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A CATEGORY ANALOGUE OF THE GENERALIZATION OF LEBESGUE DENSITY TOPOLOGY

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ABSTRACT. A notion of $\mathcal{A}_I$-topology, a generalization of Wilczyński's I-density topology (see [Wilczyński, W.: *A generalization of the density topology*, Real. Anal. Exchange **8** (1982-1983), 16–20] is introduced. The notion is based on his reformulation of the definition od Lebesgue density point. We consider a category version of the topology, which is a category analogue of the notion of an $\mathcal{A}_d$--density topology on the real line given in [Wojdowski, W.: *A generalization of density topology*, Real. Anal. Exchange **32** (2006/2007), 1–10]. We also discuss the properties of continuous functions with respect to the topology.

Let S be a σ-algebra of subsets of the real line $\mathbb{R}$, and $I \subset S$ a proper σ-ideal. We shall say that the sets $A, B \in S$ are equivalent ($A \sim B$) if and only if $A \triangle B \in I$. We will denote by λ the Lebesgue measure on the real line.

Let us recall that the point $x \in \mathbb{R}$ is said to be a Lebesgue density point of a measurablea a set A, if

$$\lim_{h \to 0} \frac{\lambda\big(A \cap [x-h, x+h]\big)}{2h} = 1\,. \tag{$*$}$$

W. Wilczyński [W1] gave his reformulation of the notion of the density point of a measurable set A, in terms of convergence almost everywhere of the sequence of characteristic functions of dilations of a set A:

A point $x \in \mathbb{R}$ is Lebesgue density point of a measurable set A if and only if every subsequence

$$\Big\{\chi_{\big(n_m\cdot(A-x)\big)\cap[-1,1]}\Big\}_{m\in\mathbb{N}} \qquad \text{of} \qquad \Big\{\chi_{\big(n\cdot(A-x)\big)\cap[-1,1]}\Big\}_{n\in\mathbb{N}}$$

contains a subsequence

$$\Big\{\chi_{\big(n_{m_p}\cdot(A-x)\big)\cap[-1,1]}\Big\}_{p\in\mathbb{N}}$$

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which converges to $\chi_{[-1,1]}$ almost everywhere on $[-1,1]$ (which means except on a null set).

Wilczynski's approach relieved definition of the notion of a measure. His definition requires only null sets. Instead of the notion of convergence in measure of a sequence of measurable functions, he uses the convergence almost everywhere. This opened a new space for studying of more subtle properties of the notion of the Lebesgue density point and density topology, their various modifications, and, most of all, the category analogues (see [PWW1], [PWW2], [CLO]).

The reformulated definition could be considered in more general settings as follows:

A point $x \in \mathbb{R}$ is an I-density point of a set $A \in S$, if every subsequence

$$\left\{\chi_{\left(n_m\cdot(A-x)\right)\cap[-1,1]}\right\}_{m\in\mathbb{N}} \qquad \text{of} \qquad \left\{\chi_{\left(n\cdot(A-x)\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

contains a subsequence

$$\left\{\chi_{\left(n_{m_p}\cdot(A-x)\right)\cap[-1,1]}\right\}_{p\in\mathbb{N}},$$

which converges to $\chi_{[-1,1]}$ I-almost everywhere on $[-1,1]$ (which means except from a set belonging to I).

In [PWW2, Corollary 1, p. 556] in the category case, and in [W2] in the measure case, it is proved that the following conditions are equivalent:

1. x is an I-density point of a set $A \in S$.
2. For any decreasing to zero sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, there exists its subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence
$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$
of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{[-1,1]}$.
3. Given $\{t_n\}_{n\in\mathbb{N}}$, a decreasing to zero sequence of real numbers fulfilling condition $\sup_{n\to\infty}\frac{t_n}{t_{n+1}} < \infty$, every subsequence $\left\{\frac{1}{t_{n_m}}\right\}_{m\in\mathbb{N}}$ of $\left\{\frac{1}{t_n}\right\}_{n\in\mathbb{N}}$ contains a subsequence $\left\{\frac{1}{t_{n_{m_p}}}\right\}_{p\in\mathbb{N}}$ such that
$$\left\{\chi_{\left(\frac{1}{t_{n_{m_p}}}\cdot(A-x)\right)\cap[-1,1]}\right\}_{p\in\mathbb{N}}$$
converges to $\chi_{[-1,1]}$ I-almost everywhere on $[-1,1]$.

Following Wilczyński's approach in [WO1] we have introduced a notion of $\mathcal{A}_d$-density of a Lebesgue measurable set leading to a notion of $\mathcal{T}_{\mathcal{A}_d}$ topology on the real line stronger than the Lebesgue density topology. The generalization was related to a given appropriate family of subsets of $[-1,1]$, namely the family of measurable sets having density one at zero.

Now, we shall formulate a generalization of the definition of $\mathcal{A}_d$-density point and then consider its category analogue.

We shall consider the following families of sets:

a) $\mathcal{F}_{[-1,1]}$ the family of ((S,I)-residual) subsets of interval $[-1,1]$ (i.e., $\mathcal{F}_{[-1,1]} \subset S$ and for $A \in \mathcal{F}_{[-1,1]}$ we have $[-1,1] \setminus A \in I$),

b) $\mathcal{F}_{[-\alpha,\alpha]}$ the family of subsets of interval $[-1,1]$ such that $\mathcal{F}_{[-1,1]} \subset S$ and $[-\alpha,\alpha] \setminus A \in I$, for some $0 < \alpha \le 1$ (i.e., (S,I)-residual on $[-\alpha,\alpha]$, where $0 < \alpha \le 1$),

c) $\mathcal{F}_I$ the family of subsets of interval $[-1,1]$ from S having 0 as its I-density point.

We have $\mathcal{F}_{[-1,1]} \subset \mathcal{F}_{I[-\alpha,\alpha]} \subset \mathcal{F}_I$.

Definition 1. We shall say that x is an $\mathcal{F}_I$-density point of $A \in S$, if for any sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$, decreasing to zero, there exists a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $B \in \mathcal{F}_I$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on $[-1,1]$ to χ_B.

By analogy, we define a notion of $\mathcal{F}_{[-\alpha,\alpha]}$-density point and $\mathcal{F}_{[-1,1]}$-density point of $A \in S$. The family $\mathcal{F}_{I[-1,1]}$ corresponds precisely to the definition of I-density point of a set $A \in S$. The set of all $\mathcal{F}_I$-density points, $\mathcal{F}_{I[-\alpha,\alpha]}$-density points and I-density points of $A \in S$ will be denoted by $\Phi_{\mathcal{F}_I}(A)$, $\Phi_{\mathcal{F}_{I[-\alpha,\alpha]}}(A)$ and $\Phi_I(A)$, respectively. Obviously, $\Phi_{\mathcal{F}_{I[-1,1]}}(A) = \Phi_I(A)$.

Proposition 1. *Let us observe that if x is an $\mathcal{F}_I$-density point of $A \in S$, there is no decreasing to zero sequence $\{t_n\}_{n\in\mathbb{N}}$ such that the sequence*

$$\left\{\chi_{\frac{1}{t_n}\cdot(A-x)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on $[-1,1]$ to 0.

P r o o f. It is a simple consequence of the definition. □

Proposition 2. *For each $A \in S$, $\Phi_I(A) \subset \Phi_{\mathcal{F}_{I[-\alpha,\alpha]}}(A) \subset \Phi_{\mathcal{F}_I}(A)$.*

P r o o f. It is obvious. □

From now on, we shall consider a particular case; an S stands for the σ-algebra of subsets of the real line $\mathbb{R}$ with the Baire property and $I \subset S$ is the σ-ideal of the sets of first category. The families $\mathcal{F}_{[-1,1]}$, $\mathcal{F}_{I[-\alpha,\alpha]}$ and $\mathcal{F}_I$ will be denoted by $\mathcal{A}_{I[-1,1]}$, $\mathcal{A}_{I[-\alpha,\alpha]}$ and $\mathcal{A}_I$, respectively. Let us recall that we can uniquely assign a regular open set G to every $A \subset \mathbb{R}$ that has the Baire property, such that $A \triangle G \in I$. We shall call it a regular open representation of A and denote by $G(A)$ (see [O] and [WO2]).

Remark 1. Let us observe that for any sets A, $B \in S$, such that $A \triangle B \in I$, the I-a.e. convergence of the sequence of characteristic functions $\{\chi_{a_n \cdot A}\}_{n\in\mathbb{N}}$ on $[-1,1]$, for each $\{a_n\}_{n\in\mathbb{N}}$ to χ_A is equivalent to the convergence of the sequence of characteristic functions $\{\chi_{a_n \cdot B}\}_{n\in\mathbb{N}}$ on $[-1,1]$ to χ_A.

Lemma 1. *Let $A \subset [0,1]$ be a set with the Baire property and let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 1, $a_n < \frac{3}{2}$. Then the sequence of characteristic functions $\{\chi_{a_n \cdot A}\}_{n\in\mathbb{N}}$ converges I-a.e. on $[-1,1]$ to χ_A.*

P r o o f. It is clear that for an open set $A \subset [0,1]$ and a sequence of positive numbers $\{a_n\}_{n\in\mathbb{N}}$, converging to 1, $a_n < \frac{3}{2}$ the sequence of characteristic functions $\{\chi_{a_n \cdot A}\}_{n\in\mathbb{N}}$ converges I-almost everywhere on $[-1,1]$ to χ_A. By the above Remark 1 it follows that we may equivalently replace the set A in thesis with its regular open representation $G(A)$. This finishes the proof. □

Proposition 3. *There exists a set A such that $\Phi_I(A) \subsetneqq \Phi_{\mathcal{A}_I}(A)$.*

P r o o f. We shall start with the notion of density from the right. We shall define a set A such that:

1) 0 is not an I-density point of A from the right,
2) 0 is not an I-density point of $\mathbb{R} - A$ from the right,
3) 0 is an $\mathcal{A}_I$-density point of A from the right.

Let $D \in \mathcal{A}_I$ be a set such that $[0,1] \setminus D \in S \setminus I$, and $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, $c_1 < 1$, such that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. We define a set $A \in S$ as

$$A = \bigcup_{n=1}^{\infty} \left[(c_n \cdot D) \cap (c_{n+1}, c_n) \right].$$

Now, let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. We can find the subsequences $\{t_{n_r}\}_{r\in\mathbb{N}}$ and $\{c_{m_r}\}_{r\in\mathbb{N}}$ of $\{t_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$, respectively, and there are neither elements of $\{t_n\}_{n\in\mathbb{N}}$ nor of $\{c_n\}_{n\in\mathbb{N}}$ between c_{m_r} and t_{n_r}.

Consider the sequence $\{c_{m_r} \cdot \frac{1}{t_{n_r}}\}_{r\in\mathbb{N}} \subset (0,1]$. We can find a convergent subsequence $\{c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\}_{k\in\mathbb{N}}$.

There are two possibilities:

a) $\lim_{k\to\infty} \left(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}} \right) = a \neq 0$; i.e., $\lim_{k\to\infty} \left(c_{m_{r_k}} \cdot \frac{1}{a t_{n_{r_k}}} \right) = 1$ and

b) $\lim_{k\to\infty}\left(c_{m_{r_k}}\cdot\frac{1}{t_{n_{r_k}}}\right)=0.$

In case a)

$$\left\{\chi_{\frac{c_{m_{r_k}}}{a t_{n_{r_k}}}\cdot\left(\frac{c_{m_{r_k}+1}}{c_{m_{r_k}}},1\right)}\right\}_{k\in\mathbb{N}}$$

converges I-almost everywhere on $(0,1)$ to $\chi_{[0,1]}$, and by Lemma 1

$$\left\{\chi_{\frac{c_{m_{r_k}}}{a t_{n_{r_k}}}\cdot\left[\left(\frac{c_{m_{r_k}+1}}{c_{m_{r_k}}},1\right)\cap D\right]}\right\}_{k\in\mathbb{N}}$$

converges I-almost everywhere to χ_D on $[0,1]$. Equivalently,

$$\left\{\chi_{\frac{1}{a t_{n_{r_k}}}\cdot\left[\left(c_{m_{r_k}+1},c_{m_{r_k}}\right)\cap\left(c_{m_{r_k}}\cdot D\right)\right]}\right\}_{k\in\mathbb{N}}$$

converges I-almost everywhere to χ_D on $[0,1]$. Thus, since

$$\left(c_{m_{r_k}+1},c_{m_{r_k}}\right)\cap\left(c_{m_{r_k}}\cdot D\right)=\left(c_{m_{r_k}+1},c_{m_{r_k}}\right)\cap A,$$

the sequence $\chi_{\left(\frac{1}{a\cdot t_{n_{r_k}}}\cdot A\right)\cap[0,1]}$ converges I-almost everywhere to χ_D on $[0,1]$, and consequently, $\chi_{\left(\frac{1}{t_{n_{r_k}}}\cdot A\right)\cap[0,a]}$ converges I-almost everywhere to $\chi_{(a\cdot D)\cap[0,a]}$ on $[0,a]$.

Thus, we obtain B on $[0,a]$ as

$$B\cap[0,a]=(a\cdot D)\cap[0,a].$$

If $a=1$, the proof is complete; 0 is an $\mathcal{A}_I$-density point of $B=D\cap[0,1]$ from the right.

If $a<1$, we have to determine B on $(a,1]$ as well.

By definition of $\{c_n\}_{n\in\mathbb{N}}$, we have $\lim_{k\to\infty}\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}=\infty$. Actually,

$$\begin{aligned}\lim_{k\to\infty}\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}&=\lim_{k\to\infty}\left(\frac{c_{m_{r_k}-1}}{t_{n_{r_{k_p}}}}\cdot\frac{c_{m_{r_k}}}{c_{m_{r_k}}}\right)\\&=\lim_{k\to\infty}\left(\frac{c_{m_{r_k}-1}}{c_{m_{r_k}}}\cdot\frac{c_{m_{r_k}}}{t_{n_{r_k}}}\right)\\&=a\cdot\lim_{k\to\infty}\frac{c_{m_{r_k}-1}}{c_{m_{r_k}}}=\infty.\end{aligned}$$

Hence, because D has 0 as its I-density point from the right, we can find a subsequence $\{\frac{1}{t_{n_{r_{k_p}}}}\}_{p\in\mathbb{N}}$ of $\{\frac{1}{t_{n_{r_k}}}\}_{k\in\mathbb{N}}$ such that the sequence $\chi_{\left(\frac{1}{t_{n_{r_{k_p}}}}\cdot A\right)\cap(a,1]}$ converges to $\chi_{(a,1]}$ I-almost everywhere on $(a,1]$, since

$$\left(c_{m_{r_{k_p}}}, c_{m_{r_{k_p}}-1}\right)\cap\left(c_{m_{r_{k_p}}-1}\cdot D\right)=\left(c_{m_{r_{k_p}}}, c_{m_{r_{k_p}}-1}\right)\cap A$$

and $\lim_{k\to\infty}\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}=\infty$ and $\lim_{k\to\infty}\frac{c_{m_{r_{k_p}}}}{t_{n_{r_{k_p}}}}=a$.

Hence, we determine B on $[0,1]$ as

$$B\cap[0,a]=(a\cdot D)\cap[0,a] \quad \text{and} \quad B\cap(a,1]=(a,1].$$

In case b), i.e., $\lim_{k\to\infty}\left(c_{m_{r_k}}\cdot\frac{1}{t_{n_{r_k}}}\right)=0$, we have two possibilities again:

b1) The sequence $\{\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}\}_{k\in\mathbb{N}}$ is bounded from above.

We take a subsequence $\{\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}\}_{p\in\mathbb{N}}$ such that $\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}=b<\infty$, and proceed similarly to the argument in a). We have $b\geq 1$, and $\chi_{(\frac{1}{t_{n_{r_{k_p}}}}\cdot A)\cap[0,1]}$ converges I-almost everywhere to $\chi_{(b\cdot D)\cap[0,1]}$ on $[0,1]$, and we obtain B on $[0,1]$, as

$$B\cap[0,1]=(b\cdot D)\cap[0,1]\,.$$

b2) The sequence $\{\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}\}_{k\in\mathbb{N}}$ is not bounded from above. We take a subsequence $\{\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}\}_{p\in\mathbb{N}}$ such that $\lim_{p\to\infty}\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}=\infty$. As D has 0 as its I-density point from the right and $\lim_{p\to\infty}\left(c_{m_{r_{k_p}}}\cdot\frac{1}{t_{n_{r_k}}}\right)=0$, we can find a subsequence $\{\frac{1}{t_{n_{r_{k_{p_s}}}}}\}_{s\in\mathbb{N}}$ of $\{\frac{1}{t_{n_{r_{k_p}}}}\}_{p\in\mathbb{N}}$ such that the sequence $\chi_{\left(\frac{1}{t_{n_{r_{k_{p_s}}}}}\cdot A\right)\cap[0,1]}$ converges to $\chi_{[0,1]}$ I-almost everywhere on $[0,1]$, and we determine B on $[0,1]$ as

$$B\cap[0,1]=[0,1]\,.$$

and B has 0 as its I-density point from the right.

Finally, 0 is a $\mathcal{A}_I$-density point of $-A\cup A$. We shall show that it is not an I-density point of $-A\cup A$ or of $\mathbb{R}\setminus(-A\cup A)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $t_n=c_n$, $n\in\mathbb{N}$. Evidently, the sequence of characteristic functions

$$\left\{\chi_{\left(\frac{1}{t_n}\cdot A\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

converges I-almost everywhere on $[0,1]$ to χ_D.

Since $[0,1]\setminus D \in S\setminus I$, no subsequence of

$$\left\{\chi_{\left(\frac{1}{t_n}\cdot A\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

is convergent I-almost everywhere on $[0,1]$ to $\chi_{[0,1]}$, and no subsequence of

$$\left\{\chi_{\left(\frac{1}{t_n}\cdot(R\setminus A)\right)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$$

is convergent I-almost everywhere on $[0,1]$ to $\chi_{[0,1]}$.

Therefore, 0 is not an I-density point of A or of $\mathbb{R}\setminus A$ from the right. Hence 0 is not an I-density point of $-A\cup A$ or of $\mathbb{R}\setminus(-A\cup A)$. $\square$

Theorem 1. *The mapping $\Phi_{\mathcal{A}_I}: S\to 2^{\mathbb{R}}$ has the following properties:*

(0) *For each $A\in S$, $\Phi_{\mathcal{A}_I}(A)\in S$.*

(1) *For each $A\in S$, $A\sim\Phi_{\mathcal{A}_I}(A)$.*

(2) *For each $A,B\in S$, if $A\sim B$, then $\Phi_{\mathcal{A}_I}(A)=\Phi_{\mathcal{A}_I}(B)$.*

(3) *$\Phi_{\mathcal{A}_I}(\emptyset)=\emptyset$, $\Phi_{\mathcal{A}_I}(\mathbb{R})=\mathbb{R}$.*

(4) *For each $A,B\in S$, $\Phi_{\mathcal{A}_I}(A\cap B)=\Phi_{\mathcal{A}_I}(A)\cap\Phi_{\mathcal{A}_I}(B)$.*

Proof. (0) From Proposition 2 $\Phi_{\mathcal{A}_I}(A)=\Phi_I(A)\cup\big(\Phi_{\mathcal{A}_I}(A)\setminus\Phi_I(A)\big)$. The set $\big(\Phi_{\mathcal{A}_I}(A)\setminus\Phi_I(A)\big)$ is a subset of a set $\mathbb{R}\setminus\big((\Phi_I(A)\cup\Phi_I(\mathbb{R}\setminus A))\big)$ from I. Then $\Phi_{\mathcal{A}_I}(A)$ is a union of a set $\Phi_I(A)$ with the property of Baire and of a first category set, hence a set from S.

(1) It is clear, in view of $A\sim\Phi_I(A)$ (see [PWW1]) and the fact that $\Phi_{\mathcal{A}_I}(A)$ and $\Phi_I(A)$ differ by a set from I.

(2) It is a simple consequence of the fact that in the definition of $\Phi_{\mathcal{A}_I}(A)$ the I-almost everywhere convergence is involved.

(3) It is obvious.

(4) Observe first that if $A\subset B$, A, $B\in S$, then $\Phi_{\mathcal{A}_I}(A)\subset\Phi_{\mathcal{A}_I}(B)$, so $\Phi_{\mathcal{A}_I}(A\cap B)\subset\Phi_{\mathcal{A}_I}(A)\cap\Phi_{\mathcal{A}_I}(B)$. To prove the opposite inclusion, assume $x\in\Phi_{\mathcal{A}_I}(A)\cap\Phi_{\mathcal{A}_I}(B)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to zero. From $x\in\Phi_{\mathcal{A}_I}(A)$, by definition, there exist a subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ and a set $A_1\in\mathcal{A}_I$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_m}}\cdot(A-x)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{A_1\cap[-1,1]}$. Similarly, for $\{t_{n_m}\}_{m\in\mathbb{N}}$ from $x\in\Phi_{\mathcal{A}_I}(B)$, by definition, there is a subsequence $\{t_{n_{m_k}}\}_{k\in\mathbb{N}}$ and a set $B_1\in\mathcal{A}_I$ such that the sequence

$$\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot(B-x)\cap[-1,1]}\right\}_{k\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on $[-1,1]$ to $\chi_{B_1 \cap [-1,1]}$. It is clear that the sequence

$$\left\{\chi_{\frac{1}{t_{n_{m_k}}}\cdot\left((A\cap B)-x\right)\cap[-1,1]}\right\}_{k\in\mathbb{N}}$$

converges I-almost everywhere on $[-1,1]$ to $\chi_{(A_1\cap B_1)\cap[-1,1]}$, and x is a $\Phi_{\mathcal{A}_I}$-density point of $A \cap B$ since $A_1 \cap B_1 \in \mathcal{A}_I$ (see [PWW1]). □

Proposition 4. *If x is an $\mathcal{A}_I$-density point of a set A, then there does not exist a decreasing to zero sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ such that the sequence $\left\{\chi_{\frac{1}{t_n}\cdot(A-x)\cap[-1,1]}\right\}_{n\in\mathbb{N}}$ of characteristic functions converges I-almost everywhere, neither on $[-1,0]$ nor on $[0,-1]$ to 0.*

Proof. It is a simple consequence of Definition 1. □

Remark 2. It is an immediate consequence of (0), (1) and (2) from Theorem 1 that $\Phi_{\mathcal{A}_I}$ is idempotent, i.e., $\Phi_{\mathcal{A}_I}(A) = \Phi_{\mathcal{A}_I}\left(\Phi_{\mathcal{A}_I}(A)\right)$. We also have $\Phi_{\mathcal{A}_I}(A) \cap \Phi_{\mathcal{A}_I}(\mathbb{R}\setminus A) = \emptyset$.

Theorem 2. *The family $\mathcal{T}_{\mathcal{A}_I} = \left\{A \in S : A \subset \Phi_{\mathcal{A}_I}(A)\right\}$ is a stronger topology than the I-density topology $\mathcal{T}_I$.*

Proof. From Theorem 1 (3), $\emptyset$ and $\mathbb{R} \in \mathcal{T}_{\mathcal{A}_I}$, and the family is closed under finite intersections according to (4). To prove that $\mathcal{T}_{\mathcal{A}_I}$ is closed under arbitrary unions, observe that from Theorem 1, $\Phi_{\mathcal{A}_I}(A) \setminus A$ is a set from I for each $A \in S$, and then we follow the proof in [W2]. Take a family $\{A_t\}_{t\in T} \subset \mathcal{T}_{\mathcal{A}_I}$. We have $A_t \subset \Phi_{\mathcal{A}_I}(A_t)$ for each t. Choose a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that for each $t \in T$ we have $A_t \setminus \bigcup_{n=1}^{\infty} A_{t_n} \in I$. It is possible because (S, I) CCC property. Then $\Phi_{\mathcal{A}_I}(A_t) = \Phi_{\mathcal{A}_I}\left((A_t \cap \bigcup_{n=1}^{\infty} A_{t_n}) \cup (A_t - \bigcup_{n=1}^{\infty} A_{t_n})\right) \subset \Phi_{\mathcal{A}_I}(\bigcup_{n=1}^{\infty} A_{t_n})$ for each $t \in T$.

Hence

$$\bigcup_{n=1}^{\infty} A_{t_n} \subset \bigcup_{t\in T} A_t \subset \bigcup_{t\in T} \Phi_{\mathcal{A}_I}(A_t) \subset \Phi_{\mathcal{A}_I}\left(\bigcup_{n=1}^{\infty} A_{t_n}\right).$$

The first and the last set in the above sequence of inclusions differ on a set from I and both sets have the property of Baire, so $\bigcup_{t\in T} A_t \in S$. Also, $\bigcup_{t\in T} A_t \subset \Phi_{\mathcal{A}_I}\left(\bigcup_{t\in T} A_t\right)$ according to central inclusion and monotonicity of $\Phi_{\mathcal{A}_I}$ implied by (4) of Theorem 1. Hence, finally $\bigcup_{t\in T}^{\infty} A_t \in \mathcal{T}_{\mathcal{A}_I}$.

The set $(-A \cup A) \cup \{0\}$, where A is defined in Proposition 2, with D additionally open, belongs to $\mathcal{T}_{\mathcal{A}_I}$, but not to $\mathcal{T}_I$ topology. □

Remark 3. Like the I-density topology, the $\mathcal{A}_I$-density topology can be described in the form: $\mathcal{T}_{\mathcal{A}_I} = \left\{\Phi_{\mathcal{A}_I}(A) \setminus P : A \in S \text{ and } P \in \mathcal{I}\right\}$, as if $A \in \mathcal{T}_{\mathcal{A}_I}$, then $A \subset \Phi_{\mathcal{A}_I}(A)$. Consequently, $A = \Phi_{\mathcal{A}_I}(A) \setminus (\Phi_{\mathcal{A}_I}(A) \setminus A)$, and we take

$P = \Phi_{\mathcal{A}_I}(A) \setminus A \in I$. Now, if $B = \Phi_{\mathcal{A}_I}(A) \setminus P$, for some $A \in S$ and $P \in I$, then we get

$$\begin{aligned} \Phi_{\mathcal{A}_I}(B) &= \Phi_{\mathcal{A}_I}\big(\Phi_{\mathcal{A}_I}(A) \setminus P\big) \\ &= \Phi_{\mathcal{A}_I}\big(\Phi_{\mathcal{A}_I}(A)\big) \\ &= \Phi_{\mathcal{A}_I}(A) \supset \Phi_{\mathcal{A}_I}(A) \setminus P = B \end{aligned}$$

from Theorem 1 (1), (2) and the above remark.

The $\mathcal{A}_I$-density topology $\mathcal{T}_{\mathcal{A}_I}$ has similar properties to those of the I-density topology $\mathcal{T}_I$.

Theorem 3. *For an arbitrary set $A \subset \mathbb{R}$*

$$\mathrm{Int}_{\mathcal{T}_{\mathcal{A}_I}}(A) = A \cap \Phi_{\mathcal{A}_I}(B),$$

where B is an S-measurable kernel of A (i.e., $B \in S$ and $D \setminus B \in I$ for any $D \in S$, $D \subset A$).

Proof. We can follow the proof of Theorem 2.5 from [W2] here, where Φ is replaced with $\Phi_{\mathcal{A}_I}$. □

Theorem 4. *A set $A \in \mathcal{T}_{\mathcal{A}_I}$ is $\mathcal{T}_{\mathcal{A}_I}$-regular open if and only if $A = \Phi_{\mathcal{A}_I}(A)$.*

Proof. Here we can adopt the proof of Theorem 2.6 from [W2]. The inclusion $\Phi_{\mathcal{A}_I}(A) \subset Cl_{\mathcal{A}_I}(A)$ in the first part of the proof can now be verified as follows

$$\begin{aligned} Cl_{\mathcal{A}_I}(A) &= \mathbb{R} \setminus \mathrm{Int}_{\mathcal{T}_{\mathcal{A}_I}}(\mathbb{R}\setminus A) \\ &= \mathbb{R} \setminus \big((\mathbb{R} \setminus A) \cap \Phi_{\mathcal{A}_I}(\mathbb{R} \setminus A)\big) \\ &= A \cup \big(\mathbb{R} \setminus \Phi_{\mathcal{A}_I}(\mathbb{R} \setminus A)\big) \supset A \cup \Phi_{\mathcal{A}_I}(A), \end{aligned}$$

since $\Phi_{\mathcal{A}_I}(A) \subset \mathbb{R} \setminus \Phi_{\mathcal{A}_I}(\mathbb{R} \setminus A)$ by Remark 2. □

Theorem 5.

$$\begin{aligned} \mathcal{I} &= \{A \subset \mathbb{R} : A \text{ is } \mathcal{T}_{\mathcal{A}_I} \text{ — nowhere dense set}\} \\ &= \{A \subset \mathbb{R} : A \text{ is } \mathcal{T}_{\mathcal{A}_I} \text{ — first category set}\} \\ &= \{A \subset \mathbb{R} : A \text{ is } \mathcal{T}_{\mathcal{A}_I} \text{ — closed } \mathcal{T}_{\mathcal{A}_I} \text{ — discrete set}\}. \end{aligned}$$

Proof. We can follow the proofs of Theorems 2 and 4 from [PWW2] or Theorem 2.8 from [W2]. To prove the second equality we recall that every set of the second category has a subset that lacks the property of Baire [see [O]]. □

Theorem 6. *The σ-algebra of $\mathcal{T}_{\mathcal{A}_I}$-Borel sets coincides with S.*

If $E \subset \mathbb{R}$ is $\mathcal{T}_{\mathcal{A}_I}$-compact set, then E is finite.

The space $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_I})$ is neither first countable nor second countable, Lindelöf, and separable.

$(\mathbb{R}, \mathcal{T}_{\mathcal{A}_I})$ is a Baire space.

P r o o f. We can follow the proofs of Theorem 3 of [PWW2] and Theorems 2.9–2.12 from [W2]. □

Remark 4. In the proof of the above theorems we have used a classical argument referring only to results for Lebesgue density topology from [W1] and [W2] and for I-density topology from [PWW2] and [PWW1]. However, since $\mathcal{T}_{\mathcal{A}_I} \subset S$ and $\Phi_{\mathcal{A}_I}$ is a closed lower density operator (i.e., $\Phi_{\mathcal{A}_I}(A) \in S$) we could rely on more recent results from [RJH] given in more general settings.

We shall consider some properties of continuous functions from $(\mathbb{R}, \mathcal{T}_{\mathcal{A}_I})$ into $(\mathbb{R}, \mathcal{T}_n)$ now.

Definition 2. We say that a real variable function f is topologically $\mathcal{T}_{\mathcal{A}_I}$-approximately continuous at a point x_0 if and only if for every number $\varepsilon > 0$, $\{x : |f(x) - y| < \epsilon\}$ there is a $\mathcal{T}_{\mathcal{A}_I}$-neighborhood of x, i.e., there exists a set $A_x \in \mathcal{S}$, $A_x \subset \{x : |f(x) - y| < \epsilon\}$ such that x is an $\mathcal{A}_I$-density point of A_x.

Definition 3. We say that a real variable function f is restrictively $\mathcal{T}_{\mathcal{A}_I}$-approximately continuous at a point x_0 if and only if there exists a set $A_{x_0} \in S$ such that

$$x_0 \in \Phi_{\mathcal{A}_I}(A_{x_0}) \quad \text{and} \quad f(x_0) = \lim_{x \to x_0 x \in A_{x_0}} f(x).$$

Theorem 7. *(i) A real function f defined on $\mathbb{R}$ has the property of Baire if and only if it is $\mathcal{T}_{\mathcal{A}_I}$-topologically continuous I-almost everywhere on $\mathbb{R}$.*

(ii) Every $\mathcal{T}_I$-topologically continuous function is $\mathcal{T}_{\mathcal{A}_I}$-topologically continuous, the converse does not hold.

P r o o f. (i) Suppose that f defined on $\mathbb{R}$ has the property of Baire. Then, by Theorem 7 from [PWW1], it is $\mathcal{T}_I$-continuous almost everywhere on $\mathbb{R}$; hence $\mathcal{T}_{\mathcal{A}_I}$-continuous almost everywhere on $\mathbb{R}$ since $\mathcal{T}_{\mathcal{I}} \subset \mathcal{T}_{\mathcal{A}_I}$.

Now, suppose that f is $\mathcal{T}_{\mathcal{A}_I}$-continuous I-almost everywhere. Let $a, b \in \mathbb{R}$, and $B = \{x : a < f(x) < b\}$. We shall show that B has the property of Baire. Let C be the set of $\mathcal{T}_{\mathcal{A}_I}$-continuity points of f.

We have $B = (B \cap C) \cup (B \setminus C)$ and $B \setminus C \in I$. The proof is completed by showing that $B \cap C$ has the property of Baire. If $x \in B \cap C$, and $y = f(x)$, we take $\epsilon > 0$, $\epsilon < \min(b - y, y - a)$. Then $\{x : |f(x) - y| < \epsilon\}$ is a $\mathcal{T}_{\mathcal{A}_I}$-neighborhood of x, i.e., there exists a set $A_x \in \mathcal{S}$, $A_x \subset \{x : |f(x) - y| < \epsilon\}$ such that x is a $\mathcal{A}_I$-density point of A_x. Of course, $A_x \subset B$, and we may assume $A_x \subset (B \cap C)$, by Theorem 1 (2), since $B \setminus C \in I$.

Finally, we obtain $B \cap C = \bigcup_{x \in B \cap C} A_x \in \mathcal{T}_{\mathcal{A}_I} \subset S$.

(ii) The first part is again a consequence of $\mathcal{T} \subset \mathcal{T}_{\mathcal{A}_d}$. The characteristic function of the set $(-A \cup A) \cup \{0\}$, where A is defined as in Proposition 2, is $\mathcal{T}_{\mathcal{A}_I}$-topologically continuous but not $\mathcal{T}_I$-topologically continuous at 0. □

Corollary 1. *For every measurable real function f, the set of $\mathcal{T}_{\mathcal{A}_I}$-topological continuity points and the set of $\mathcal{T}_I$-topological continuity points may differ by a set from I.*

Remark 5. In the proof of part (i) of the above theorem we have used a classical argument referring only to results from [PWW1] and to the inclusion $\mathcal{T}_I \subset \mathcal{T}_{\mathcal{A}_I}$. However, since $\mathcal{T}_{\mathcal{A}_I} \subset S$ and $\Phi_{\mathcal{A}_I}$ is a lower density operator, we could rely on Theorem 6.39 from [LMZ] or use the recent results of Bartoszewicz and Kotlicka given in more general settings (see [BK] Theorem 2.2).

Proposition 5. *There exists a function that is (right) $\mathcal{T}_{\mathcal{A}_I}$-topologically, but not $\mathcal{T}_{\mathcal{A}_I}$-restrictively continuous at zero.*

Proof. We shall start with the continuity at zero from the right. Let $\{c_n\}_{n\in\mathbb{N}}$ be a decreasing to zero sequence of real numbers such that $c_{n+1} < \frac{1}{4^n}c_n$, $c_1 = 1$.

Define

$$f(x) = \sum_{n=1}^{\infty}\left(\sum_{i=0}^{\infty}\frac{1}{2^i}\chi_{\left(\frac{c_n}{2^{i+1}},\frac{c_n}{2^i}\right]}(x)\right)\chi_{(c_{n+1},c_n]}(x)$$

for $x \in (0,1]$ and $f(0) = 0$.

Equivalently, put

$$g(x) = \sum_{i=0}^{\infty}\frac{1}{2^i}\chi_{\left(\frac{1}{2^{i+1}},\frac{1}{2^i}\right]}(x)$$

and define

$$f(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{c_n}x\right)\chi_{(c_{n+1},c_n]}(x)$$

for $x \in (0,1]$ and $f(0) = 0$.

The function f is right $\mathcal{T}_{\mathcal{A}_I}$-topologically continuous at zero.

Really, consider the set

$$E_k = \left\{x \in [0,1] : |f(x) - 0| \le \frac{1}{2^k}\right\}.$$

By definition of f,

$$E_k = \bigcup_{n\in\mathbb{N}}\left(c_{n+1}, \frac{c_n}{2^k}\right]$$

and it is a simple observation that for every $k \in \mathbb{N}$, E_k has 0 as the $\mathcal{A}_I$-density point (even $\mathcal{A}_{I[-1,1]}$-density point).

The function f is not right $\mathcal{T}_{\mathcal{A}_I}$-restrictively continuous at 0.

Suppose, on the contrary to our claim, that there exists a set $E \in \mathcal{A}_I$ such that $\lim_{x\in E,\ x\to 0} f(x) = 0$.

Let $t_n = c_n$, $n \in \mathbb{N}$. Then, since $E \in \mathcal{A}_I$, we can find its subsequence $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that the sequence

$$\left\{\chi_{\left(\frac{1}{t_{n_m}}\cdot E\right)\cap[-1,1]}\right\}_{m\in\mathbb{N}}$$

of characteristic functions converges I-almost everywhere on $[0,1]$ to $\chi_{[0,1]}$.

On the other side, since $\lim_{x\in E,\ x\to 0} f(x) = 0$, we can find $c > 0$ such that $f(x) < \frac{1}{2^k}$ for all $x \in E \cap (0,c)$. Hence, $E \cap (0,c) \subset \left\{x \ : \ f(x) < \frac{1}{2^k}\right\}$.

We take a subsequence $\left\{t_{n_{m_k}}\right\}_{k\in\mathbb{N}}$ of $\{t_{n_m}\}_{m\in\mathbb{N}}$ such that $f(x) < \frac{1}{2^k}$, $k \in \mathbb{N}$ on $E \cap \left(0, t_{n_{m_k}}\right)$. By the definition of f, for every $k \in \mathbb{N}$ we have $\left(\frac{1}{t_{n_{m_k}}} \cdot E\right) \subset (0, \frac{1}{2^{k+1}})$. Thus, it is clear that the sequence

$$\left\{\chi_{\left(\frac{1}{t_{n_{m_k}}}\cdot E\right)\cap[0,1]}\right\}_{k\in\mathbb{N}}$$

of characteristic functions converges even everywhere on $(0,1]$ to 0. Since $E \in \mathcal{A}_I$, this is in contradiction with Proposition 1.

Now, the function

$$h(x) = \begin{cases} f(x), & x > 0, \\ 0, & x = 0, \\ f(-x). & x < 0 \end{cases}$$

is $\mathcal{T}_{\mathcal{A}_I}$-topologically but not $\mathcal{T}_{\mathcal{A}_I}$-restrictively continuous at zero. □

Definition 4. We say that the sets A and B are essentially different if for every $t, \alpha \in R$ the set $\left(A \triangle (t \cdot B)\right) \cap [0, \alpha]$ is not from I.

Proposition 6. *There exists a set $A \subset [0,1]$ such that zero is an $\mathcal{A}_I$-right density point of A and such that there are: a decreasing to zero sequence of real numbers $\{t_n\}_{n\in\mathbb{N}}$ and* **c** *essentially different sets associated with different subsequences $\left\{\frac{1}{t_{n_m}}\right\}_{m\in\mathbb{N}}$ in Definition 1.*

P r o o f. Let $\{w_i\}_{i\in\mathbb{N}}$ be a sequence of all rational numbers from interval $\left(\frac{1}{2}, 1\right)$. Let $\{c_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers decreasing to 0, $c < 1$, such that $\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$. Put $D_i = [0, \frac{1}{2}] \cup (w_i, 1]$. We define a set A:

$$A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} \left(c_{\frac{n(n-1)}{2}+i} \cdot D_i\right) \cap \left(c_{\frac{n(n-1)}{2}+i+1}, c_{\frac{n(n-1)}{2}+i}\right).$$

Each natural number k can be uniquely presented as a sum $k = \frac{n(n-1)}{2} + i$, where $n \in \mathbb{N}$ and $i = 1, 2, \ldots, \left(\frac{(n+1)n}{2} - \frac{n(n-1)}{2}\right) = n$. We denote i as a function of k, i.e., $i(k)$. In particular, we have $\frac{(n+1)n}{2} = \frac{n(n-1)}{2} + n$ and $i\big(\frac{(n+1)n}{2}\big) = n$. We may rewrite the definition of the set A as

$$A = \bigcup_{k=1}^{\infty} \big(c_k \cdot D_{i(k)}\big) \cap (c_{k+1}, c_k)\,.$$

The $i(k)$, as a function of k, takes the following values, consecutively: 1,1,2, 1,2,3, 1,2,3,4,... We shall show that zero is an $\mathcal{A}_I$-density point of $A \cup (-A)$.

Suppose that $\{t_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence of real numbers decreasing to zero. Similarly, as it in described in the proof of Proposition 2, we choose two subsequences $\{t_{n_r}\}_{m\in\mathbb{N}}$ and $\{c_{m_r}\}_{m\in\mathbb{N}}$ such that $c_{m_r} \le t_{n_r}$, $r \in \mathbb{N}$ and there are neither elements of $\{c_m\}_{m\in\mathbb{N}}$ nor of $\{t_n\}_{n\in\mathbb{N}}$ between c_{m_r} and t_{n_r}. Again, we consider the sequence $\{\frac{c_{m_r}}{t_{n_r}}\}_{r\in\mathbb{N}}$ and find a subsequence $\{\frac{c_{m_{r_k}}}{t_{n_{r_k}}}\}_{k\in\mathbb{N}}$ convergent to some $a \in [0,1]$.

There are two possible situations:

a) $\lim_{k\to\infty}\Big(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\Big) = a \neq 0$, i.e., $\lim_{k\to\infty}\Big(c_{m_{r_k}} \cdot \frac{1}{a t_{n_{r_k}}}\Big) = 1$.
In this case, we consider the behaviour of the sequence $c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}} w_{i(m_{r_k})}$. Since it is bounded, it contains a subsequence $c_{m_{r_{k_p}}} \cdot \frac{1}{t_{n_{r_{k_p}}}} w_{i(m_{r_{k_p}})}$ convergent to some $c \in a \cdot \big[\frac{1}{2}, 1\big]$, and $\chi_{\big(\frac{1}{t_{n_{r_{k_p}}}} \cdot A\big) \cap [0,a]}$ converges I-a.e. to $\chi_{a\cdot\left([0,\frac{1}{2}] \cup [c,1]\right)}$.

Thus, we obtain B on $[0,a]$ as

$$B \cap [0,a] = a \cdot \left(\left[0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right]\right).$$

If $a = 1$, the proof is complete; $B \in \mathcal{A}_{I[-\alpha,\alpha]} \subset \mathcal{A}_I$.
If $a < 1$, like in the proof of Proposition 2 we obtain

$$\begin{aligned} B \cap [0,a] &= a \cdot \left(\left[0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right]\right), \\ B \cap (a,1] &= (a,1]. \end{aligned}$$

And again, $B \in \mathcal{A}_{I[-\alpha,\alpha]} \subset \mathcal{A}_I$.

b) $\lim_{k\to\infty}\big(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\big) = 0$.
In this case, we have two possible situations again:

b1) the sequence $\{\frac{c_{m_{r_k}-1}}{t_{n_{r_k}}}\}_{k\in\mathbb{N}}$ is bounded from above. We take its subsequence $\{\frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}}\}_{p\in\mathbb{N}}$ such that $\lim_{p\to\infty} \frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}} = b < \infty$, and proceed similarly, as in a). We find a subsequence $c_{m_{r_{k_{p_s}}}-1} \cdot \frac{1}{t_{n_{r_{k_{p_s}}}}} w_{i(m_{r_{k_{p_s}}}-1)}$

of $c_{m_{r_{k_p}}-1} \cdot \frac{1}{t_{n_{r_{k_p}}}} w_{i(m_{r_{k_p}}-1)}$ convergent to some $c \leq b$ and obtain I-a.e. convergence of $\chi_{\left(\frac{1}{t_{n_{r_{k_{p_s}}}}} \cdot A\right) \cap [0,b]}$ to $\chi_{b \cdot \left([0,\frac{1}{2}] \cup [\frac{c}{b},1]\right)}$. Thus, we obtain the set B as

$$B = \left[b \cdot \left(\left[0, \frac{1}{2} \right] \cup \left[\frac{c}{b}, 1 \right] \right) \right] \cap [0,1] .$$

And again, $B \in \mathcal{A}_{I[-\alpha,\alpha]} \subset \mathcal{A}_I$.

b2) the sequence $\left\{ \frac{c_{m_{r_k}-1}}{t_{n_{r_k}}} \right\}_{k \in \mathbb{N}}$ is not bounded from above. We take its subsequence $\left\{ \frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}} \right\}_{p \in \mathbb{N}}$ such that $\lim_{p \to \infty} \frac{c_{m_{r_{k_p}}-1}}{t_{n_{r_{k_p}}}} = \infty$. As every D_i contains the interval $\left[0, \frac{1}{2}\right]$, therefore we have $[0,1] \subset \frac{1}{t_{n_{r_{k_p}}}} \cdot A$, for p appropriately large, and the sequence $\chi_{\left(\frac{1}{t_{n_{r_{k_p}}}} \cdot A\right)}$ converges to $\chi_{[0,1]}$ a.e. on $[0,1]$. And we obtain B on $[0,1]$ as

$$B \cap [0,1] = [0,1] .$$

And, again, $B \in \mathcal{A}_{I[-\alpha,\alpha]} \subset \mathcal{A}_I$.

Finally, zero is an $\mathcal{A}_I$-density point of $(-A \cup A)$.

Now, let $d \in \left[\frac{1}{2}, 1\right]$ and $\{w_{n_i}\}_{i \in \mathbb{N}}$ be a subsequence of $\{w_n\}_{n \in \mathbb{N}}$ convergent to d. As a sequence $\{t_n\}_{n \in \mathbb{N}}$ we take $\{c_n\}_{n \in \mathbb{N}}$. The set $[0, \frac{1}{2}] \cup (d,1] \in \mathcal{A}_I$ is associated with the subsequence $\left\{ c_{\frac{(n_i+1)n_i}{2}} \right\}_{i \in \mathbb{N}}$ and we obtain the sequence of characteristic functions

$$\chi_{\left(\left(\left(\frac{1}{c_{\frac{(n_i+1)n_i}{2}}} \right) \cdot A \right) \cap [0,1] \right)}$$

convergent I-a.e. to $\chi_{\left[0,\frac{1}{2}\right] \cup [d,1]}$ on $[0,1]$. □

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