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ON STRONGLY COUNTABLY CONTINUOUS FUNCTIONS

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ABSTRACT. A real-valued function f on $\mathbb{R}$ is strongly countably continuous provided that there is a sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ such that the graph of f is contained in the union of the graphs of f_n.

Some examples of interesting strongly countably continuous functions are given: one for which the inverse function is not strongly countably continuous, another which is an additive discontinuous function with a big image and a function which is approximately and I-approximately continuous, but it is not strongly countably continuous.

Definition 1. A function $f\colon \mathbb{R} \to \mathbb{R}$ is countably continuous if there exists a family of sets $\{A_n\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} A_n = \mathbb{R}$ and $f|A_n$, the restriction of f to A_n is continuous.

Definition 2 ([GFG])**.** A function $f\colon \mathbb{R} \to \mathbb{R}$ is strongly countably continuous if there is a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ such that $f \subset \bigcup_{n=1}^{\infty} f_n$.

Of course, if g is strongly countably continuous, then it is countably continuous (because $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, where $A_n = \{x \in \mathbb{R} : f(x) = f_n(x)\}$ and $f|A_n = f_n|A_n$ is continuous for $n \in \mathbb{N}$).

But, the class of strongly countably continuous functions is essentially weaker than the class of the countably continuous functions. For example, each monotone function is countably continuous since it has a countable set of points of discontinuity, but if this set is dense, then the function cannot be strongly countably continuous ([GFG], [CL]).

For basic properties of the class of countably continuous functions, we refer the reader to [GFG].

Theorem 1. *If functions $f, g\colon \mathbb{R} \to \mathbb{R}$ are strongly countably continuous and $F\colon \mathbb{R}^2 \to \mathbb{R}$ is also strongly countably continuous, then the function $x \to F\big(f(x), g(x)\big)$ is also strongly countably continuous.*

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We omit the definition of a strong countable continuity for the function defined on the plane but it is an obvious generalization.

P r o o f. By assumption there exist sequences of continuous functions $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$, $\{F_n\}_{n\in\mathbb{N}}$ such that

$$f_n, g_n\colon \mathbb{R}\to\mathbb{R}, \quad F_n\colon \mathbb{R}^2\to\mathbb{R}, \qquad n\in\mathbb{N}$$

and

$$f\subset\bigcup_{n\in\mathbb{N}} f_n, \quad g_n\subset\bigcup_{n\in\mathbb{N}} g_n, \quad F\subset\bigcup_{n\in\mathbb{N}} F_n.$$

Let

$$\begin{aligned} A_n &= \big\{x\in\mathbb{R} : f_n(x)=f(x)\big\},\\ B_n &= \big\{x\in\mathbb{R} : g_n(x)=g(x)\big\},\\ Z_n &= \big\{(x,y)\in\mathbb{R}^2 : F_n(x)=F(x)\big\}, \qquad n\in\mathbb{N}. \end{aligned}$$

Then

$$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n = \mathbb{R} \qquad \text{and} \qquad \bigcup_{n\in\mathbb{N}} Z_n = \mathbb{R}^2.$$

Let $G\colon \mathbb{R}\to\mathbb{R}^2$ be a function given by the following formula:

$$G(x) = \big(f(x), g(x)\big), \qquad\qquad x\in\mathbb{R},$$

and let

$$C_{n,k,i} = A_n\cap B_k\cap G^{-1}(Z_i), \qquad n,k,i\in\mathbb{N}.$$

Obviously,

$$\bigcup_{n,k,i\in\mathbb{N}} C_{n,k,i} = \mathbb{R}$$

and if $C_{n,k,i}\neq\emptyset$, then for $x\in C_{n,k,i}$ we have $f(x)=f_n(x)$, $g(x)=g_k(x)$ and $\big(f(x),g(x)\big)\in Z_i$ so,

$$F\big(f(x),g(x)\big) = F_i\big(f(x),g(x)\big) = F_i\big(f_n(x),g_k(x)\big).$$

Therefore, the function $x\to F\big(f(x),g(x)\big)$ is continuous on $C_{n,k,i}$, $n,k,i\in\mathbb{N}$ and $F(f,g)\subset\bigcup_{n,k,i\in\mathbb{N}} F_i(f_n,g_n)$. □

The last theorem is slightly stronger than an analogous one in [GFG], but the following corollary comes from [GFG].

Corollary 1. *If $a,b\in\mathbb{R}$ are constants, $f,g:\mathbb{R}\to\mathbb{R}$ are strongly countably continuous functions then the functions $af+bg$, $f\cdot g$, $\max(f,g)$, $\min(f,g)$ and $f\circ g$ are also strongly countably continuous. If $g(x)\neq 0$ for all $x\in\mathbb{R}$, then the quotient $f|g$ is also strongly countably continuous.*

EXAMPLE 1. There is a strongly countably continuous function for which the inverse function is not strongly countably continuous.

Let $g\colon [0,1] \to [0,1]$ be an increasing, left-hand continuous function satisfying the condition $g(1) = 1$ and $g(0) = 0$, with a countable dense set of points of discontinuity denoted by $D = \{y_1, y_2, \dots\}$.

Let $\{I_n\}_{n\in\mathbb{N}}$ be a sequence of intervals, where $I_n = \big(g(y_n), \lim_{y\to y_n^+} g(y)\big)$ for $n \in \mathbb{N}$. We see, at once, that $I_i \cap I_j = \emptyset$ for $i \neq j$ and $g\big([0,1]\big) = [0,1] \setminus \bigcup_{n=1}^{\infty} I_n$.

Let $f\colon [0,1] \to \big[0, 1 + \sum_{n=1}^{\infty} \lambda(I_n)\big]$ be defined by a formula:

$$f(x) = \begin{cases} g^{-1}(x) & \text{for } \ x \in [0,1] \setminus \bigcup_{n=1}^{\infty} I_n, \\ 1 + \sum\limits_{i=1}^{n-1} \lambda(I_i) + (x - \inf I_n) & \text{for } \ x \in I_n, n \in \mathbb{N}, \end{cases}$$

where $\lambda(I)$ denotes the length of an interval I.

It is easy to check out that f is strongly countably continuous. We have to construct a sequence of continuous functions such that $f \subset \bigcup_{n=0}^{\infty} f_n$. Let f_0 be an extension of the function f from $[0,1] \setminus \bigcup_{n=1}^{\infty} I_n$ to $[0,1]$ given by the formula

$$f_0(x) = \begin{cases} g^{-1}(x) & \text{for } x \in [0,1] \setminus \bigcup_{n=1}^{\infty} I_n, \\ y_n & \text{for } x \in I_n, n \in \mathbb{N} \end{cases}$$

and for each $n \in \mathbb{N}$, let f_n be an extension of the function f from I_n to $[0,1]$ given by the formula

$$f_n(x) = 1 + \sum_{i=1}^{n-1} \lambda(I_n) + (x - \inf I_n) \qquad \text{for} \quad x \in [0,1].$$

This sequence fulfils the desired conditions, so f is strongly countably continuous.

Suppose, contrary to our claim, that the inverse function f^{-1} is strongly countably continuous. Since $g \subset f^{-1}$, it follows that the function $g = f^{-1}_{|[0,1]}$ should be also strongly countably continuous, which is not the case (see [GFG, Example 1]). □

Now, we want to look at our class of functions in the context of additive functions.

DEFINITION 3 ([K]). A function $f\colon \mathbb{R} \to \mathbb{R}$ is additive if it satisfies Cauchy's equation

$$f(x + y) = f(x) + f(y) \qquad \text{for all} \quad x, y \in \mathbb{R}.$$

We wish to investigate how large an image for a strongly countably continuous additive but discontinuous function can be. We know that an image for such a function can be uncountable ([GFG]).

Example 2. There is a strongly countably continuous additive discontinuous function with an image which intersects every uncountable Borel subset of $\mathbb{R}$. Consequently, its image is a set of the full outer measures and of the second category everywhere.

Let $H \subset \mathbb{R}$ be a Burstin basis, it means, a Hamel basis which intersects every uncountable Borel subset of $\mathbb{R}$ (such a basis exists — see [K]). The construction of the desired function appears in [GFG].

Fix a point $h_0 \in H$ and let

$$f(x) = \begin{cases} x & \text{for } x \in H \setminus \{h_0\}, \\ 0 & \text{for } x = h_0. \end{cases}$$

Then $f(H) = \big(H \setminus \{h_0\}\big) \cup \{0\}$. Since $\frac{f(x)}{x}$ is not a constant function, the additive extension $g\colon \mathbb{R} \to \mathbb{R}$ of $f\colon H \to \mathbb{R}$ is discontinuous (see [K]). The function g is strongly countably continuous (see [GFG] for more details). The image $g(\mathbb{R})$ contains $H \setminus \{h_0\}$. It is easy to check out that if a set intersects every uncountable Borel subset of $\mathbb{R}$, then this intersection contains at least two points, so the image $g(\mathbb{R})$ is as we want it to be.

Suppose, the set $g(\mathbb{R})$ is not of the full outer measure. Then its complement has a positive inner measure, hence it contains a F_σ set with a positive measure (an uncountable Borel set), a contradiction. Similarly, if there exists an interval I, such that $I \setminus g(\mathbb{R})$ is residual on I, then it contains a G_δ set of the second category (an uncountable Borel set), which is impossible. □

In [GFG], we can find an example of a bounded approximately continuous function which is not strongly countably continuous.

Example 3. There exists a bounded function which is approximately and I--approximately continuous but it is not strongly countably continuous.

For relevant definitions, look in [CLO]

We use here a function from [CLO, p. 84].

Let $\{q_n\}_{n\in\mathbb{N}}$ be a sequence of all rational numbers and $E = \bigcup_{n\in\mathbb{N}}[a_n, b_n]$ be an interval set for which 0 is a right-side dispersion and I-dispersion point.

Put

$$f(x) = \begin{cases} 0 & \text{for } x \notin E, \\ \frac{\operatorname{dist}(\{x\},(a_n,b_n)^c)}{b_n - a_n}, & x \in [a_n, b_n] \end{cases}$$

and let $h_n(x) = \frac{2}{3^n} f(x - q_n)$ for $n \in \mathbb{N}$.

Then $h_n(x) \in \big[0, \frac{1}{3^n}\big]$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\operatorname{osc}(h_n, q_n) = \frac{1}{3^n}$. We define $g_n = \sum_{i \le n} h_i$ for $n \in \mathbb{N}$ and $g = \lim_{n\to\infty} g_n$. Since $|g(x) - g_n(x)| < \frac{1}{3^{n+1}}$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, the sequence converges uniformly.

Each function h_n is approximately and I-approximately continuous since it is a linear function on intervals except the point q_n and it is a constant function on the neighbourhood of q_n in density and I-density topology. Since the families of approximately and I-approximately continuous functions are closed under addition and uniform convergence, the function g is also approximately and I-approximately continuous function.

Assume that g is strongly countably continuous, i.e, there exists a sequence of continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$ such that $g \subset \bigcup_{n=1}^{\infty} f_n$.

Let $A_n = \{x : f(x) = f_n(x)\}$ for $n \in \mathbb{N}$. Then $\mathbb{R} = \bigcup_{n=1}^{\infty} A_n$, therefore there exist $n_0 \in \mathbb{N}$ and an open interval J such that $J \subset \overline{A_{n_0}}$. Define $k_o = \min\{k \in \mathbb{N} : q_k \in J\}$. Since $\operatorname{osc}(g, q_{k_0}) = \frac{1}{3^{k_0}}$, and g is not continuous from the right-side (see [CLO, p. 83]), so there exist two decreasing sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{z_i\}_{i\in\mathbb{N}}$ of points from J tending to q_{k_0} such that

$$\Big|\lim_{i\to\infty} g(x_i) - \lim_{i\to\infty} g(z_i)\Big| = \frac{1}{3^{k_0}}.$$

Fix $i \in \mathbb{N}$. If $x_i \notin Q$, then $\operatorname{osc}(g, x_i) = 0$. If $x_i \in Q$, then there exists an index $k > k_0$ such that $x_i = q_k$, hence $\operatorname{osc}(g, x_i) = \frac{1}{3^k} \le \frac{1}{3} \cdot \operatorname{osc}(g, q_{k_0}) = \frac{1}{3} \cdot \frac{1}{3^{k_0}}$. In both cases, $\operatorname{osc}(g, x_i) \le \frac{1}{3^k} \le \frac{1}{3} \operatorname{osc}(g, q_{k_0})$. Similarly, we obtain $\operatorname{osc}(g, z_i) \le \frac{1}{3} \operatorname{osc}(g, q_{k_0})$.

Since A_{n_0} is dense on J, then for every $i \in \mathbb{N}$ there exists a point $\hat{x}_i \in J \cap A_{n_0}$ such that

$$|\hat{x}_i - x_i| < \frac{1}{i} \quad \text{and} \quad |g(x_i) - g(\hat{x}_i)| < \frac{3}{7} \operatorname{osc}(g, q_{k_0})$$

and there exists a point $\hat{z}_i \in J \cap A_{n_0}$ such that

$$|\hat{z}_i - z_i| < \frac{1}{i} \quad \text{and} \quad |g(z_i) - g(\hat{z}_i)| < \frac{3}{7} \operatorname{osc}(g, q_{k_0}).$$

Obviously,

$$\lim_{i\to\infty} \hat{x}_i = q_{k_0} \quad \text{and} \quad \lim_{i\to\infty} \hat{z}_i = q_{k_0}$$

and

$$\liminf_{i\to\infty} |g(\hat{x}_i) - g(\hat{z}_i)| \ge \big(1 - \frac{3}{7} - \frac{3}{7}\big) \operatorname{osc}(g, q_{k_0}) = \frac{1}{7} \operatorname{osc}(g, q_{k_0}),$$

but simultaneously,

$$g(\hat{x}_i) = f_{n_0}(\hat{x}_i) \quad \text{and} \quad g(\hat{z}_i) = f_{n_0}(\hat{z}_i),$$

since $\hat{x}_i, \hat{z}_i \in A_{n_0}$.

Therefore, the function f_{n_0} is not continuous at the point q_{k_0} – a contradiction. □

Remark 1. The function presented above is countably continuous since its set of discontinuity is countable.

This function is of the first Baire class but not Baire*1, since to be Baire*1, a function should be continuous on a dense open set.

The following characterization of Baire*1 function was given by O' M a l l e y:

THEOREM 2 ([O'M]). *A function $f\colon [0,1] \to \mathbb{R}$ is Baire*1 if and only if there is a sequence of closed sets E_n such that $\bigcup_{n\in\mathbb{N}} E_n = [0,1]$ and $f|E_n$ is continuous for each $n \in \mathbb{N}$.*

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