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SOME CONVERGENCE THEOREMS FOR BK-INTEGRAL IN LOCALLY CONVEX SPACES

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ABSTRACT. In this paper, we present some convergence theorems for Bk-integral of functions taking values in a locally convex topological vector space. These theorems are involved with the notion of equi-Bk-integrability.

1. Introduction

The Bk-integral defined in this paper is an extension to a locally convex topological vector space of Birkhoff integral in the Fremlin sense (see [7]). Our Bk-integral definition is equivalent to Bk-integral definition in [6] and to $\mathcal{V}$-integral definition in [8].

The aim of our paper is to give sufficient conditions for the interchange of the limit and the integration, for a sequence (f_n) of Bk-integrable functions from a complete probability space to a locally convex topological vector space (V, τ). We assume equi-Bk-integrability in (V, τ) of the sequence (f_n) converging pointwise in the topology τ or in the weak topology $\sigma(V, V')$. For the case when functions take values in a Banach space, our notion of equi-Bk-integrability is equivalent to the equi-Birkhoff-integrability in [2] or [10]. Convergence theorems of this type of functions taking values in a Banach space have been shown in [2], [9], [10].

In the second section, we present three convergence theorems for Bk-integral which are involved with the notion of equi-Bk-integrability of a sequence (f_n); Theorem 2.1 assumes the pointwise convergence of the sequence (f_n) in (V, τ); Theorem 2.6 and Theorem 2.8 assume the pointwise convergence of the sequence (f_n) in $(V, \sigma(V, V'))$. Theorem 2.1 is a generalization of Theorem 7 in [2]. Theorem 2.6 is a generalization of Theorem 2.12 in [10].

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The main result in this paper is Theorem 2.8. Theorem 2.1 and Theorem 2.6 are used in the proof of Theorem 2.8. For the case when locally convex topological vector space is weakly sequential complete but not complete, we use Theorem 2.8 instead of Theorem 2.6. Do locally convex topological vector spaces of this type exist? The following example shows that the answer to this question is positive.

Let V be a Lebesgue space $L^1([0,1])$ endowed with its weak topology. This space is locally convex and Hausdorff, and its dual is linearly isometric to $L^\infty([0,1])$ (see [3, Th.IV.8.5; p. 289]), so the weak topology $\sigma(V, V')$ coincides with the original one. Sequential completeness follows from the Vitali-Hahn-Saks Theorem (see [4, p. 29]). Indeed, the elements of $L^1([0,1])$ can be identified with the countably additive measure μ on $\mathcal{B}$, the Borel σ-field of $[0,1]$, that are absolutely continuous with respect to the Lebesgue measure λ. The weak convergence of a sequence (μ_n) is clearly equivalent to pointwise convergence on all elements of $\mathcal{B}$. To show that V is not complete, it is sufficient to construct a net (f_r) of non-negative L^1 functions, with unit norm, such that the limit

$$\lim_r \int_0^1 f_r(x) g(x) dx$$

exists in $\mathbb{R}$ for all $g \in L^\infty$, but defines a finitely additive measure on $\mathcal{B}$ which is not countably additive. To this aim, one can define a sequence (x_n) in $[0,1]^{\mathcal{B}}$. Since $[0,1]^{\mathcal{B}}$ is compact for the product topology, the sequence (x_n) has a convergent subnet $(x_{n_r})_r$. Now, all elements x_{n_r} can be thought of as measures in $\mathcal{B}$, absolutely continuous with respect to λ, and their Radon-Nikodym derivatives f_{n_r} form the requested net.

Throughout this paper, (Ω, Σ, μ) is a complete probability space and (V, τ) is a locally convex topological vector space, which is Hausdorff (or separated) space. We set P the family of all continuous semi-norms in this space; for every $p \in P$, $\widetilde{V}^p$ denotes a quotient vector space of the vector space V with respect to the equivalence relation $x \sim^p y \Leftrightarrow p(x-y) = 0$; the map $\phi_p : V \to \widetilde{V}^p$ is a canonical quotient map, thus $\phi_p(x)$ is an equivalence class of an element $x \in V$ with respect to the relation " $\sim^p$ "; a quotient normed space $(\widetilde{V}^p, \widetilde{p})$ is called a normed component of the space (V, τ), where $\widetilde{p}(\phi_p(x)) = p(x)$, for each $x \in V$; a Banach space $(\overline{V}^p, \overline{p})$, which is the completion of the space $(\widetilde{V}^p, \widetilde{p})$, is called a Banach component of the space (V, τ); V', V'_p, $\widetilde{V}'_p$ and $\overline{V}'_p$ are topological duals of (V, τ), (V, p), $(\widetilde{V}^p, \widetilde{p})$ and $(\overline{V}^p, \overline{p})$, respectively; $\sigma(V, V')$ is the a topology of (V, τ). It is easy to see that

$$V' = \left\{ \widetilde{v}'_p \circ \phi_p / \widetilde{v}'_p \in \widetilde{V}'_p, p \in P \right\}, \tag{1.1}$$

because for every $v' \in V'$, we have that $|v'(.)| \in P$.

The function

$$\phi : V \to \prod_p (\widetilde{V}^p, \widetilde{p}), \phi(x) = (\phi_p(x)), x \in V,$$

is clearly linear, and since (V, τ) is Hausdorff, it is also one to one. Moreover, the function ϕ is readily seen to be a homeomorphism, and hence, an isomorphism of V onto $\phi(V)$ (for the isomorphic definition of topological vector spaces see, [11, p. 11].

For every $p, q \in P$ such that $p \leq q$, we define the map

$$\widetilde{g}_{pq} : \widetilde{V}^q \to \widetilde{V}^p, \; \widetilde{g}_{pq}(w_q) = w_p, w_q \in \widetilde{V}^q,$$

where $w_p = \phi_p(x)$, for some vector $x \in w_q$. Since for every $y \in w_q$, we have $\phi_p(y) = w_p$, the map $\widetilde{g}_{pq}$ is well defined. It is easy to prove that $\widetilde{g}_{pq}$ is a continuous linear map. We also define the map $\overline{g}_{pq} : \overline{V}^q \to \overline{V}^p$ as a continuous linear extension of $\widetilde{g}_{pq}$, for every $p, q \in P$ such that $p \leq q$.

The projective limit of the family $\{(\widetilde{V}^p, \widetilde{p})/p \in P\}$ with respect to the family $\{\widetilde{g}_{pq}/p, q \in P, p \leq q\}$ is denoted

$$\lim_{\leftarrow}(\widetilde{V}^p, \widetilde{p}), \widetilde{g}_{pq}$$

and the projective limit of the family $\{(\overline{V}^p, \overline{p})/p \in P\}$ with respect to the family $\{\overline{g}_{pq}/p, q \in P, p \leq q\}$ is denoted

$$\lim_{\leftarrow}(\overline{V}^p, \overline{p}), \overline{g}_{pq}\,,$$

for the projective limit concept (see [11, p. 52].

The following theorem is proved in a similar way as that of Theorem II. 5.4 in [11, p. 53]. The symbol $V \equiv L$ is used to mean that topological vector spaces V and L are isomorphic.

Theorem 1.1. *If (V, τ) is a complete locally convex topological vector space, then we have*

$$(V, \tau) \equiv \lim_{\leftarrow}(\widetilde{V}^p, \widetilde{p}), \widetilde{g}_{pq} \equiv \lim_{\leftarrow}(\overline{V}^p, \overline{p}), \overline{g}_{pq}.$$

Let p be an element of P and let $\Gamma = (E_n)$ be a countable partition of Ω in Σ. A series $\sum_n x_n$ of elements $x_n \in V, n \in N$ is unconditionally convergent in (V, p) if and only if $\sum_n \phi_p(x_n)$ is unconditionally convergent in $(\widetilde{V}^p, \widetilde{p})$; moreover, the series $\sum_n \phi_p(x_n)$ is unconditionally convergent to $w \in \widetilde{V}^p$, if and only if $\sum_n x_n$ is unconditionally convergent to $x \in V$, for any $x \in w$. The function $f : \Omega \to V$ is called p-summable with respect to Γ, if satisfying conditions:

(1) the function $f|_{E_n}$ is bounded in (V, p) whenever $\mu(E_n) > 0$;

(2) the set of sums

$$J_p(f, \Gamma) = \left\{ \sum_n \mu(E_n) f(t_n)/t_n \in E_n, n \in N \right\},$$

is made up of unconditionally convergent series in (V, p).

Definition 1.2. A function $f : \Omega \to V$ is called Bk-integrable on Ω in (V, τ) if there exists a vector $I_f \in V$ satisfying the following property: for every $p \in P$ and for every $\epsilon > 0$ there exists a countable partition

$$\Gamma_\epsilon^{(p)} = (E_n)$$

of Ω in Σ such that the function f is p-summable with respect to $\Gamma_\epsilon^{(p)}$ and the inequality

$$p\left(I_f - \sum_n \mu(E_n).f(t_n)\right) < \epsilon,$$

holds true for each choice of points $t_n \in E_n, n \in N$.

By virtue of Definition 1.2, we obtain Theorem 1.3. This theorem guarantees a simple and important relation of Bk-integral in a locally convex spaces and Birkhoff integral in its components.

Theorem 1.3. *A function $f : \Omega \to V$ is Bk-integrable on Ω in (V, τ) if and only if there exists a vector $I_f \in V$ such that for every $p \in P$ the function $\phi_p \circ f$ is Birkhoff integrable on Ω in the normed component $(\widetilde{V}^p, \widetilde{p})$ while*

$$(Bk)\int_\Omega \phi_p \circ f = \phi_p(I_f).$$

Definition 1.4. A sequence $(f_n : \Omega \to V)$ is called equi-Bk-integrable in (V, τ) if every function f_n is Bk-integrable on Ω in (V, τ) and for every $p \in P$ and for every $\epsilon > 0$ there exists a countable partition

$$\Gamma_\epsilon^{(p)} = (A_m)$$

of Ω in Σ such that

(1) every function f_n is p-summable with respect to $\Gamma_\epsilon^{(p)}$;
(2) for any choice of points $t_m \in A_m, m \in N$, we have that for each $\delta > 0$ there exists $k \in N$ such that

$$p\left(\sum_{m \in M} \mu(A_m) f_n(t_m)\right) \leq \delta,$$

for every finite set $M \subset N$ disjoint from $\{1, \ldots, k\}$ and for all $n \in N$;
(3) the inequality

$$p\left(\sum_m \mu(A_m) f_n(t_m) - (Bk)\int_\Omega f_n\right) \leq \epsilon$$

holds for every $n \in N$.

2. Convergence theorems

The first convergence theorem assumes that the sequence (f_n) converges point-wise to a function f in the topology τ.

Theorem 2.1. *Let (V,τ) be a sequentially complete locally convex topological vector space. If a sequence $(f_n : \Omega \to V)$ is equi-Bk-integrable in (V,τ) and converges to a function $f : \Omega \to V$ in (V,τ), then the function f is Bk-integrable on Ω in (V,τ) and we have*

$$\lim_{n\to\infty} (Bk)\int_\Omega f_n = (Bk)\int_\Omega f$$

in (V,τ).

P r o o f. The sequence (f_n) converges to the function f in (V,τ), if and only if for each $p \in P$ the sequence $(\phi_p \circ f_n)$ converges to the function $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$. Therefore, the sequence (f_n) converges to the function f in (V,τ), if and only if the sequence $(\phi_p \circ f_n)$ converges to the function $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$, for each $p \in P$.

By Definition 1.4, the sequence $(\phi_p \circ f_n)$ is equi-Birkhoff-integrable in the Banach component $(\overline{V}^p, \overline{p})$, for each $p \in P$.

Thus, we are in conditions of Theorem 7 in [2], for each $p \in P$. Therefore, for each $p \in P$ the function $\phi_p \circ f$ is Birkhoff integrable on Ω in the Banach component $(\overline{V}^p, \overline{p})$ and

$$\lim_{n\to\infty} (Bk)\int_\Omega \phi_p \circ f_n = (Bk)\int_\Omega \phi_p \circ f \tag{2.1}$$

in $(\overline{V}^p, \overline{p})$.

According to Theorem 1.3, we have that $(Bk)\int_\Omega \phi_p \circ f_n \in \widetilde{V}^p$, for each $n \in N$ and $p \in P$. Therefore, for each $p \in P$, the sequence $(\phi_p((Bk)\int_\Omega f_n\ ))$ is a Cauchy sequence in $(\widetilde{V}^p, \widetilde{p})$. This yields that $((Bk)\int_\Omega f_n)$ is a Cauchy sequence in (V,τ) and as a consequence it converges to $I_f \in V$ in (V,τ). By (2.1), this implies that

$$\phi_p(I_f) = (Bk)\int_\Omega \phi_p \circ f,$$

for every $p \in P$. Consequently, by Theorem 1.3, the function f is Bk-integrable on Ω in (V,τ) and

$$\lim_{n\to\infty} (Bk)\int_\Omega f_n = (Bk)\int_\Omega f$$

in (V,τ). □

Now, we present the convergence theorems which assume that a sequence (f_n) converges pointwise to a function f in the weak topology $\sigma(V,V')$. The following lemmas prepare the proof of Theorem 2.6.

Lemma 2.2. *Assume that a locally convex topological vector space W which is Hausdorff is given and let $T : V \to W$ be a continuous linear function. If a function $f : \Omega \to V$ is Bk-integrable on Ω in V, then the function $T(f)$ is Bk-integrable on Ω in W and*

$$(Bk)\int_{\Omega} T(f) = T\left((Bk)\int_{\Omega} f\right).$$

P r o o f. Let S be the family of all continuous semi-norms in W. Let s be an arbitrary element of S. Since T is a continuous function, for a given $\epsilon > 0$, there exists $p \in P$ and $\epsilon' > 0$ such that

$$p(x) < \epsilon' \Rightarrow s(T(x)) < \epsilon, \quad x \in V. \tag{2.2}$$

According to Definition 1.2, there exists a countable partition $\Gamma_{\epsilon'}^{(p)} = (E_n)$ of Ω in Σ such that the function f is p-summable with respect to $\Gamma_{\epsilon'}^{(p)}$ and the inequality

$$p\left(I_f - \sum_n \mu(E_n).f(t_n)\right) < \epsilon',$$

holds for each choice of points $t_n \in E_n, n \in N$. Therefore, since T is a continuous linear function, the function $T(f)$ is s-summable with respect to $\Gamma_{\epsilon'}^{(p)}$ and, because of (2.2), the inequality

$$s\left(T(I_f) - \sum_n \mu(E_n).(T \circ f)(t_n)\right) < \epsilon,$$

holds for each choice of points $t_n \in E_n, n \in N$. Because of arbitrariness of s, the function $T(f)$ is Bk-integrable on Ω in W and

$$(Bk)\int_{\Omega} T(f) = T\left((Bk)\int_{\Omega} f\right).$$

□

Lemma 2.3. *Let (V, τ) be a complete locally convex topological vector space. A function $f : \Omega \to V$ is Bk-integrable on Ω in (V, τ) if and only if for every $p \in P$ the function $\phi_p \circ f$ is Birkhoff integrable on Ω in the Banach component $(\overline{V}^p, \overline{p})$. In this case, we have that the equality*

$$\phi_p\left((Bk)\int_{\Omega} f\right) = (Bk)\int_{\Omega} \phi_p \circ f,$$

holds for every $p \in P$.

P r o o f. The direct part of theorem is easy to prove by applying Theorem 1.3.

Conversely, assume that for every $p \in P$ the function $\phi_p \circ f$ is Birkhoff integrable on S in $(\overline{V}^p, \overline{p})$. We set $(Bk)\int_{\Omega} \phi_p \circ f = \overline{I}_p$, for $p \in P$.

Assume that two arbitrary continuous semi-norms p and q such that $p \leq q$ are given. According to Lemma 2.2, we have

$$\overline{g}_{pq}\left((Bk)\int\limits_{\Omega}\phi_q \circ f\right) = (Bk)\int\limits_{\Omega}(\widetilde{g}_{pq} \circ \phi_q) \circ f = (Bk)\int\limits_{\Omega}\phi_p \circ f$$

or $\overline{g}_{pq}(\overline{I}_q) = \overline{I}_p$. Consequently, we obtain

$$(\overline{I}_p) \in \lim_{\leftarrow}(\overline{V}^p, \overline{p}), \overline{g}_{pq}$$

and therefore, by Theorem 1.1, it follows that there exists $I_f \in V$ such that $\phi_p(I_f) = \overline{I}_p$, for each $p \in P$. By virtue of Theorem 1.3, this means that the function f is Bk-integrable on Ω in (V, τ) and the proof is finished. □

In the following lemma, the symbol $f|_E$ stands for the restriction of the function f on E. This lemma has been shown in a different way in [6].

Lemma 2.4. *Let (V, τ) be a complete locally convex topological vector space. If a function $f : \Omega \to V$ is Bk-integrable on Ω in (V, τ), then, for each $E \in \Sigma$, the function $f|_E$ is Bk-integrable with respect to (E, Σ_E, μ_E) in (V, τ), where $\Sigma_E = \{E \cap F / F \in \Sigma\}$ and μ_E stands for the restriction of μ to Σ_E .*

(In this case, Bk-integral of the function $f|_E$ with respect to (E, Σ_E, μ_E) in (V, τ) is said to be Bk-integral of the function f on E in (V, τ) and it is denoted

$$(Bk)\int\limits_{E} f = (Bk)\int\limits_{E} f|_E).$$

P r o o f. Let E be an arbitrary element of Σ. By Lemma 2.3, the function $\phi_p \circ f$ is Birkhoff integrable on Ω in the Banach component $(\overline{V}^p, \overline{p})$, for each $p \in P$. Hence, by Theorem 14 in [1, p. 367], the function $(\phi_p \circ f)|_E$ is Birkhoff integrable with respect to (E, Σ_E, μ_E) in the Banach component $(\overline{V}^p, \overline{p})$, and since

$$(\phi_p \circ f)|_E = \phi_p \circ (f|_E),$$

by Lemma 2.3, we obtain that the function $f|_E$ is Bk-integrable with respect to (E, Σ_E, μ_E) in (V, τ). □

According to Lemma 2.3 and Lemma 2.4, we obtain the following

Corollary 2.5. *Let (V, τ) be a complete locally convex topological vector space and let E be an element of Σ. A function $f : \Omega \to V$ is Bk-integrable on E in (V, τ), if and only if for every $p \in P$ the function $\phi_p \circ f$ is Birkhoff integrable on E in $(\overline{V}^p, \overline{p})$. In this case, we have that the equality*

$$\phi_p\left((Bk)\int\limits_{E} f\right) = (Bk)\int\limits_{E}\phi_p \circ f,$$

holds for each $p \in P$.

Now, we are ready to show the second convergence theorem.

Theorem 2.6. *Let (V, τ) be a complete locally convex topological vector space. If a sequence $(f_n : \Omega \to V)$ is equi-Bk-integrable in (V, τ) and converges to a function $f : \Omega \to V$ in the weak topology $\sigma(V, V')$, then the function f is Bk-integrable on Ω in (V, τ) and, for every $E \in \Sigma$, we have*

$$\lim_{n\to\infty} (Bk)\int_E f_n = (Bk)\int_E f,$$

in the weak topology $\sigma(V, V')$.

P r o o f. According to Definition 1.4, the sequence $(\phi_p \circ f_n)$ is equi-Birkhoff-integrable in the Banach component $(\overline{V}^p, \overline{p})$, for each $p \in P$.

By virtue of (1.1), the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$ in the weak topology, for every $p \in P$. Therefore, for every $p \in P$, the sequence $(\phi_p \circ f_n)$ converges also to $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$ in the weak topology.

Thus, we are in conditions of Theorem 2.12 in [10], for every $p \in P$. This yields that for every $p \in P$ the function $\phi_p \circ f$ is Birkhoff integrable on Ω in $(\overline{V}^p, \overline{p})$ and, for every $E \in \Sigma$, we have that the equality

$$\lim_{n\to\infty} \overline{v}'_p \left((Bk)\int_E \phi_p \circ f_n \right) = \overline{v}'_p \left((Bk)\int_E \phi_p \circ f \right), \tag{2.3}$$

holds for every $\overline{v}'_p \in \overline{V}'_p$.

Let E be an arbitrary element of Σ. We have that, for every $p \in P$, the function $\phi_p \circ f$ is Birkhoff integrable on E in the Banach component $(\overline{V}^p, \overline{p})$. Then, by Corollary 2.5, the function f is Bk-integrable on E in (V, τ) and

$$(Bk)\int_E \phi_p \circ f = \phi_p \left((Bk)\int_E f \right) \in \widetilde{V}^p, \tag{2.4}$$

for every $p \in P$. By the same corollary, we also have

$$(Bk)\int_E \phi_p \circ f_n = \phi_p \left((Bk)\int_E f_n \right) \in \widetilde{V}^p, \tag{2.5}$$

for every $p \in P$ and $n \in N$. Hence, by (2.4),(2.5) and (2.3), we obtain that for every $p \in P$, the equality

$$\lim_{n\to\infty} \left(\widetilde{v}'_p \circ \phi_p\right) \left((Bk)\int_E f_n\right) = \left(\widetilde{v}'_p \circ \phi_p\right) \left((Bk)\int_E f\right), \tag{2.6}$$

holds for every $\widetilde{v}'_p \in \widetilde{V}'_p$.

Now, let v' be an arbitrary element of V'. According to (1.1), there exist $p \in P$ and $\widetilde{v}'_p \in \widetilde{V}'_p$ such that $v' = \widetilde{v}'_p \circ \phi_p$. The latter equality together with (2.6) imply

$$\lim_{n\to\infty} v' \left((Bk)\int_E f_n\right) = v' \left((Bk)\int_E f\right).$$

Because of arbitrariness of v' and E, this equality holds for every $v' \in V'$ and for every $E \in \Sigma$. The proof is finished. □

Finally, we can present the third convergence theorem. Let us begin with following lemma.

Lemma 2.7. *Let (V, τ) be a locally convex topological vector space which is sequentially complete with respect to the weak topology $\sigma(V, V')$. If the sequence $(f_n : \Omega \to V)$ is equi-Bk-integrable in (V, τ) and converges pointwise to the function $f : \Omega \to V$ in the weak topology, then there exists $I_f \in V$ such that the equality*

$$\lim_{n\to\infty} v' \left((Bk)\int_\Omega f_n\right) = v'(I_f),$$

holds for every $v' \in V'$.

Proof. The locally convex topological vector space $(V, \sigma(V, V'))$ is Hausdorff (see [12, Cor.IV.6.1; p. 107]). Let denote by P' the family of all continuous semi norms in $(V, \sigma(V, V'))$. Since $P' \subset P$, then the sequence (f_n) is equi-Bk-integrable in $(V, \sigma(V, V'))$ and converges to the function f in this space. So that, we are in conditions of Theorem 2.1. Hence there exists $I_f \in V$ such that:

$$\lim_{n\to\infty} p' \left((Bk)\int_\Omega f_n - I_f\right) = 0,$$

for every $p' \in P'$ and, as a consequence, we have

$$\lim_{n\to\infty} v' \left((Bk)\int_\Omega f_n\right) = v'(I_f),$$

for every $v' \in V'$, because $|v'(.)| \in P'$. □

Theorem 2.8. *Let (V,τ) be a locally convex topological vector space which is sequentially complete with respect to the weak topology $\sigma(V,V')$. If the sequence of functions $(f_n : \Omega \to V)$ is equi-Bk-integrable in (V,τ) and converges to $f : \Omega \to V$ in the weak topology, then f is Bk-integrable on Ω in (V,τ) and*

$$\lim_{n\to\infty}(Bk)\int_{\Omega} f_n = (Bk)\int_{\Omega} f,$$

in the weak topology.

P r o o f. Let p be any continuous semi-norm in (V,τ). By virtue of (1.1), the sequence $(\phi_p \circ f_n)$ converges to $\phi_p \circ f$ in the normed component $(\widetilde{V}^p, \widetilde{p})$ with respect to the weak topology, and consequently, the sequence $(\phi_p \circ f_n)$ converges also to the function $\phi_p \circ f$ in the Banach component $(\overline{V}^p, \overline{p})$ with respect to the weak topology. According to Definition 1.4, the sequence $(\phi_p \circ f_n)$ is also equi-Bk-integrable in $(\overline{V}^p, \overline{p})$. Then, by the Banach version of Theorem 2.6, the function $\phi_p \circ f$ is Bk-integrable in $(\overline{V}^p, \overline{p})$ and the equality

$$\lim_{n\to\infty}\overline{v}'_p\left((Bk)\int_{\Omega}\phi_p \circ f_n\right) = \overline{v}'_p\left((Bk)\int_{\Omega}\phi_p \circ f\right),$$

holds for every $\overline{v}'_p \in \overline{V}'_p$, and since every $\overline{v}'_p \in \overline{V}'_p$ is the continuous extension of an element $\widetilde{v}'_p \in \widetilde{V}'_p$, it follows that the equality

$$\lim_{n\to\infty}\widetilde{v}'_p\left((Bk)\int_{\Omega}\phi_p \circ f_n\right) = \overline{v}'_p\left((Bk)\int_{\Omega}\phi_p \circ f\right) \tag{2.7}$$

holds for every $\widetilde{v}'_p \in \widetilde{V}'_p$, where $\overline{v}'_p$ is a continuous extension of $\widetilde{v}'_p$.

By applying Lemma 2.2, for every $\phi_p \circ f_n$, we obtain

$$\widetilde{v}'_p\left((Bk)\int_{\Omega}\phi_p \circ f_n\right) = (Bk)\int_{\Omega}\big(\widetilde{v}'_p \circ (\phi_p \circ f_n)\big) = (Bk)\int_{\Omega} v'_p \circ f_n,$$

where $v'_p = \widetilde{v}'_p \circ \phi_p$, and again, by applying Lemma 2.2, for every f_n, we obtain

$$(Bk)\int_{\Omega} v'_p \circ f_n = v'_p\left((Bk)\int_{\Omega} f_n\right)$$

and consequently,

$$\widetilde{v}'_p\left((Bk)\int_{\Omega}\phi_p \circ f_n\right) = v'_p\left((Bk)\int_{\Omega} f_n\right). \tag{2.8}$$

Hence, by (2.8) and (2.7), we get

$$\lim_{n\to\infty} v_p'\left((Bk)\int\limits_{\Omega} f_n\right) = \overline{v}_p'\left((Bk)\int\limits_{\Omega}\phi_p \circ f\right) \tag{2.9}$$

Also, according to Lemma 2.7, there exists $I_f \in V$ such that the equality

$$\lim_{n\to\infty} v_p'\left((Bk)\int\limits_{\Omega} f_n\right) = v_p'(I_f) = \widetilde{v}_p'\big(\phi_p(I_f)\big) \tag{2.10}$$

holds for every $v_p' \in V_p'$. Hence, by (2.10) and (2.9), we obtain

$$\widetilde{v}_p'\big(\phi_p(I_f)\big) = \overline{v}_p'\left((Bk)\int\limits_{\Omega}\phi_p \circ f\right),$$

for every $\widetilde{v}_p' \in \widetilde{V}_p'$, where $\overline{v}_p'$ is a continuous extension of $\widetilde{v}_p'$. Consequently,

$$\overline{v}_p'\big(\phi_p(I_f)\big) = \overline{v}_p'\left((Bk)\int\limits_{\Omega}\phi_p \circ f\right),$$

for every $\overline{v}_p' \in \overline{V}_p'$, and according to [12, Cor.IV.6.2; p. 108], this means that

$$(Bk)\int\limits_{\Omega}\phi_p \circ f = \phi_p(I_f) \in \widetilde{V}^p.$$

Therefore, by Theorem 1.3, the function f is Bk-integrable and

$$(Bk)\int\limits_{\Omega} f = I_f.$$

□

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