)| < \varepsilon$ for each $x \in U$. Hence, f is measurable and has Denjoy property. Simultaneously, for each measurable set A such that $\bar{d}(A, 0) > 0$, the restiction $f_{|(A\cup\{0\})}$ is not continuous at 0. Consequently, $f \notin \mathcal{UC}_\rho$ for each $\rho \in [0, 1)$. □

Let $\mathcal{A}$ denote the family of all approximately continuous functions. Obviously, each approximately continuous function is ρ-upper continuous for each $\rho \in [0, 1)$, so

$$\mathcal{A} \subset \bigcap_{\rho\in[0,1)} \mathcal{UC}_\rho.$$

We will prove that this inclusion is also proper.

Theorem 8. *There exists a function $f\colon \mathbb{R} \to [0, 1]$ such that $f \in \mathcal{UC}_\rho$ for each $\rho \in [0, 1)$ and f is not approximately continuous.*

Proof. Let f be a function defined in [B, Proposition 1]. Then f is continuous at each point different from zero and ρ-upper continuous at zero for each $\rho \in [0, 1)$. Simultaneously, f is not approximately continuous at zero. □

The next theorem is an improvement of the result obtained by Kowalczyk and Nowakowska for $\rho \in \left(0, \frac{1}{2}\right)$.

Theorem 9. *There exists a function $f\colon \mathbb{R} \to [0, 1]$ such that $f \in \mathcal{UC}_\rho$ for each $\rho \in [0, 1)$ and f is not in the first class of Baire.*

Proof. We will construct a Cantor-like set in the following way:

In the first step, we remove a concentric open interval $I_{1,1}$ of the length $\frac{1}{3}$ from the interval $J_{0,1} = [0, 1]$. Let us denote the components of the set $J_{0,1} \setminus I_{1,1}$ by $J_{1,1}$, $J_{1,2}$, respectively. Next, from each of the intervals $J_{1,1}$, $J_{1,2}$, we remove concentric open intervals $I_{2,1}$, $I_{2,2}$ of the length $\frac{2}{2+2} \cdot m(J_{1,1})$. Denote the components of the set $[0, 1] \setminus (I_{1,1} \cup I_{2,1} \cup I_{2,2})$ by $J_{2,1}$, $J_{2,2}$, $J_{2,3}$, $J_{2,4}$, respectively. Assume that we have already removed concentric open intervals $I_{i,k}$ for $i \in \{1, \dots, n\}$, $k \in \{1, \dots, 2^{i-1}\}$, each of the length $\frac{i}{i+2} \cdot m(J_{i-1,1})$, where $J_{i-1,1}$ is the first component from the left of the set $[0, 1] \setminus \bigcup_{l=1}^{n} \bigcup_{p=1}^{2^{l-1}} I_{l,p}$. Denote the components of the set $[0, 1] \setminus \bigcup_{l=1}^{n} \bigcup_{p=1}^{2^{l-1}} I_{l,p}$ by $J_{n,k}$, for $k \in \{1, \dots, 2^n\}$. From each of the intervals $J_{n,k}$, for $k \in \{1, \dots, 2^n\}$, we remove a concentric open interval $I_{n+1,k}$ of the length $\frac{n+1}{(n+1)+2} \cdot m(J_{n,1})$.

Let us put

$$C = [0, 1] \setminus \bigcup_{l=1}^{\infty} \bigcup_{p=1}^{2^{l-1}} I_{l,p}$$

and let H be the set of the endpoints of all intervals $I_{l,p}$, where $l \in \mathbb{N}$, $p \in \{1,\dots,2^{l-1}\}$. Put

$$A = \bigcup_{s=1}^{\infty} \bigcup_{p=1}^{2^{(2s-1)-1}} I_{2s-1,p} \cup H$$

and

$$B = \bigcup_{s=1}^{\infty} \bigcup_{p=1}^{2^{2s-1}} I_{2s,p} \cup (C \setminus H) \cup \big(\mathbb{R} \setminus [0,1]\big).$$

For $x \in \mathbb{R}$, put

$$f(x) = \chi_A(x).$$

We will prove that $f \in \mathcal{UC}_\rho$ for each $\rho \in [0,1)$. It is sufficient to show that $\bar{d}(A,x) = 1$ for each $x \in A$ and $\bar{d}(B,x) = 1$ for each $x \in B$. Let $x \in C$. Then, for each $l \in \mathbb{N}$, there exists a number $p(l) \in \{1,\dots,2^l\}$ such that $x \in J_{l,p(l)}$. Define two sequences $\{k_i\}_{i\in\mathbb{N}}$, $\{t_i\}_{i\in\mathbb{N}}$ in the following way. For each $i \in \mathbb{N}$, we have $x \in J_{i,p(i)}$. In the case when the interval $I_{i+1,p(i)} = \big(a_{i+1,p(i)}, b_{i+1,p(i)}\big)$ removed from $J_{i,p(i)}$ is on the right side of the point x, put $k_i = b_{i+1,p(i)} - x$ and $t_i = 0$. In another case, let us put $k_i = 0$ and $t_i = x - a_{i+1,p(i)}$. Then, $[x - t_i, x + k_i] \subset J_{i,p(i)}$.

Consider subsequences $\{k_{2s-1}\}_{s\in\mathbb{N}}$ and $\{t_{2s-1}\}_{s\in\mathbb{N}}$ of the sequences $\{k_i\}_{i\in\mathbb{N}}$ and $\{t_i\}_{i\in\mathbb{N}}$, respectively. Then,

$$I_{2s,p(2s-1)} \subset [x - t_{2s-1}, x + k_{2s-1}],$$

so

$$\begin{aligned} \frac{m(B \cap [x - t_{2s-1}, x + k_{2s-1}])}{t_{2s-1} + k_{2s-1}} &\geq \frac{m(I_{2s,p(2s-1)})}{m(J_{2s-1,p(2s-1)})} \\ &= \frac{\frac{2s}{2s+2} \cdot m(J_{2s-1,p(2s-1)})}{m(J_{2s-1,p(2s-1)})} \\ &= \frac{2s}{2s+2}. \end{aligned}$$

Hence,

$$\lim_{s\to\infty} \frac{m(B \cap [x - t_{2s-1}, x + k_{2s-1}])}{t_{2s-1} + k_{2s-1}} = 1.$$

Since $\lim_{s\to\infty} t_{2s-1} = 0$ and $\lim_{s\to\infty} k_{2s-1} = 0$, we obtain $\bar{d}(B,x) = 1$. Considering similarly subsequences $\{k_{2s}\}_{s\in\mathbb{N}}$ and $\{t_{2s}\}_{s\in\mathbb{N}}$ of the sequences $\{k_i\}_{i\in\mathbb{N}}$ and $\{t_i\}_{i\in\mathbb{N}}$, respectively, we obtain

$$\frac{m(A \cap [x - t_{2s}, x + k_{2s}])}{t_{2s} + k_{2s}} \geq \frac{m(I_{2s+1,p(2s)})}{m(J_{2s,p(2s)})} = \frac{\frac{2s+1}{(2s+1)+2} \cdot m(J_{2s,p(2s)})}{m(J_{2s,p(2s)})} = \frac{2s+1}{2s+3}.$$

Hence, $\bar{d}(A,x) = 1$.

If $x \in A \setminus C$, then obviously $\bar{d}(A, x) = 1$. Analogously, if $x \in B \setminus C$, then $\bar{d}(B, x) = 1$. Hence, $f \in \mathcal{UC}_\rho$ for each $\rho \in [0, 1)$.

Let us note that $f_{|C}$ has no point of continuity. In fact, for each $x \in C$, there exist two sequences $\{x_n\}_{n\in\mathbb{N}} \subset A \cap C$ and $\{y_n\}_{n\in\mathbb{N}} \subset B \cap C$ such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = x$. So, $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = 0$. Thus f is not in the first class of Baire. □

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