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THE STRUCTURE OF THE FRÉCHET SPACE s REGARDING THE SERIES $\sum f_n(x_n)$

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This paper is dedicated to the legacy of the late professor Tibor Šalát, to whom the second author is indebted greatly.

ABSTRACT. We investigate the subsets of the Fréchet space s of all sequences of real numbers equipped with the Fréchet metric ρ from the Baire category point of view. In particular, we concentrate on the "convergence" sets of the series $\sum f_n(x_n)$ that is, sets of sequences $x = (x_n)$ for which the series converges, or has a sum (perhaps infinite), or oscillates. Provided all f_n are continuous real functions, sufficient conditions are given for the "convergence" sets to be of the first Baire category or residual in s.

1. Introduction

Nowadays, there is an amount of papers exploring the properties of the Fréchet space s of sequences of real numbers equipped with the metric ρ,

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}, \qquad x = (x_n), \quad y = (y_n) \in s,$$

especially regarding the convergence of series composed of the elements of the space s.

For example, in [LŠ] it was shown that the set of all elements $x = (x_n)$ in s for which the series $\sum a_n x_n$ converges (for a fixed $a = (a_n)$ with $a_n \neq 0$ for infinitely many n) is of first Baire category in s. Perhaps even stronger statements can be found in [Ne] with generalizations in [EŠ1] and [EŠ2]. For more results see for example [GL], [La], [LG], etc.

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Our approach, however, will be a more general one. Given a sequence $(f_n)_{n=1}^{\infty}$ of real functions, $f_n : \mathbb{R} \to \mathbb{R}$, we make the following notation:

$$A = A\ (f_1, f_2, \dots) := \Big\{x \in s;\ \exists \sum\nolimits_{n=1}^{\infty} f_n(x_n)\ \text{finite or infinite}\Big\},$$

$$A_0 = A_0(f_1, f_2, \dots) := \Big\{x \in s;\ \exists \sum\nolimits_{n=1}^{\infty} f_n(x_n)\ \text{finite}\Big\},$$

$$O = O\ (f_1, f_2, \dots) := \Big\{x \in s;\ \sum\nolimits_{n=1}^{\infty} f_n(x_n)\ \text{oscillates}\Big\},$$

$$O^* = O^*(f_1, f_2, \dots) := \Big\{x \in s;\ \sum\nolimits_{n=1}^{\infty} f_n(x_n)\ \text{oscillates from } -\infty \text{ to } +\infty\Big\}.$$

Obviously,

$$A_0 \subseteq A \qquad \text{and} \qquad O^* \subseteq O,$$

and we have a decomposition of the Fréchet space s:

$$s = A \cup O.$$

It is easy to see that in general case one of the sets A, O in the foregoing union may be empty: for example, if all functions f_n are non-negative (or non--positive), then $O = \emptyset$ and $s = A$. It is also easy to see that with a suitable choice of f_n we can get $s = A^+$ or $s = A^-$ (see the definition below). Moreover, if, for example $|f_n(t)| \le n^{-2}$ for each $t \in \mathbb{R}$, $n \in \mathbb{N}$, then $s = A_0$.

On the other hand, if $f_n(t) = (-1)^n$, for each $t \in \mathbb{R}$, $n \in \mathbb{N}$, then $A = \emptyset$ (and $s = O \setminus O^*$). Similarly, if $f_n(t) = n.(-1)^n$, then $O^* = s$.

The reference results mentioned earlier can be viewed as special cases of our situation if we put $f_n(t) = a_n.t$, $t \in \mathbb{R}$, $n \in \mathbb{N}$, or if we put $f_n \equiv f$, $n \in \mathbb{N}$.

2. Results

Consider two more sets:

$$A^+ = A^+(f_1, f_2, \dots) := \Big\{x \in s;\ \sum\nolimits_{n=1}^{\infty} f_n(x_n) = +\infty\Big\},$$

$$A^- = A^-(f_1, f_2, \dots) := \Big\{x \in s;\ \sum\nolimits_{n=1}^{\infty} f_n(x_n) = -\infty\Big\}.$$

Then, of course, $A = A_0 \cup A^+ \cup A^-$.

Proposition 2.1. *If any of the sets A_0, A^+, A^-, O, O^* is nonempty, then the set is dense in the space s.*

P r o o f. The result follows from the properties of the Fréchet metric ρ and from the fact that the convergence character of a series remains the same if only a finite number of its terms has been changed. □

Evidently, more specific properties of these sets can be shown if the functions f_n are continuous.

Theorem 2.1. *Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous real functions.*

(i) *If A^+ (A^-) is nonempty, then the set $A_0 \cup A^-$ ($A_0 \cup A^+$) is of the first Baire category in s. In particular, A_0 is of first Baire category.*

(ii) *If both A^+ and A^- are nonempty, then the set A is of the first Baire category in s.*

Proof. For each $k \in \mathbb{N}$ and $x \in s$ put $s_k(x) := \sum_{n=1}^{k} f_n(x_n)$ and define the function $F_k\colon s \to \mathbb{R}$ by

$$F_k(x) = \frac{s_k(x)}{1 + |s_k(x)|}\,.$$

Then each F_k is continuous (this follows from the continuity of the functions f_n and from the fact that the convergence in s is equivalent to the convergence along the coordinates). Thus, the restriction $F_k\restriction_A$ is continuous on A and for each $x \in A$ there is the limit

$$F(x) = \lim_{k\to\infty} F_k(x)$$

such that $F\colon A \to [-1, 1]$ where

$$F^{-1}(1) = A^+, \quad F^{-1}(-1) = A^- \quad \text{and} \quad F^{-1}((-1,1)) = A_0.$$

Moreover, the function F belongs to the first Baire class of functions on A.

Suppose now that $A^+ \neq \emptyset$ (the case for A^- is symmetric). Thus A^+ is dense in A. Since $A^+ = F^{-1}(1)$, it follows that any point $x \in A_0 \cup A^-$ is a point of discontinuity of F, or $x \in D(F)$, since F assumes the value 1 arbitrarily close to x. Therefore, $A_0 \cup A^- \subseteq D(F)$.

It is known (see [Si, p. 185]) that $D(f)$ is a first Baire category set whenever f is a first Baire class function. Hence, $A_0 \cup A^-$ is of the first Baire category in A and in s.

If both sets A^+ and A^- are nonempty (and hence dense), then both $F^{-1}(1)$ and $F^{-1}(-1)$ are dense in A; that is, every point of A is a discontinuity point of F, so $A = D(F)$ is of first Baire category in A and in s. $\square$

Note 2.1. Consider for a second the situation where $f_n \equiv f$ for each $n \in \mathbb{N}$. Then $A(f_1, f_2, \dots) = A(f)$, etc. The assumption (i) of the previous theorem is satisfied provided f has at least one positive (or negative) value that is, if $\mathbb{R} \neq Z_f := \{t \in \mathbb{R};\ f(t) = 0\}$. The assumption (ii) holds if f has a positive and a negative value. Hence, we obtain the following two corollaries:

Corollary 2.1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function.*

(i) *If $f \not\equiv 0$ (or $Z_f \neq \mathbb{R}$), then $A_0(f)$ is of the first Baire category in s.*

(ii) *If there are* $t_1, t_2 \in \mathbb{R}$ *with* $f(t_1) < 0 < f(t_2)$, *then* $A(f)$ *is of the first Baire category in* s.

Corollary 2.2. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be continuous. Then either* $A_0(f) = s$ *or* $A_0(f)$ *is of first Baire category in* s.

Proof. If $f \equiv 0$, then $Z_f = \mathbb{R}$ and $A_0(f) = s$. Otherwise, $Z_f \neq \mathbb{R}$ and the preceding corollary implies the result. □

The latter statement resembles the well-known 0–1 *theorem* from the *classical measure and probability theory.* Now, let us turn our attention to the "oscillation" sets O and O^* instead of the "convergence" sets A and A_0.

Theorem 2.2. *Let* $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, *be continuous functions. Then the set* $O^*(f_1, f_2, \dots)$ *is of* G_δ*-type.*

Proof. Recall that $s_k(x)$ denotes the k-th partial sum of our series $\sum f_n(x_n)$ and for $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, consider the set $H(t_1, t_2)$ of the form:

$$\begin{aligned} H(t_1, t_2) = \big\{x \in s;\ &\exists\, I_1 \subseteq \mathbb{N} \text{ infinite } \forall\, k \in I_1 : s_k(x) < t_1, \\ &\exists\, I_2 \subseteq \mathbb{N} \text{ infinite } \forall\, k \in I_2 : s_k(x) > t_2\big\}. \end{aligned}$$

It follows from the definition of the set $O^*(f_1, f_2, \dots)$ that

$$O^*(f_1, f_2, \dots) = \bigcap_{t=1}^{\infty} H(-t, t)\,.$$

Now, for each $k \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, define the sets

$$\begin{aligned} H_k^-(t_1) &= \big\{x \in s;\ s_k(x) < t_1\big\}, \\ H_k^+(t_2) &= \big\{x \in s;\ s_k(x) > t_2\big\}. \end{aligned}$$

These sets are open (see the proof of Theorem 2.1), so for each $t \in \mathbb{N}$, the set

$$H(-t, t) = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} H_k^-(-t)\right) \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} H_k^+(t)\right)$$

is of G_δ-type. It follows that also $O^*(f_1, f_2, \dots)$ is of G_δ-type. □

Theorem 2.3. *Let* $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, *be continuous functions. If the set* $O^*(f_1, f_2, \dots)$ *is nonempty, then it is residual in* s. *In that case,* $O(f_1, f_2, \dots)$ *is also residual in* s.

Proof. If $O^*(f_1, f_2, \dots)$ is nonempty, then it is dense in s (see Proposition 2.1) and, by the preceding theorem, it is of G_δ-type. According to Kuratowski [Ku, p. 49] a dense G_δ set is residual (i.e., is a complement of a first category set). The second statement obviously folows from $O^* \subseteq O$. □

Corollary 2.3. *Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions infinitely many of which are surjective (i.e., for infinitely many $n \in \mathbb{N}$ it is $f_n[\mathbb{R}] = \mathbb{R}$). Then the sets $O^*(f_1, f_2, \dots)$ and $O(f_1, f_2, \dots)$ are residual in s.*

Proof. Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of natural numbers with $n_k < n_{k+1}$ and $f_{n_k}[\mathbb{R}] = \mathbb{R}$. Define the element $d = (d_i)$ of s in the following way: choose $d_1, \dots d_{n_1-1} \in \mathbb{R}$ arbitrarily. By the surjectivity of f_{n_1}, we can find d_{n_1} in $\mathbb{R}$ such that

$$f_{n_1}(d_{n_1}) < -\big(f_1(d_1) + \dots + f_{n_1-1}(d_{n_1-1})\big) - 1.$$

Choose again $d_{n_1+1}, \dots d_{n_2-1} \in \mathbb{R}$ arbitrarily and by the surjectivity of f_{n_2}, find d_{n_2} in $\mathbb{R}$ such that

$$f_{n_2}(d_{n_2}) > -\big(f_1(d_1) + \dots + f_{n_2-1}(d_{n_2-1})\big) + 1.$$

Alternating in this way, we construct the sequence $d = (d_i) \in s$ by means of induction such that the partial sums of $\sum f_n(d_n)$ for each $k \in \mathbb{N}$ satisfy

$$\begin{aligned} s_{n_{2k}}(d) &> k, \\ s_{n_{2k-1}}(d) &< -k. \end{aligned}$$

For each $k \in \mathbb{N}$ the element d belongs to the set $H(-k, k)$ defined in the proof of Theorem 2.2, that is,

$$d \in \bigcap \{\, H(-k, k);\ k \in \mathbb{N}\} = O^*$$

and hence O^* is nonempty. The rest follows from the preceding theorem. □

Given a sequence $(d_n)_{n=1}^{\infty}$, we denote the set of all its accumulation points by $(d_n)'_n$. The following theorem improves a similar result from [Ne].

Theorem 2.4. *Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. Let M denote the set of those $x \in s$ such that each number in $\mathbb{R}^* = [-\infty, \infty]$, the extended reals, is an accumulation point of the sequence $\big(s_k(x)\big)_{k=1}^{\infty}$, that is,*

$$M = \Big\{x \in s;\ \big(s_k(x)\big)'_k = \mathbb{R}^*\Big\}.$$

If the set M is nonempty, then it is residual in s.

Proof. Let $t \in \mathbb{R}$, $m, k \in \mathbb{N}$. Denote by $M(t, m, k)$ the set

$$M(t, m, k) = \Big\{x \in s;\ |s_k(x) - t| < \frac{1}{m}\Big\}.$$

Because of the continuity of the k-th partial sum $s_k\colon s \to \mathbb{R}$, each set $M(t, m, k)$ is open in s. Therefore, the set

$$M(t) = \bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} M(t, m, k)$$

is of G_δ type. It is also easy to see that if $\mathbb{Q}$ denotes the set of all rationals, then

$$M = \bigcap_{t\in\mathbb{Q}} M(t)$$

and hence, M is also a G_δ set. In view of our previous remarks, it suffices to show that M is dense in s, for a dense G_δ set is residual.

So, let $d = (d_i) \in M$ and let $z = (z_i) \in s$ and $\varepsilon > 0$ be arbitrary. It suffices to show that M contains a point from the open ρ-ball $B_\rho(z,\varepsilon)$ with the centre z and radius ε.

Choose $n \in \mathbb{N}$ such that $2^{-n} < \varepsilon$ and define the element $u = (u_i) \in s$ in the following way

$$u_i = \begin{cases} z_i, & i = 1, 2, \dots n, \\ d_i, & i \geq n+1. \end{cases}$$

Then $\rho(u,z) < \sum_{i=n+1}^{\infty} 2^{-i} = 2^{-n} < \varepsilon$, so $u \in B_\rho(z,\varepsilon)$. Moreover, since u differs from $d \in M$ only in a finite number of terms, we have $u \in M$. Therefore, M is dense, G_δ and hence residual. □

THEOREM 2.5. *Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. If the set $s\setminus A_0(f_1, f_2, \dots)$ is nonempty, then it is residual in s, i.e., the set $A_0(f_1, f_2, \dots)$ is then of first Baire category in s.*

Proof. For $m, k, p \in \mathbb{N}$ put

$$C(m,k,p) = \left\{ x \in s;\ |f_{k+1}(x_{k+1}) + \dots + f_{k+p}(x_{k+p})| \leq \frac{1}{m} \right\}.$$

By the Cauchy-Bolzano criterion for the convergence of series we have

$$A_0(f_1, f_2, \dots) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{\substack{p\in\mathbb{N}\\ k\geq n}} C(m,k,p).$$

For our convenience put

$$C(m) = \bigcup_{n=1}^{\infty} \bigcap_{\substack{p\in\mathbb{N}\\ k\geq n}} C(m,k,p),$$

so,

$$A_0(f_1, f_2, \dots) = \bigcap_{m\in\mathbb{N}} C(m).$$

Because of the continuity of all f_n functions, each set $C(m,k,p)$ is closed, i.e., each set $C(m)$ is of F_σ-type.

Next, $s \setminus A_0 = \bigcup_{m\in\mathbb{N}} \big(s \setminus C(m)\big) = \bigcup_{m\in\mathbb{N}} D(m)$ if we put $D(m) = s \setminus C(m)$. If $s \setminus A_0$ is nonempty, then there is $d = (d_j) \in s$ and $m_0 \in \mathbb{N}$ such that $d \in D(m_0)$. We want to show that $D(m_0)$ is dense in s.

Let $z = (z_j) \in s$ and $\varepsilon > 0$ be arbitrary fixed. It suffices to show that $D(m_0) \cap B_\rho(z, \varepsilon) \neq \emptyset$. Since $d \in D(m_0)$, for each $n \in \mathbb{N}$ there is $k \in \mathbb{N}$, $k \geq n$, and $p \in \mathbb{N}$ such that

$$|f_{k+1}(d_{k+1}) + \cdots + f_{k+p}(d_{k+p})| > \frac{1}{m_0}. \tag{1}$$

Now, choose $l \in \mathbb{N}$ with $2^{-l} < \varepsilon$ and define the element $u = (u_j) \in s$ by

$$u_j = \begin{cases} z_j, & j = 1, 2, \dots l, \\ d_j, & j \geq l + 1. \end{cases}$$

Then, $\rho(u, z) < \sum_{j=l+1}^{\infty} 2^{-j} = 2^{-l} < \varepsilon$, that is, $u \in B_\rho(z, \varepsilon)$. Moreover, if $j \geq l + 1$, then $u_j = d_j$ and it follows from (1) that if $n \geq l + 1$, then there is $k \geq n$ and $p \in \mathbb{N}$ with

$$|f_{k+1}(u_{k+1}) + \cdots + f_{k+p}(u_{k+p})| > \frac{1}{m_0}$$

and consequently, $u \in D(m_0)$.

Therefore $D(m_0)$ is dense and G_δ, i.e., residual. Hence $s \setminus A_0 = \bigcup_{m \in \mathbb{N}} D(m)$ is residual and A_0 is of first Baire category in s. $\square$

The result of the preceding theorem can be reformulated to resemble Corollary 2.2 and the 0–1 *measure theorem* mentioned earlier. This corollary can also be deduced from Theorem 2.1.

COROLLARY 2.4. *Let $f_n \colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be continuous functions. Then either $A_0(f_1, f_2, \dots) = s$ or the set $A_0(f_1, f_2, \dots)$ is of first Baire category in s.*

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