A CATEGORY ANALOGUE OF THE GENERALIZATION OF LEBESGUE DENSITY TOPOLOGY

WOJCIECH WOJDOWSKI

ABSTRACT. A notion of $A_I$-topology, a generalization of Wilczyński’s $I$-density topology (see [Wilczyński, W.: A generalization of the density topology, Real. Anal. Exchange 8 (1982-1983), 16–20] is introduced. The notion is based on his reformulation of the definition of Lebesgue density point. We consider a category version of the topology, which is a category analogue of the notion of an $A_d$-density topology on the real line given in [Wojdowski, W.: A generalization of density topology, Real. Anal. Exchange 32 (2006/2007), 1–10]. We also discuss the properties of continuous functions with respect to the topology.

Let $S$ be a $\sigma$-algebra of subsets of the real line $\mathbb{R}$, and $I \subset S$ a proper $\sigma$-ideal. We shall say that the sets $A, B \in S$ are equivalent ($A \sim B$) if and only if $A \Delta B \in I$. We will denote by $\lambda$ the Lebesgue measure on the real line.

Let us recall that the point $x \in \mathbb{R}$ is said to be a Lebesgue density point of a measurable set $A$, if

$$\lim_{h \to 0} \frac{\lambda(A \cap [x-h, x+h])}{2h} = 1. \quad (*)$$

W. Wilczyński [W1] gave his reformulation of the notion of the density point of a measurable set $A$, in terms of convergence almost everywhere of the sequence of characteristic functions of dilations of a set $A$:

A point $x \in \mathbb{R}$ is Lebesgue density point of a measurable set $A$ if and only if every subsequence

$$\left\{ \chi_{(n \cdot (A-x)) \cap [-1,1]} \right\}_{m \in \mathbb{N}} \quad \text{of} \quad \left\{ \chi_{(n \cdot (A-x)) \cap [-1,1]} \right\}_{n \in \mathbb{N}}$$

contains a subsequence

$$\left\{ \chi_{(n_{mp} \cdot (A-x)) \cap [-1,1]} \right\}_{p \in \mathbb{N}}$$

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which converges to \( \chi[-1,1] \) almost everywhere on \([-1,1]\) (which means except on a null set).

Wilczynski’s approach relieved definition of the notion of a measure. His definition requires only null sets. Instead of the notion of convergence in measure of a sequence of measurable functions, he uses the convergence almost everywhere. This opened a new space for studying of more subtle properties of the notion of the Lebesgue density point and density topology, their various modifications, and, most of all, the category analogues (see \[PWW1\], \[PWW2\], \[CLO\]).

The reformulated definition could be considered in more general settings as follows:

A point \( x \in \mathbb{R} \) is an \( I \)-density point of a set \( A \in S \), if every subsequence

\[
\left\{ \chi\left( n \cdot (A-x) \right) \cap [-1,1] \right\}_{n \in \mathbb{N}}
\]
contains a subsequence

\[
\left\{ \chi\left( n \cdot (A-x) \right) \cap [-1,1] \right\}_{n \in \mathbb{N}}
\]

which converges to \( \chi[-1,1] \) \( I \)-almost everywhere on \([-1,1]\) (which means except from a set belonging to \( I \)).

In \[PWW2\] Corollary 1, p. 556 in the category case, and in \[W2\] in the measure case, it is proved that the following conditions are equivalent:

1. \( x \) is an \( I \)-density point of a set \( A \in S \).
2. For any decreasing to zero sequence of real numbers \( \{t_n\}_{n \in \mathbb{N}} \), there exists its subsequence \( \{t_{n_m}\}_{m \in \mathbb{N}} \) such that the sequence

\[
\left\{ \chi\left( t_{n_m} \cdot (A-x) \cap [-1,1] \right) \right\}_{m \in \mathbb{N}}
\]

of characteristic functions converges \( I \)-almost everywhere on \([-1,1]\) to \( \chi[-1,1] \).
3. Given \( \{t_n\}_{n \in \mathbb{N}} \), a decreasing to zero sequence of real numbers fulfilling condition \( \sup_{n \to \infty} \frac{t_{n+1}}{t_{n}} < \infty \), every subsequence \( \left\{ \frac{1}{t_{n_m}} \right\}_{m \in \mathbb{N}} \) of \( \left\{ \frac{1}{t_n} \right\}_{n \in \mathbb{N}} \) contains a subsequence \( \left\{ \frac{1}{t_{n_m}} \right\}_{m \in \mathbb{N}} \) such that

\[
\left\{ \chi\left( t_{n_m} \cdot (A-x) \cap [-1,1] \right) \right\}_{m \in \mathbb{N}}
\]

converges to \( \chi[-1,1] \) \( I \)-almost everywhere on \([-1,1]\).

Following Wilczynski’s approach in \[WO1\] we have introduced a notion of \( \mathcal{A}_d \)-density of a Lebesgue measurable set leading to a notion of \( T_{\mathcal{A}_d} \) topology on the real line stronger than the Lebesgue density topology. The generalization
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was related to a given appropriate family of subsets of \([-1, 1]\), namely the family of measurable sets having density one at zero.

Now, we shall formulate a generalization of the definition of \(A_d\)-density point and then consider its category analogue.

We shall consider the following families of sets:

a) \(F_{[-1,1]}\) the family of ((\(S, I\))-residual) subsets of interval \([-1, 1]\) (i.e., \(F_{[-1,1]} \subset S\) and for \(A \in F_{[-1,1]}\) we have \([-1, 1] \setminus A \in I\)),

b) \(F_{[-\alpha, \alpha]}\) the family of subsets of interval \([-1, 1]\) such that \(F_{[-1,1]} \subset S\) and \([-\alpha, \alpha] \setminus A \in I\), for some \(0 < \alpha \leq 1\) (i.e., \((S, I)\)-residual on \([-\alpha, \alpha]\), where \(0 < \alpha \leq 1\)),

c) \(F_I\) the family of subsets of interval \([-1, 1]\) from \(S\) having 0 as its \(I\)-density point.

We have \(F_{[-1,1]} \subset F_{[-\alpha, \alpha]} \subset F_I\).

**Definition 1.** We shall say that \(x\) is an \(F_I\)-density point of \(A \in S\), if for any sequence of real numbers \(\{t_n\}_{n \in \mathbb{N}}\), decreasing to zero, there exists a subsequence \(\{t_{n_m}\}_{m \in \mathbb{N}}\) and a set \(B \in F_I\) such that the sequence

\[
\left\{ \chi_{\frac{1}{t_{n_m}}(A-x) \cap [-1,1]} \right\}_{m \in \mathbb{N}}
\]

of characteristic functions converges \(I\)-almost everywhere on \([-1, 1]\) to \(\chi_B\).

By analogy, we define a notion of \(F_{[-\alpha, \alpha]}\)-density point and \(F_{[-1,1]}\)-density point of \(A \in S\). The family \(F_{[-1,1]}\) corresponds precisely to the definition of \(I\)-density point of a set \(A \in S\). The set of all \(F_I\)-density points, \(F_{[-\alpha, \alpha]}\)-density points and \(I\)-density points of \(A \in S\) will be denoted by \(\Phi_{F_I}(A)\), \(\Phi_{F_{[-\alpha, \alpha]}}(A)\) and \(\Phi_I(A)\), respectively. Obviously, \(\Phi_{F_{[-1,1]}}(A) = \Phi_I(A)\).

**Proposition 1.** Let us observe that if \(x\) is an \(F_I\)-density point of \(A \in S\), there is no decreasing to zero sequence \(\{t_n\}_{n \in \mathbb{N}}\) such that the sequence

\[
\left\{ \chi_{\frac{1}{n}(A-x) \cap [-1,1]} \right\}_{n \in \mathbb{N}}
\]

of characteristic functions converges \(I\)-almost everywhere on \([-1, 1]\) to 0.

**Proof.** It is a simple consequence of the definition. \(\square\)

**Proposition 2.** For each \(A \in S\), \(\Phi_I(A) \subset \Phi_{F_{[-\alpha, \alpha]}}(A) \subset \Phi_{F_I}(A)\).

**Proof.** It is obvious. \(\square\)

From now on, we shall consider a particular case; an \(S\) stands for the \(\sigma\)-algebra of subsets of the real line \(\mathbb{R}\) with the Baire property and \(I \subset S\) is the \(\sigma\)-ideal of the sets of first category. The families \(F_{[-1,1]}, F_{[-\alpha, \alpha]}\) and \(F_I\) will be denoted
Proof. We shall start with the notion of density from the right. We shall define a sequence of positive numbers converging to 1 for each \(a_n\) decreasing to 0.

Remark 1. Let us observe that for any sets \(A, B \in S\), such that \(A \Delta B \in I\), the I-a.e. convergence of the sequence of characteristic functions \(\{\chi_{a_n \cdot A}\}_{n \in \mathbb{N}}\) on \([-1, 1]\), for each \(a_n\), \(n \in \mathbb{N}\), to \(\chi_A\) is equivalent to the convergence of the sequence of characteristic functions \(\{\chi_{a_n \cdot B}\}_{n \in \mathbb{N}}\) on \([-1, 1]\) to \(\chi_A\).

**Remark 1**. It follows that we may equivalently replace the set \(A\) in thesis with its regular open representation \(G(A)\). This finishes the proof. \(\Box\)

**Proposition 3.** There exists a set \(A\) such that \(\Phi_I(A) \subseteq \Phi_{A_I}(A)\).

**Proof.** We shall start with the notion of density from the right. We shall define a set \(A\) such that:

1) 0 is not an \(I\)-density point of \(A\) from the right,
2) 0 is not an \(I\)-density point of \(\mathbb{R} - A\) from the right,
3) 0 is an \(A_I\)-density point of \(A\) from the right.

Let \(D \in A_I\) be a set such that \([0, 1] \setminus D \in S \setminus I\), and \(\{c_n\}_{n \in \mathbb{N}}\) be an arbitrary sequence of real numbers decreasing to 0, \(c_1 < 1\), such that \(\lim_{n \to \infty} \frac{c_{n+1}}{c_n} = 0\). We define a set \(A \in S\) as

\[
A = \bigcup_{n=1}^{\infty} [(c_n \cdot D) \cap (c_{n+1}, c_n)].
\]

Now, let \(\{t_n\}_{n \in \mathbb{N}}\) be an arbitrary sequence of real numbers decreasing to zero. We can find the subsequences \(\{t_{n_r}\}_{r \in \mathbb{N}}\) and \(\{c_{m_r}\}_{r \in \mathbb{N}}\) of \(\{t_n\}_{n \in \mathbb{N}}\) and \(\{c_n\}_{n \in \mathbb{N}}\), respectively, and there are neither elements of \(\{t_n\}_{n \in \mathbb{N}}\) nor of \(\{c_n\}_{n \in \mathbb{N}}\) between \(c_{m_r}\) and \(t_{n_r}\).

Consider the sequence \(\{c_{m_r} \cdot \frac{1}{t_{n_r}}\}_{r \in \mathbb{N}} \subset (0, 1]\). We can find a convergent subsequence \(\{c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\}_{k \in \mathbb{N}'}\).

There are two possibilities:

a) \(\lim_{k \to \infty} \left(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\right) = a \neq 0\); i.e., \(\lim_{k \to \infty} \left(c_{m_{r_k}} \cdot \frac{1}{t_{n_{r_k}}}\right) = 1\) and
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b) \( \lim_{k \to \infty} \left( c_{m_k} \cdot \frac{1}{t_{m_k}} \right) = 0 \).

In case a)

\[
\left\{ \chi_{c_{m_k}t_{m_k}} \cdot \left( \frac{c_{m_k}+1}{c_{m_k}} \cdot 1 \right) \right\}_{k \in \mathbb{N}}
\]
converges \( I \)-almost everywhere on \((0, 1)\) to \( \chi_{[0,1]} \), and by Lemma 1

\[
\left\{ \chi_{c_{m_k}t_{m_k}} \cdot \left( \frac{c_{m_k}+1}{c_{m_k}} \cdot 1 \right) \cap D \right\}_{k \in \mathbb{N}}
\]
converges \( I \)-almost everywhere to \( \chi_D \) on \([0,1]\). Equivalently,

\[
\left\{ \chi_{\frac{1}{t_{m_k}}} \cdot \left( \left( c_{m_k+1}c_{m_k} \right) \cap \left( c_{m_k} \cdot D \right) \right) \right\}_{k \in \mathbb{N}}
\]
converges \( I \)-almost everywhere to \( \chi_D \) on \([0,1]\). Thus, since

\[
\left( c_{m_k}+1, c_{m_k} \right) \cap \left( c_{m_k} \cdot D \right) = \left( c_{m_k}+1, c_{m_k} \right) \cap A,
\]
the sequence \( \chi_{\frac{1}{t_{m_k}}} \cdot \left( \left( c_{m_k+1}c_{m_k} \right) \cap \left( c_{m_k} \cdot A \right) \right) \cap [0,1] \) converges \( I \)-almost everywhere to \( \chi_D \) on \([0,1]\), and consequently, \( \chi_{\frac{1}{t_{m_k}}} \cdot \left( \left( c_{m_k+1}c_{m_k} \right) \cap \left( c_{m_k} \cdot A \right) \right) \cap [0,a] \) converges \( I \)-almost everywhere to \( \chi_{(a \cdot D)} \cap [0,a] \) on \([0,a]\).

Thus, we obtain \( B \) on \([0,a]\) as

\[
B \cap [0,a] = (a \cdot D) \cap [0,a].
\]

If \( a = 1 \), the proof is complete; 0 is an \( A_I \)-density point of \( B = D \cap [0,1] \) from the right.

If \( a < 1 \), we have to determine \( B \) on \((a, 1]\) as well.

By definition of \( \{c_n\}_{n \in \mathbb{N}} \), we have \( \lim_{k \to \infty} \frac{c_{m_k}-1}{t_{n_k}} = \infty \). Actually,

\[
\lim_{k \to \infty} \frac{c_{m_k}-1}{t_{n_k}} = \lim_{k \to \infty} \left( \frac{c_{m_k}-1}{t_{n_k}} \cdot \frac{c_{m_k}}{c_{m_k}} \right)
\]

\[
= \lim_{k \to \infty} \left( \frac{c_{m_k}-1}{c_{m_k}} \cdot \frac{c_{m_k}}{t_{n_k}} \right)
\]

\[
= a \cdot \lim_{k \to \infty} \frac{c_{m_k}-1}{c_{m_k}} = \infty.
\]
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Hence, because $D$ has 0 as its $I$-density point from the right, we can find a subsequence $\left\{ \frac{1}{t_{nrkp}} \right\}_{p \in \mathbb{N}}$ of $\left\{ \frac{1}{t_{nk}} \right\}_{k \in \mathbb{N}}$ such that the sequence $\chi(\frac{1}{t_{nrkp}}) \cap (a,1]$ converges to $\chi(a,1]$ $I$-almost everywhere on $(a,1]$, since

$$
(c_{m_{rkp}}, c_{m_{rkp}} - 1) \cap (c_{m_{rkp}} - 1 \cdot D) = (c_{m_{rkp}}, c_{m_{rkp}} - 1) \cap A
$$

and $\lim_{k \to \infty} \frac{c_{m_{rkp}} - 1}{t_{nrkp}} = \infty$ and $\lim_{k \to \infty} \frac{c_{m_{rkp}}}{t_{nrkp}} = a$.

Hence, we determine $B$ on $[0,1]$ as

$$
B \cap [0,a] = (a \cdot D) \cap [0,a] \quad \text{and} \quad B \cap (a,1] = (a,1].
$$

In case b), i.e., $\lim_{k \to \infty} (c_{m_{rk}} \cdot \frac{1}{t_{nrk}}) = 0$, we have two possibilities again:

b1) The sequence $\left\{ \frac{c_{m_{rkp}} - 1}{t_{nrkp}} \right\}_{k \in \mathbb{N}}$ is bounded from above.

We take a subsequence $\left\{ \frac{c_{m_{rkp}} - 1}{t_{nrkp}} \right\}_{p \in \mathbb{N}}$ such that $\lim_{p \to \infty} \frac{c_{m_{rkp}} - 1}{t_{nrkp}} = b < \infty$, and proceed similarly to the argument in a). We have $b \geq 1$, and $\chi(\frac{1}{t_{nrkp}}) \cap [0,1]$ converges $I$-almost everywhere to $\chi(b \cdot D) \cap [0,1]$ on $[0,1]$, and we obtain $B$ on $[0,1]$, as

$$
B \cap [0,1] = (b \cdot D) \cap [0,1].
$$

b2) The sequence $\left\{ \frac{c_{m_{rkp}} - 1}{t_{nrkp}} \right\}_{k \in \mathbb{N}}$ is not bounded from above. We take a subsequence $\left\{ \frac{c_{m_{rkp}} - 1}{t_{nrkp}} \right\}_{p \in \mathbb{N}}$ such that $\lim_{p \to \infty} \frac{c_{m_{rkp}} - 1}{t_{nrkp}} = \infty$. As $D$ has 0 as its $I$-density point from the right and $\lim_{p \to \infty} (c_{m_{rkp}} \cdot \frac{1}{t_{nrkp}}) = 0$, we can find a subsequence $\left\{ \frac{1}{t_{nrkp_p}} \right\}_{p \in \mathbb{N}}$ of $\left\{ \frac{1}{t_{nrkp}} \right\}_{p \in \mathbb{N}}$ such that the sequence $\chi(\frac{1}{t_{nrkp_p}}) \cap [0,1]$ converges to $\chi[0,1]$ $I$-almost everywhere on $[0,1]$, and we determine $B$ on $[0,1]$ as

$$
B \cap [0,1] = [0,1].
$$

and $B$ has 0 as its $I$-density point from the right.

Finally, 0 is a $A_t$-density point of $-A \cup A$. We shall show that it is not an $I$-density point of $-A \cup A$ or of $\mathbb{R} \setminus (-A \cup A)$. Let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers such that $t_n = c_n$, $n \in \mathbb{N}$. Evidently, the sequence of characteristic functions

$$
\left\{ \chi(\frac{1}{t_n} \cdot A) \cap [-1,1] \right\}_{n \in \mathbb{N}}
$$

converges $I$-almost everywhere on $[0,1]$ to $\chi_D$. 16
Since \([0, 1] \setminus D \subseteq S \setminus I\), no subsequence of
\[
\left\{ \chi\left(\frac{1}{n^2}A\right) \cap [-1, 1] \right\}_{n \in \mathbb{N}}
\]
is convergent \(I\)-almost everywhere on \([0, 1]\) to \(\chi_{[0,1]}\), and no subsequence of
\[
\left\{ \chi\left(\frac{1}{n}(R \setminus A)\right) \cap [-1, 1] \right\}_{n \in \mathbb{N}}
\]
is convergent \(I\)-almost everywhere on \([0, 1]\) to \(\chi_{[0,1]}\).

Therefore, 0 is not an \(I\)-density point of \(A\) or of \(\mathbb{R} \setminus A\) from the right. Hence 0 is not an \(I\)-density point of \(\overline{A} \cup A\) or of \(\mathbb{R} \setminus (\overline{A} \cup A)\).

\[\square\]

**Theorem 1.** The mapping \(\Phi_{A_I} : S \to 2^\mathbb{R}\) has the following properties:

(0) For each \(A \in S\), \(\Phi_{A_I} (A) \in S\).

(1) For each \(A \in S\), \(A \sim \Phi_{A_I} (A)\).

(2) For each \(A, B \in S\), if \(A \sim B\), then \(\Phi_{A_I} (A) = \Phi_{A_I} (B)\).

(3) \(\Phi_{A_I} (\emptyset) = \emptyset\), \(\Phi_{A_I} (\mathbb{R}) = \mathbb{R}\).

(4) For each \(A, B \in S\), \(\Phi_{A_I} (A \cap B) = \Phi_{A_I} (A) \cap \Phi_{A_I} (B)\).

**Proof.** (0) From Proposition 2 \(\Phi_{A_I} (A) = \Phi_I (A) \cup \left(\Phi_{A_I} (A) \setminus \Phi_I (A)\right)\). The set \(\left(\Phi_{A_I} (A) \setminus \Phi_I (A)\right)\) is a subset of a set \(\mathbb{R} \setminus \left(\left(\Phi_I (A) \cup \Phi_I (\mathbb{R} \setminus A)\right)\right)\) from \(I\). Then \(\Phi_{A_I} (A)\) is a union of a set \(\Phi_I (A)\) with the property of Baire and of a first category set, hence a set from \(S\).

(1) It is clear, in view of \(A \sim \Phi_I (A)\) (see \([PWW1]\)) and the fact that \(\Phi_{A_I} (A)\) and \(\Phi_I (A)\) differ by a set from \(I\).

(2) It is a simple consequence of the fact that in the definition of \(\Phi_{A_I} (A)\) the \(I\)-almost everywhere convergence is involved.

(3) It is obvious.

(4) Observe first that if \(A \subset B\), \(A, B \in S\), then \(\Phi_{A_I} (A) \subset \Phi_{A_I} (B)\), so \(\Phi_{A_I} (A \cap B) \subset \Phi_{A_I} (A) \cap \Phi_{A_I} (B)\). To prove the opposite inclusion, assume \(x \in \Phi_{A_I} (A) \cap \Phi_{A_I} (B)\). Let \(\{t_n\}_{n \in \mathbb{N}}\) be an arbitrary sequence of real numbers decreasing to zero. From \(x \in \Phi_{A_I} (A)\), by definition, there exist a subsequence \(\{t_{n_m}\}_{m \in \mathbb{N}}\) and a set \(A_1 \in A_I\) such that the sequence
\[
\left\{ \chi_{\left[\frac{1}{t_{n_m}}(A-x)\right]} \cap [-1, 1] \right\}_{m \in \mathbb{N}}
\]
of characteristic functions converges \(I\)-almost everywhere on \([-1, 1]\) to \(\chi_{A_1} \cap [-1, 1]\). Similarly, for \(\{t_{n_m}\}_{m \in \mathbb{N}}\) from \(x \in \Phi_{A_I} (B)\), by definition, there is a subsequence \(\{t_{n_{m_k}}\}_{k \in \mathbb{N}}\) and a set \(B_1 \in A_I\) such that the sequence
\[
\left\{ \chi_{\left[\frac{1}{t_{n_{m_k}}}(B-x)\right]} \cap [-1, 1] \right\}_{k \in \mathbb{N}}
\]
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of characteristic functions converges \( I \)-almost everywhere on \([-1, 1]\) to \( \chi_{B_1 \cap [-1,1]} \).
It is clear that the sequence
\[
\left\{ \chi_{\frac{1}{n_m_k}} ((A \cap B) - x) \cap [-1,1] \right\}_{k \in \mathbb{N}}
\]
converges \( I \)-almost everywhere on \([-1,1]\) to \( \chi_{(A_1 \cap B_1) \cap [-1,1]} \), and \( x \) is a \( \Phi_{A_t} \)-density point of \( A \cap B \) since \( A_1 \cap B_1 \in A_t \) (see [PWW1]).

**Proposition 4.** If \( x \) is an \( A_t \)-density point of a set \( A \), then there does not exist a decreasing to zero sequence of real numbers \( \{t_n\}_{n \in \mathbb{N}} \) such that the sequence \( \left\{ \chi_{\frac{1}{n}} (A-x) \cap [-1,1] \right\}_{n \in \mathbb{N}} \) of characteristic functions converges \( I \)-almost everywhere, neither on \([-1,0]\) nor on \([0,1]\) to 0.

**Proof.** It is a simple consequence of Definition 1. \( \square \)

**Remark 2.** It is an immediate consequence of (0), (1) and (2) from Theorem 1 that \( \Phi_{A_t} \) is idempotent, i.e., \( \Phi_{A_t} (A) = \Phi_{A_t} (\Phi_{A_t} (A)) \). We also have \( \Phi_{A_t} (A) \cap \Phi_{A_t} (\mathbb{R} \setminus A) = \emptyset \).

**Theorem 2.** The family \( \mathcal{T}_{A_t} = \{ A \in S : A \subset \Phi_{A_t} (A) \} \) is a stronger topology than the \( I \)-density topology \( \mathcal{T}_I \).

**Proof.** From Theorem 1 (3), \( \emptyset \) and \( \mathbb{R} \in \mathcal{T}_{A_t} \), and the family is closed under finite intersections according to (4). To prove that \( \mathcal{T}_{A_t} \) is closed under arbitrary unions, observe that from Theorem 1, \( \Phi_{A_t} (A) \setminus A \) is a set from \( I \) for each \( A \in S \), and then we follow the proof in [W2]. Take a family \( \{A_t\}_{t \in T} \subset \mathcal{T}_{A_t} \). We have \( A_t \subset \Phi_{A_t} (A_t) \) for each \( t \). Choose a sequence \( \{t_n\}_{n \in \mathbb{N}} \) such that for each \( t \in T \) we have \( A_t \setminus \bigcup_{n=1}^{\infty} A_{t_n} \in I \). It is possible because \( (S,I) \) CCC property. Then \( \Phi_{A_t} (A_t) = \Phi_{A_t} ((A_t \cap \bigcup_{n=1}^{\infty} A_{t_n}) \cup (A_t - \bigcup_{n=1}^{\infty} A_{t_n})) \subset \Phi_{A_t} (\bigcup_{n=1}^{\infty} A_{t_n}) \)

Hence
\[
\bigcup_{n=1}^{\infty} A_{t_n} \subset \bigcup_{t \in T} A_t \subset \bigcup_{t \in T} \Phi_{A_t} (A_t) \subset \Phi_{A_t} \left( \bigcup_{n=1}^{\infty} A_{t_n} \right).
\]

The first and the last in the above sequence of inclusions differ on a set from \( I \) and both sets have the property of Baire, so \( \bigcup_{t \in T} A_t \in S \). Also, \( \bigcup_{t \in T} A_t \subset \Phi_{A_t} \left( \bigcup_{t \in T} A_t \right) \) according to central inclusion and monotonicity of \( \Phi_{A_t} \), implied by (4) of Theorem 1. Hence, finally \( \bigcup_{t \in T} A_t \in \mathcal{T}_{A_t} \).

The set \((-A \cup A) \cup \{0\}\), where \( A \) is defined in Proposition 2, with \( D \) additionally open, belongs to \( \mathcal{T}_{A_t} \), but not to \( \mathcal{T}_I \) topology. \( \square \)

**Remark 3.** Like the \( I \)-density topology, the \( A_t \)-density topology can be described in the form: \( \mathcal{T}_{A_t} = \{ \Phi_{A_t} (A) \setminus P : A \in S \and P \in \mathcal{I} \} \), as if \( A \in \mathcal{T}_{A_t} \), then \( A \subset \Phi_{A_t} (A) \). Consequently, \( A = \Phi_{A_t} (A) \setminus (\Phi_{A_t} (A) \setminus A) \), and we take
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\[ P = \Phi_{A_I}(A) \setminus A \in I. \]

Now, if \( B = \Phi_{A_I}(A) \setminus P \), for some \( A \in S \) and \( P \in I \), then we get

\[
\begin{align*}
\Phi_{A_I}(B) &= \Phi_{A_I}(\Phi_{A_I}(A) \setminus P) \\
&= \Phi_{A_I}(\Phi_{A_I}(A)) \\
&= \Phi_{A_I}(A) \supset \Phi_{A_I}(A) \setminus P = B
\end{align*}
\]

from Theorem 1 (1), (2) and the above remark.

The \( A_I \)-density topology \( T_{A_I} \) has similar properties to those of the \( I \)-density topology \( T_I \).

**Theorem 3.** For an arbitrary set \( A \subset \mathbb{R} \)

\[
\text{Int}_{T_{A_I}}(A) = A \cap \Phi_{A_I}(B),
\]

where \( B \) is an \( S \)-measurable kernel of \( A \) (i.e., \( B \in S \) and \( D \setminus B \in I \) for any \( D \in S \), \( D \subset A \)).

**Proof.** We can follow the proof of Theorem 2.5 from \[W2\] here, where \( \Phi \) is replaced with \( \Phi_{A_I} \).

\[ \square \]

**Theorem 4.** A set \( A \in T_{A_I} \) is \( T_{A_I} \)-regular open if and only if \( A = \Phi_{A_I}(A) \).

**Proof.** Here we can adopt the proof of Theorem 2.6 from \[W2\]. The inclusion \( \Phi_{A_I}(A) \subset \text{Cl}_{A_I}(A) \) in the first part of the proof can now be verified as follows

\[
\begin{align*}
\text{Cl}_{A_I}(A) &= \mathbb{R} \setminus \text{Int}_{T_{A_I}}(\mathbb{R} \setminus A) \\
&= \mathbb{R} \setminus (\mathbb{R} \setminus A) \cap \Phi_{A_I}(\mathbb{R} \setminus A)) \\
&= A \cup (\mathbb{R} \setminus \Phi_{A_I}(\mathbb{R} \setminus A)) \supset A \cup \Phi_{A_I}(A),
\end{align*}
\]

since \( \Phi_{A_I}(A) \subset \mathbb{R} \setminus \Phi_{A_I}(\mathbb{R} \setminus A) \) by Remark 2.

\[ \square \]

**Theorem 5.**

\[
\mathcal{I} = \{ A \subset \mathbb{R} : A \text{ is } T_{A_I} \text{ — nowhere dense set} \}
\]

\[
= \{ A \subset \mathbb{R} : A \text{ is } T_{A_I} \text{ — first category set} \}
\]

\[
= \{ A \subset \mathbb{R} : A \text{ is } T_{A_I} \text{ — closed } T_{A_I} \text{ — discrete set} \}.
\]

**Proof.** We can follow the proofs of Theorems 2 and 4 from \[PWW2\] or Theorem 2.8 from \[W2\]. To prove the second equality we recall that every set of the second category has a subset that lacks the property of Baire [see \[O\]].

\[ \square \]

**Theorem 6.** The \( \sigma \)-algebra of \( T_{A_I} \)-Borel sets coincides with \( S \).

If \( E \subset \mathbb{R} \) is \( T_{A_I} \)-compact set, then \( E \) is finite.

The space \((\mathbb{R}, T_{A_I})\) is neither first countable nor second countable, Lindelöf, and separable.

\((\mathbb{R}, T_{A_I})\) is a Baire space.
Proof. We can follow the proofs of Theorem 3 of \cite{PWW2} and Theorems 2.9–2.12 from \cite{W2}.

\textbf{Remark 4.} In the proof of the above theorems we have used a classical argument referring only to results for Lebesgue density topology from \cite{W1} and \cite{W2} and for I-density topology from \cite{PWW2} and \cite{PWW1}. However, since \( T_{A_x} \subset S \) and \( \Phi_{A_x} \) is a closed lower density operator (i.e., \( \Phi_{A_x} (A) \in S \)) we could rely on more recent results from \cite{RJH} given in more general settings.

We shall consider some properties of continuous functions from \((\mathbb{R}, T_{A_x})\) into \((\mathbb{R}, T_n)\) now.

\textbf{Definition 2.} We say that a real variable function \( f \) is topologically \( T_{A_x} \)-approximately continuous at a point \( x_0 \) if and only if for every number \( \varepsilon > 0 \),
\[
\{ x : |f(x) - y| < \varepsilon \}
\]
there is a \( T_{A_x} \)-neighborhood of \( x \), i.e., there exists a set \( A_x \in S, A_x \subset \{ x : |f(x) - y| < \varepsilon \} \) such that \( x \) is an \( A_x \)-density point of \( A_x \).

\textbf{Definition 3.} We say that a real variable function \( f \) is restrictively \( T_{A_x} \)-approximately continuous at a point \( x_0 \) if and only if there exists a set \( A_{x_0} \in S \) such that
\[
x_0 \in \Phi_{A_x} (A_{x_0}) \quad \text{and} \quad f(x) = \lim_{x \to x_0 \in A_{x_0}} f(x).
\]

\textbf{Theorem 7.} (i) A real function \( f \) defined on \( \mathbb{R} \) has the property of Baire if and only if it is \( T_{A_x} \)-topologically continuous \( I \)-almost everywhere on \( \mathbb{R} \).

(ii) Every \( T_I \)-topologically continuous function is \( T_{A_x} \)-topologically continuous, the converse does not hold.

Proof. (i) Suppose that \( f \) defined on \( \mathbb{R} \) has the property of Baire. Then, by Theorem 7 from \cite{PWW1}, it is \( T_I \)-continuous almost everywhere on \( \mathbb{R} \); hence \( T_{A_x} \)-continuous almost everywhere on \( \mathbb{R} \) since \( T_I \subset T_{A_x} \).

Now, suppose that \( f \) is \( T_{A_x} \)-continuous \( I \)-almost everywhere. Let \( a, b \in \mathbb{R} \), and \( B = \{ x : a < f(x) < b \} \). We shall show that \( B \) has the property of Baire. Let \( C \) be the set of \( T_{A_x} \)-continuity points of \( f \).

We have \( B = (B \cap C) \cup (B \setminus C) \) and \( B \setminus C \in I \). The proof is completed by showing that \( B \cap C \) has the property of Baire. If \( x \in B \cap C \), and \( y = f(x) \), we take \( \varepsilon > 0 \), \( \varepsilon < \min (b - y, y - a) \). Then \( \{ x : |f(x) - y| < \varepsilon \} \) is a \( T_{A_x} \)-neighborhood of \( x \), i.e., there exists a set \( A_x \in S, A_x \subset \{ x : |f(x) - y| < \varepsilon \} \) such that \( x \) is an \( A_x \)-density point of \( A_x \). Of course, \( A_x \subset B \), and we may assume \( A_x \subset (B \cap C) \), by Theorem 1 (2), since \( B \setminus C \in I \).

Finally, we obtain \( B \cap C = \bigcup_{x \in B \cap C} A_x \in T_{A_x} \subset S \).

(ii) The first part is again a consequence of \( T \subset T_{A_x} \). The characteristic function of the set \( (A \cup A) \cup \{ 0 \} \), where \( A \) is defined as in Proposition 2, is \( T_{A_x} \)-topologically continuous but not \( T_I \)-topologically continuous at 0. \( \square \)
Corollary 1. For every measurable real function $f$, the set of $T_{A_I}$-topological continuity points and the set of $T_I$-topological continuity points may differ by a set from $I$.

Remark 5. In the proof of part (i) of the above theorem we have used a classical argument referring only to results from [PWW1] and to the inclusion $T_I \subset T_{A_I}$. However, since $T_{A_I} \subset S$ and $\Phi_{A_I}$ is a lower density operator, we could rely on Theorem 6.39 from [LMZ] or use the recent results of Bartoszewicz and Kotlicka given in more general settings (see [BK] Theorem 2.2).

Proposition 5. There exists a function that is (right) $T_{A_I}$-topologically, but not $T_{A_I}$-restrictively continuous at zero.

Proof. We shall start with the continuity at zero from the right. Let $\{c_n\}_{n \in \mathbb{N}}$ be a decreasing to zero sequence of real numbers such that $c_{n+1} < \frac{1}{2^k} c_n, c_1 = 1$.

Define

$$f(x) = \sum_{n=1}^{\infty} \left( \sum_{i=0}^{\infty} \frac{1}{2^i} \chi\left(\frac{c_{n+1}}{2^i} + \frac{c_n}{2^i}\right)(x) \right) \chi(c_{n+1}, c_n)(x)$$

for $x \in (0, 1]$ and $f(0) = 0$.

Equivalently, put

$$g(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} \chi\left(\frac{1}{2^i + \frac{1}{2^i}}\right)(x)$$

and define

$$f(x) = \sum_{n=1}^{\infty} g\left(\frac{1}{c_n} x\right) \chi(c_{n+1}, c_n)(x)$$

for $x \in (0, 1]$ and $f(0) = 0$.

The function $f$ is right $T_{A_I}$-topologically continuous at zero.

Really, consider the set

$$E_k = \left\{ x \in [0, 1] : |f(x) - 0| \leq \frac{1}{2^k} \right\}.$$

By definition of $f$,

$$E_k = \bigcup_{n \in \mathbb{N}} \left( c_{n+1}, \frac{c_n}{2^k} \right]$$

and it is a simple observation that for every $k \in \mathbb{N}$, $E_k$ has 0 as the $A_I$-density point (even $A_{I[-1,1]}$-density point).

The function $f$ is not right $T_{A_I}$-restrictively continuous at zero.

Suppose, on the contrary to our claim, that there exists a set $E \in A_I$ such that $\lim_{x \in E, x \to 0} f(x) = 0$. 

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Let \( t_n = c_n, \ n \in \mathbb{N} \). Then, since \( E \in \mathcal{A}_I \), we can find its subsequence \( \{t_{n_m}\}_{m \in \mathbb{N}} \) such that the sequence

\[
\left\{ \chi\left( \frac{1}{t_{n_m}} \cdot E \right) \cap [-1,1] \right\}_{m \in \mathbb{N}}
\]

does not converge \( I \)-almost everywhere on \([0, 1]\) to \( \chi_{[0, 1]} \).

On the other side, since \( \lim_{x \to 0} f(x) = 0 \), we can find \( c > 0 \) such that \( f(x) < \frac{1}{2c} \) for all \( x \in E \cap (0, c) \). Hence, \( E \cap (0, c) \subset \{ x : f(x) < \frac{1}{2c} \} \).

We take a subsequence \( \{t_{n_{mk}}\}_{k \in \mathbb{N}} \) of \( \{t_{n_m}\}_{m \in \mathbb{N}} \) such that \( f(x) < \frac{1}{2c} \), \( k \in \mathbb{N} \) on \( E \cap (0, t_{n_{mk}}) \). By the definition of \( f \), for every \( k \in \mathbb{N} \) we have \( \left( \frac{1}{t_{n_{mk}}} \cdot E \right) \subset (0, \frac{1}{2c + 1}) \). Thus, it is clear that the sequence

\[
\left\{ \chi\left( \frac{1}{t_{n_{mk}}} \cdot E \right) \cap [0,1] \right\}_{k \in \mathbb{N}}
\]

does not converge even everywhere on \((0, 1]\) to 0. Since \( E \in \mathcal{A}_I \), this is in contradiction with Proposition 1.

Now, the function

\[
h(x) = \begin{cases} 
  f(x), & x > 0, \\
  0, & x = 0, \\
  f(-x), & x < 0
\end{cases}
\]

is \( \mathcal{T}_{\mathcal{A}_I} \)-topologically but not \( \mathcal{T}_{\mathcal{A}_I} \)-restrictively continuous at zero. \( \square \)

**DEFINITION 4.** We say that the sets \( A \) and \( B \) are essentially different if for every \( t, \alpha \in R \) the set \( (A \triangle (t \cdot B)) \cap [0, \alpha] \) is not from \( I \).

**PROPOSITION 6.** There exists a set \( A \subset [0,1] \) such that zero is an \( \mathcal{A}_I \)-right density point of \( A \) and such that there are: a decreasing to zero sequence of real numbers \( \{t_n\}_{n \in \mathbb{N}} \) and \( c \) essentially different sets associated with different subsequence \( \{\frac{1}{t_{n_m}}\}_{m \in \mathbb{N}} \) in Definition 1.

**Proof.** Let \( \{w_i\}_{i \in \mathbb{N}} \) be a sequence of all rational numbers from interval \((\frac{1}{2}, 1)\). Let \( \{c_n\}_{n \in \mathbb{N}} \) be an arbitrary sequence of real numbers decreasing to 0, \( c < 1 \), such that \( \limsup_{n \to \infty} \frac{c_{n+1}}{c_n} = 0 \). Put \( D_i = [0, \frac{1}{2}] \cup (w_i, 1) \). We define a set \( A \):

\[
A = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{n} \left( \frac{c_{n(n-1)}}{2} + i \cdot D_i \right) \cap \left( \frac{c_{n(n-1)}}{2} + i + 1, \frac{c_{n(n-1)}}{2} + i \right)
\]

Each natural number \( k \) can be uniquely presented as a sum \( k = \frac{n(n-1)}{2} + i \), where \( n \in \mathbb{N} \) and \( i = 1, 2, \ldots, \left( \frac{n+1}{n} \right) - \frac{n(n-1)}{2} = n \). We denote \( i \) as a function
of $k$, i.e., $i(k)$. In particular, we have $\frac{(n+1)n}{2} = \frac{n(n-1)}{2} + n$ and $i\left(\frac{(n+1)n}{2}\right) = n$.

We may rewrite the definition of the set $A$ as

$$A = \bigcup_{k=1}^{\infty} (c_k \cdot D_i(k)) \cap (c_{k+1}, c_k).$$

The $i(k)$, as a function of $k$, takes the following values, consecutively: 1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ... We shall show that zero is an $A_f$-density point of $A \cup (-A)$.

Suppose that $\{t_n\}_{n \in \mathbb{N}}$ is an arbitrary sequence of real numbers decreasing to zero. Similarly, as it is described in the proof of Proposition 2, we choose two subsequences $\{t_{n_r}\}_{m \in \mathbb{N}}$ and $\{c_{m_r}\}_{m \in \mathbb{N}}$ such that $c_{m_r} \leq t_{n_r}$, $r \in \mathbb{N}$ and there are neither elements of $\{c_m\}_{m \in \mathbb{N}}$ nor of $\{t_n\}_{n \in \mathbb{N}}$ between $c_{m_r}$ and $t_{n_r}$. Again, we consider the sequence $\{\frac{c_{m_r}}{t_{n_r}}\}_{r \in \mathbb{N}}$ and find a subsequence $\{\frac{c_{m_r}}{t_{n_r}}\}_{k \in \mathbb{N}}$ convergent to some $a \in [0, 1]$.

There are two possible situations:

a) $\lim_{k \to \infty} \left(\frac{c_{m_{r_k}}}{t_{n_{r_k}}} \cdot \frac{1}{w_i(m_{r_k})}\right) = a \neq 0$, i.e., $\lim_{k \to \infty} \left(\frac{c_{m_{r_k}}}{t_{n_{r_k}}} \cdot \frac{1}{w_i(m_{r_k})}\right) = 1$.

In this case, we consider the behaviour of the sequence $\frac{c_{m_{r_k}}}{t_{n_{r_k}}} \cdot \frac{1}{w_i(m_{r_k})}$.

Since it is bounded, it contains a subsequence $\frac{c_{m_{r_k}}}{t_{n_{r_k}}} \cdot \frac{1}{w_i(m_{r_k})}$ convergent to some $c \in a \cdot \left[\frac{1}{2}, 1\right]$, and $\chi_{a \cdot \left[0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right]}$ converges $I$-a.e. to $\chi_a([0, \frac{1}{2}] \cup [c, 1])$.

Thus, we obtain $B$ on $[0, a]$ as

$$B \cap [0, a] = a \cdot \left(0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right].$$

If $a = 1$, the proof is complete; $B \in A_{I[-a, a]} \subset A_I$.

If $a < 1$, like in the proof of Proposition 2 we obtain

$$B \cap [0, a] = a \cdot \left(0, \frac{1}{2}\right] \cup \left[\frac{c}{a}, 1\right],$$

$$B \cap (a, 1] = (a, 1].$$

And again, $B \in A_{I[-a, a]} \subset A_I$.

b) $\lim_{k \to \infty} \left(\frac{c_{m_{r_k}}}{t_{n_{r_k}}} \cdot \frac{1}{w_i(m_{r_k})}\right) = 0$.

In this case, we have two possible situations again:

b1) the sequence $\{\frac{c_{m_{r_k}}}{t_{n_{r_k}}} - 1\}_{k \in \mathbb{N}}$ is bounded from above. We take its subsequence $\{\frac{c_{m_{r_{k_p}}}}{t_{n_{r_{k_p}}}} - 1\}_{p \in \mathbb{N}}$ such that $\lim_{p \to \infty} \frac{c_{m_{r_{k_p}}}}{t_{n_{r_{k_p}}}} = b < \infty$, and proceed similarly, as in a). We find a subsequence $\frac{c_{m_{r_{k_p}}}}{t_{n_{r_{k_p}}}} - 1 \cdot \frac{1}{t_{n_{r_{k_p}}}} w_i(m_{r_{k_p}} - 1)$.
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of \( c_{m_{rk} - 1} \cdot \frac{1}{t_{n_{rk}}} w_i(m_{rk} - 1) \) convergent to some \( c \leq b \) and obtain I-a.e. convergence of \( \chi\left(\frac{1}{t_{n_{rk}}}, A\right) \cap [0, b] \) to \( \chi_b\left([0, \frac{1}{2}] \cup [\frac{c}{b}, 1]\right) \). Thus, we obtain the set \( B \) as

\[
B = \left[b \cdot \left(\left[0, \frac{1}{2}\right] \cup \left[\frac{c}{b}, 1\right]\right)\right] \cap [0, 1].
\]

And again, \( B \in \mathcal{A}_{I[-\alpha, \alpha]} \subset \mathcal{A}_I \).

b2) the sequence \( \left\{ \frac{c_{m_{rk} - 1}}{t_{n_{rk}}} \right\}_{k \in \mathbb{N}} \) is not bounded from above. We take its subsequence \( \left\{ \frac{c_{m_{rk} - 1}}{t_{n_{rk}}} \right\}_{p \in \mathbb{N}} \) such that \( \lim_{p \to \infty} \frac{c_{m_{rk} - 1}}{t_{n_{rk}}} = \infty \). As every \( D_i \) contains the interval \([0, \frac{1}{2}]\), therefore we have \([0, 1] \subset \frac{1}{t_{n_{rk}}} \cdot A\), for \( p \) appropriately large, and the sequence \( \chi\left(\frac{1}{t_{n_{rk}}} \cdot A\right) \) converges to \( \chi_{[0,1]} \) a.e. on \([0, 1]\). And we obtain \( B \) on \([0, 1]\) as

\[
B \cap [0, 1] = [0, 1].
\]

And, again, \( B \in \mathcal{A}_{I[-\alpha, \alpha]} \subset \mathcal{A}_I \).

Finally, zero is an \( \mathcal{A}_I \)-density point of \((-A \cup A)\).

Now, let \( d \in \left[\frac{1}{2}, 1\right] \) and \( \{w_n\}_{n \in \mathbb{N}} \) be a subsequence of \( \{w_n\}_{n \in \mathbb{N}} \) convergent to \( d \). As a sequence \( \{t_n\}_{n \in \mathbb{N}} \) we take \( \{c_n\}_{n \in \mathbb{N}} \). The set \([0, \frac{1}{2}] \cup (d, 1] \in \mathcal{A}_I \) is associated with the subsequence \( \{c_{(n+1)n}\}_{n \in \mathbb{N}} \) and we obtain the sequence of characteristic functions

\[
\chi\left(\left(\frac{1}{c_{(n+1)n}}, \cdot \right) \cdot A \right) \cap [0, 1]
\]

convergent I-a.e. to \( \chi_{[0, \frac{1}{2}] \cup [d, 1]} \) on \([0, 1]\). □

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Institute of Mathematics
Technical University of Łódź
Wólczańska 215
PL–90-924 Łódź
POLAND
E-mail: wojwoj@gmail.com