A NEW APPROACH TO $\psi$-CONTINUITY

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ABSTRACT. Let $T_\psi$ be a $\psi$-density topology for a fixed function $\psi$. For any topological space $X$ with the topology $\tau$ we will consider the family $C(X, \mathbb{R}_\psi)$ of all continuous functions $f$ from $(X, \tau)$ into $(\mathbb{R}, T_\psi)$. The main aim of this paper is to investigate when $C(X, \mathbb{R}_\psi)$ is a ring. This article is based on the results achieved by M. Knox [A characterization of rings of density continuous functions, Real Anal. Exchange 31 (2005), 165–177].

We will use the following notations: $\mathbb{R}$ will denote the set of real numbers, $S$ – the $\sigma$-algebra of Lebesgue measurable subsets on $\mathbb{R}$, $m$ – the Lebesgue measure on $\mathbb{R}$. For $A \subset \mathbb{R}$ let $A'$ be its complement. Let $C$ be a family of nondecreasing continuous functions $\psi: (0, \infty) \to (0, \infty)$ such that $\lim_{t \to 0^+} \psi(t) = 0$.

Fix $\psi \in C$. We say that $x \in \mathbb{R}$ is a $\psi$-density point of a measurable set $A \subset \mathbb{R}$ if

$$\lim_{h \to 0^+} \frac{m(A' \cap (x-h, x+h))}{2h\psi(2h)} = 0.$$ 

For any $A \in S$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}.$$ 

From [TW-B, Theorem 1.4] we obtain that the family $T_\psi = \{A \in S : A \subset \Phi_\psi(A)\}$ is a topology stronger than the natural topology $T_e$ and weaker than the density topology $T_d$. We will say that a set is $T_\psi$-open ($T_\psi$-closed) when it is open (closed) in topology $T_\psi$. Some properties of the topology $T_\psi$ are very similar to those of density topology. $T_\psi$ is invariant under translation, but it does not need be invariant under multiplication.

**Theorem 1 (F).** Let $\psi \in C$. $T_\psi$ is invariant under multiplication if and only if

$$\limsup_{x \to 0^+} \frac{\psi(\alpha x)}{\psi(x)} < \infty$$

for any $\alpha \in \mathbb{R}_+$. 

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To shorten the notation, we will use the following abbreviations: \( \mathbb{R}_e = (\mathbb{R}, \mathcal{T}_e) \), \( \mathbb{R}_d = (\mathbb{R}, \mathcal{T}_d) \), \( \mathbb{R}_\psi = (\mathbb{R}, \mathcal{T}_\psi) \), \( \mathbb{R}_{\text{discr}} = (\mathbb{R}, \mathcal{T}_{\text{discr}}) \), where \( \mathcal{T}_{\text{discr}} \) stands for discrete topology on the real line. Let \( C(X,Y) \) be the family of all continuous functions \( f: X \to Y \), where \( X, Y \) are topological spaces. If \( Y = \mathbb{R}_e \), then we write \( C(X) \) instead of \( C(X, \mathbb{R}_e) \).

In the articles \([\text{FT1}]\) and \([\text{FT2}]\) there were discussed some properties of families \( C(X,Y) \), where \( X, Y \in \{ \mathbb{R}_e, \mathbb{R}_d, \mathbb{R}_\psi \} \). It was shown that the family \( C(\mathbb{R}_\psi, \mathbb{R}_e) \) consists of constant functions only. It was also proved that the family \( C(\mathbb{R}_\psi, \mathbb{R}_\psi) \) (the family of so-called \( \psi \)-continuous functions) is not closed under pointwise addition: there exists a \( \psi \)-continuous function \( f \) such that \( f + x \) is not \( \psi \)-continuous.

Hence, \( C(\mathbb{R}_\psi, \mathbb{R}_\psi) \) is not a ring for any function \( \psi \in \mathcal{C} \). If the condition \((\Delta2)\) is not fulfilled for a certain \( \alpha > 1 \), then even linear function \( f(x) = \alpha x \) is not \( \psi \)-continuous. In this paper, we will concentrate on the family \( C(X, \mathbb{R}_\psi) \), where \( X = (X, \tau) \) is an arbitrary topological space. If \( f \in C(X, \mathbb{R}_\psi) \), then \( f \) is said to be \( \tau \psi \)-continuous. There a natural question arose—for which topological space \( X \) the family \( C(X, \mathbb{R}_\psi) \) can be a ring? We will also discuss the problem if it is a lattice.

Let \( \mathbb{R}^X \) denote the space of all real-valued functions on \( X \). With the operations of pointwise addition and multiplication, it is a ring. We can define the partial order on \( \mathbb{R}^X \) by \( f \leq g \) if

\[
\forall x \in X, \quad f(x) \leq g(x).
\]

For any two elements of \( \mathbb{R}^X \), there exist infimum and supremum of them. We will denote them by \( f \land g \) and \( f \lor g \), respectively, and for any \( x \in X \) we have

\[
(f \land g)(x) = f(x) \land g(x),
\]

\[
(f \lor g)(x) = f(x) \lor g(x).
\]

Then \( \mathbb{R}^X \) is a lattice. The space \( C(X) \) is a subring and a sublattice of \( \mathbb{R}^X \) for any space \( X \).

**Definition 2.** Let \( f \in \mathbb{R}^X \). The set

(a) \( Z(f) = \{ x \in X : f(x) = 0 \} \) is called a zero set of the function \( f \),

(b) \( Coz(f) = \{ x \in X : f(x) \neq 0 \} \) is called a cozero set of the function \( f \).

For each function \( f \in C(X) \) the set \( Z(f) \) is closed and \( Coz(f) \) is open in the space \( X \).

If \( Z \) is a subset of \( X \) such that it is a zero set for a certain function \( f \in C(X) \), then we call it a zero set of a space \( X \). The collection of such sets is denoted by \( Z(X) \). It is closed under countable intersection. If \( Z \subset X \) satisfies \( Z = Z(f) \) for a certain function \( f \in C(X, \mathbb{R}_\psi) \), then \( Z \) will be called a \( \psi \)-zero set of a space \( X \) and the family of such sets will be denoted by \( Z_\psi(X) \). So, \( Z_\psi(X) = \{ Z(f) : f \in C(X, \mathbb{R}_\psi) \} \). The families \( Coz(X) \) and \( Coz_\psi(X) \) are defined analogously.
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Example 3. If $X = \mathbb{R}_e$, then $C(X, \mathbb{R}_\psi) = \text{Const } ([\mathbb{PT}]$) and $Z_\psi(X) = \{\emptyset, \mathbb{R}\}$. 

Now, we will discuss some properties of the family $C(X, \mathbb{R}_\psi)$ and $\tau_\psi$-continuous functions. Since $T_e \subset T_\psi \subset T_d$, we obtain $C(X, \mathbb{R}_d) \subset C(X, \mathbb{R}_\psi) \subset C(X)$.

Lemma 4. Let $X, Y$ be topological spaces and suppose that $X = A \cup B$, where $A, B$ are closed. We assume, that $f: A \to Y$ and $g: B \to Y$ are continuous. If $f(x) = g(x)$ for any $x \in A \cap B$, then a function $h: X \to Y$ defined by the formula

$$h(x) = \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B \end{cases}$$

is also continuous.

Proposition 5. For any topological space, $X$ the space $C(X, \mathbb{R}_\psi)$ is a sublattice of $C(X)$.

Proof. Let $f, g \in C(X, \mathbb{R}_\psi)$. From inclusion $C(X, \mathbb{R}_\psi) \subset C(X)$ we obtain that $f, g$ are continuous functions from $X$ to $\mathbb{R}_e$, hence the sets

$$A = \{x \in X: f(x) \geq g(x)\} \quad \text{and} \quad B = \{x \in X: f(x) \leq g(x)\}$$

are closed in $X$ and $A \cup B = X$. Let us define the function

$$f \vee g = \begin{cases} f & \text{on } A, \\ g & \text{on } B. \end{cases}$$

As $f$ is $\tau_\psi$-continuous on $A$, and $g$ is $\tau_\psi$-continuous on $B$, so $f \vee g$ is $\tau_\psi$-continuous on both sets $A$ and $B$. Moreover, $f = g$ on $A \cap B$, so from Lemma 4 we have that $f \vee g \in C(X, \mathbb{R}_\psi)$. Analogously, we show that the function

$$f \wedge g = \begin{cases} f & \text{on } B, \\ g & \text{on } A \end{cases}$$

is $\tau_\psi$-continuous on $X$. We obtain that for any $f, g \in C(X, \mathbb{R}_\psi)$ the functions $f \vee g$ and $f \wedge g$ are the elements of the space $C(X, \mathbb{R}_\psi)$, so $C(X, \mathbb{R}_\psi)$ is a sublattice of $C(X)$.

For any $f \in C(X, \mathbb{R}_\psi)$ we put

$$f^+ = f \vee 0, \quad f^- = -f \vee 0, \quad |f| = f^+ \vee f^-,$$

where $0$ denotes the constant function equal to zero. From Proposition 5 we obtain the following.

Corollary 6. The functions $f^+, f^-, |f|: X \to \mathbb{R}_\psi$ are $\tau_\psi$-continuous for any $f \in C(X, \mathbb{R}_\psi)$.

Theorem 7. For any topological space $X$, the family $Z_\psi(X)$ is a sublattice of $Z(X)$ (partially ordered by inclusion).
Proof. It is sufficient to show that the union and intersection of two $\psi$-zerosets of a space $X$ is also a $\psi$-zeroset of this space. Let $f, g \in C(X, \mathbb{R}_\psi)$. Then, from the above corollary, their absolute values $|f|, |g|$ are also $\tau\psi$-continuous on $X$. Hence, 

$$|f| \lor |g|, \ |f| \land |g| \in C(X, \mathbb{R}_\psi).$$

Notice that sets 

$$Z(f) \cap Z(g) = Z(|f| \lor |g|), \quad Z(f) \cup Z(g) = Z(|f| \land |g|)$$

are elements of $Z_\psi(X)$. □

**Corollary 8.** Let $f \in C(X, \mathbb{R}_\psi)$, $a, b \in \mathbb{R}$, $a \leq b$. Then, 

$$\{x \in X : f(x) \leq a\}, \ \{x \in X : f(x) \geq a\} \in Z_\psi(X),$$

$$\{x \in X : f(x) < a\}, \ \{x \in X : f(x) > a\} \in Coz_\psi(X).$$

Moreover, 

$$f^{-1}([a, b]) \in Z_\psi(X), \quad f^{-1}((a, b)) \in Coz_\psi(X).$$

Proof. Let $a$ denotes the costant function on $X$ equal to a real number $a$. From Proposition 5 we have that the functions $f \lor a$ and $f \land a$ are $\tau\psi$-continuous on $X$. The topology $T_\psi$ is invariant under translation, hence the functions $f \lor a - a$ and $f \land a - a$ are also $\tau\psi$-continuous on $X$. We obtain 

$$\{x \in X : f(x) \leq a\} = Z(f \lor a - a),$$

hence 

$$\{x \in X : f(x) \leq a\} \in Z_\psi(X).$$

The set $\{x \in X : f(x) > a\}$ is a complement of $\{x \in X : f(x) \leq a\}$, so it is the $\psi$-cozeroset of the space $X$. In the same way, we prove the next conditions. Let us consider the set 

$$f^{-1}([a, b]) = \{x \in X : f(x) \geq a\} \cap \{x \in X : f(x) \leq b\}.$$ 

As $Z_\psi(X)$ is a sublattice of $Z(X)$, then we obtain $f^{-1}([a, b]) \in Z_\psi(X)$. □

**Theorem 9.** Let $f, g \in C(X, \mathbb{R}_\psi)$ and 

$$Coz(f) \cap Coz(g) = \emptyset. \quad \quad (1)$$

Then $f + g \in C(X, \mathbb{R}_\psi)$.

Proof. Let $x$ be an arbitrary point of $X$. Let us consider two cases:

1. Let $x \in Coz(f + g)$.

From (1) we obtain that $x \in Coz(f)$ or $x \in Coz(g)$. Let us suppose that $x \in Coz(f)$. Then, $f(x) \neq 0$ and $g(x) = 0$. Hence, there exists the neighbourhood $U$ of $x$ included in $Coz(f)$ such that $f + g = f$ on $U$. From this, we have that $f + g$ is $\tau\psi$-continuous at point $x$. 

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(2) Let $x \in Z(f + g)$.

From (1) we obtain that $Z(f + g) = Z(f) \cap Z(g)$. Let $V$ be a $T_\psi$-open neighbourhood of 0. We will show that there exists a set $U \in \tau$ such that $f(U) \subset V$. From the assumption $f, g \in C(X, \mathbb{R}_\psi)$, there exist the open neighbourhoods of $x$: $U_1, U_2 \in \tau$ such that $f(U_1) \subset V$ and $g(U_2) \subset V$. We will show $(f + g)(U_1 \cap U_2) \subset V$. Let $y \in U_1 \cap U_2$. If $y \in Coz(f)$, then from (1) we have $y \in Z(g)$, so $(f + g)(y) = f(y) \in V$. If $y \in Coz(g)$, then $(f + g)(y) = g(y) \in V$. If $y \in Z(f) \cap Z(g)$, then $(f + g)(y) = 0 \in V$. Hence, for any $y \in U_1 \cap U_2$ we obtain $(f + g)(y) \in V$, so $(f + g)(U_1 \cap U_2) \subset V$ and, consequently, $f + g \in C(X, \mathbb{R}_\psi)$.

\[ \square \]

**Lemma 10.** Let $f \in C(X, \mathbb{R}_\psi)$. Then $\frac{1}{f}$ is $\tau_\psi$-continuous on $Coz(f)$.

As it was mentioned earlier, the space $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a group under addition. Is it possible to determine when $C(X, \mathbb{R}_\psi)$ is a group for any topological space $X$? The next lemmas and propositions will help us to answer this question.

Fix a function $\psi \in \mathcal{C}$. Without loss of generality, we can assume that for any natural number $n$

\[
\psi\left(\frac{1}{2^n}\right) \leq 1 \quad \text{and} \quad \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) < \frac{1}{2^n}.
\]

**Lemma 11.** Let

\[
V = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n}\right).
\]

Then 0 is a $\psi$-density point of $V$.

**Proof.** Notice that $V' = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^n} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right]$. Let $h > 0$. Then there exists a natural number $n$ for which $h \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$. We estimate the expression

\[
\frac{m(V' \cap [0, h])}{2h\psi(2h)} \leq \frac{m(V' \cap [0, \frac{1}{2^n}] )}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\left(2 \cdot \frac{1}{2^{n+1}}\right)} \leq \frac{\sum_{k-n+1}^{\infty} \frac{1}{2^n} \psi\left(\frac{1}{2^n}\right)}{\frac{1}{2^n} \cdot \psi\left(\frac{1}{2^n}\right)} \leq \frac{\psi\left(\frac{1}{2^n}\right)}{\frac{1}{2^n} \psi\left(\frac{1}{2^n}\right)} < \frac{1}{2^n}.
\]

Therefore, 0 is a $\psi$-density point of the set $V$. \[ \square \]
**Lemma 12.** Assume that $W$ is an arbitrary $T_\psi$-open neighbourhood of 0. Then the set

$$W_0 = ((-\infty, 0] \cap W) \cup \bigcup_{n=1}^{\infty} \left( \left( \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi \left( \frac{1}{2^n} \right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^{n+1}} \cdot \psi \left( \frac{1}{2^n} \right) \right) \cap W.$$  \hspace{1cm} (3)

is also a $T_\psi$-open neighbourhood of 0.

**Proof.** Assume that $V$ is the set given by formula (2). Let $W$ be an arbitrary $T_\psi$-open neighbourhood of 0. Then the set

$$V \cap W = ((-\infty, 0] \cap W) \cup \bigcup_{n=1}^{\infty} \left( \left( \frac{1}{2^n} + \frac{1}{4^{n+1}} \cdot \psi \left( \frac{1}{2^n} \right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^{n+1}} \cdot \psi \left( \frac{1}{2^n} \right) \right),$$

is also $T_\psi$-open neighbourhood of 0. We will show that 0 is a right-hand $\psi$-density point of the set

$$\bigcup_{n=1}^{\infty} \left( \left( \frac{1}{2^n} + \frac{1}{4^n} \cdot \psi \left( \frac{1}{2^n} \right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^n} \cdot \psi \left( \frac{1}{2^n} \right) \right),$$

hence 0 is a $\psi$-dispersion point of $W'$. Let $\varepsilon > 0$. From the assumption, there exists a positive number $\delta$ such that for any $x \in (0, \delta)$ we have

$$m \left( W' \cap [0, x] \right) < \frac{\varepsilon}{3}. \hspace{1cm} (4)$$

Let $N$ stands for a natural number for which $\frac{1}{2^{n+1}} \leq \min(\delta, \frac{\varepsilon}{3})$ and $h \in (0, \frac{1}{2^n})$. There is $n \geq N$ such that $h \in \left( \frac{1}{2^{n+1}}, \frac{1}{2^n} \right)$. Let us notice that for any $k \geq n + 1$ we have

$$m \left( W_0 \cap \left[ \frac{1}{2^{k+1}}, \frac{1}{2^k} \right] \right) = m \left( \left( \frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \cdot \psi \left( \frac{1}{2^k} \right), \frac{1}{2^k} \right) \cap W - \frac{1}{4^{k+1}} \cdot \psi \left( \frac{1}{2^k} \right) \right)$$

$$= m \left( \left( \frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \cdot \psi \left( \frac{1}{2^k} \right), \frac{1}{2^k} \right) \cap W \right)$$

$$= m \left( (V \cap W) \cap \left[ \frac{1}{2^{k+1}}, \frac{1}{2^k} \right] \right).$$
Hence,

\[ m\left(W_0' \cap \left[\frac{1}{2k+1}, \frac{1}{2k}\right]\right) = m\left((V' \cup W') \cap \left[\frac{1}{2k+1}, \frac{1}{2k}\right]\right) \]

\[ \leq m\left(V' \cap \left[\frac{1}{2k+1}, \frac{1}{2k}\right]\right) + m\left(W' \cap \left[\frac{1}{2k+1}, \frac{1}{2k}\right]\right). \]  \hspace{1cm} (5)

Moreover,

\[ m\left(W_0' \cap \left[\frac{1}{2n+1}, h\right]\right) \]

\[ \leq m\left(V' \cap \left[\frac{1}{2n+1}, h\right]\right) + m\left(W' \cap \left[\frac{1}{2n+1}, h + \frac{1}{4n+1} \cdot \psi\left(\frac{1}{2n}\right)\right]\right) \]

\[ \leq m\left(V' \cap \left[\frac{1}{2n+1}, h\right]\right) + m\left(W' \cap \left[\frac{1}{2n+1}, h\right]\right) + \frac{1}{4n+1} \cdot \psi\left(\frac{1}{2n}\right). \]  \hspace{1cm} (6)

From (5) and (6) we obtain

\[ m\left(W_0' \cap [0, h]\right) \leq m\left(V' \cap [0, h]\right) + m\left(W' \cap [0, h]\right) + \frac{1}{4n+1} \cdot \psi\left(\frac{1}{2n}\right). \]

From the above and (4), we have inequalities

\[ \frac{m(W_0' \cap [0, h])}{2h \cdot \psi(2h)} \leq \frac{m(W' \cap [0, h])}{2h \cdot \psi(2h)} + \frac{m(V' \cap [0, \frac{1}{2n}])}{2 \cdot \frac{1}{2n+1} \cdot \psi\left(\frac{1}{2n}\right)} + \frac{1}{4n+1} \cdot \psi\left(\frac{1}{2n}\right) \]

\[ \leq \frac{\varepsilon}{3} + \sum_{k=n}^{\infty} \frac{1}{2n} \psi\left(\frac{1}{2n}\right) + \frac{1}{2n} < \varepsilon \]

and we obtain, that 0 is a $\psi$-dispersion point of $W_0'$.

Now, we construct the functions which enable us to solve the main problem of this paper.

**Lemma 13.** Let $\psi \in C$ fulfill the condition (\H2). Suppose that $f \in C(X, \mathbb{R}_\psi)$ is a nonnegative function and its zeroset is not open. Let $g_n$ be a linear mapping from $\left[\frac{1}{2n+1} - \frac{1}{4n+2} \cdot \psi\left(\frac{1}{2n+1}\right), \frac{1}{2n+1}\right]$ onto $\left[\frac{1}{2n+1}, \frac{1}{2n+1} + \frac{1}{4n+1} \cdot \psi\left(\frac{1}{2n}\right)\right]$, $n \in \mathbb{N}$.  \hspace{1cm} \square
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If \( g: X \to \mathbb{R}_\psi \) is defined by the following formula

\[
g(x) = \begin{cases} 
\frac{1}{16} \cdot \psi\left(\frac{1}{2}\right) + f(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2} - \frac{1}{16} \cdot \psi\left(\frac{1}{2}\right), \infty\right]\right), \\
\frac{1}{2n+1} \cdot \psi\left(\frac{1}{2^n}\right) + f(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2n+1} - \frac{1}{2n+1} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2n+1}\right]\right), \quad n \geq 2 \\
(g_n \circ f)(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2n+1} - \frac{1}{2n+1} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2n+1}\right]\right), \quad n \geq 2 \\
0 & \text{for } x \in Z(f),
\end{cases}
\]

(7)

then \( g \) is \( \tau\psi \)-continuous on \( X \).

Proof. If \( f \in C(X, \mathbb{R}_\psi) \) and condition (A2) is fulfilled, then \( g_n \) is \( \psi \)-continuous, hence \( g_n \circ f \) is \( \tau\psi \)-continuous on \( X \) for any \( n \in \mathbb{N} \). From (7) we have that \( g \) is \( \tau\psi \)-continuous on \( X \setminus Z(f) \) and \( \text{Int} \ Z(f) \). We will show that \( g \) is \( \tau\psi \)-continuous at each point of the set \( Z(f) \setminus \text{Int} \ Z(f) \). Let \( W \) be any \( \mathcal{T}_\psi \)-open neighbourhood of \( 0 \). Then, the set \( W_0 \) defined in Lemma 12 is the \( \mathcal{T}_\psi \)-open neighbourhood of \( 0 \). Let \( a \in Z(f) \setminus \text{Int} \ Z(f) \). Function \( f \) is \( \tau\psi \)-continuous, so there exists a neighbourhood \( U \in \tau \) of \( a \) such that \( f(U) \subset W_0 \). We will prove that \( g(U) \subset W \).

(a) If \( x \in Z(f) \), then \( g(x) = 0 \). So, \( g(x) \in W \).

(b) If \( x \notin Z(f) \), then \( f(x) \in W_0 \) and there exists \( N \in \mathbb{N} \) such that

\[
f(x) \in \left(\frac{1}{2^{N+1}} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right), \frac{1}{2^N}\right) \cap W - \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right).
\]

Hence,

\[
f(x) + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right) \in \left(\frac{1}{2^{N+1}} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right), \frac{1}{2^N}\right) \cap W \subset W. \quad (8)
\]

On the other hand,

\[
f(x) \in \left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right)\right) \cap \left(W - \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right)\right) \subset \left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right)\right).
\]

Therefore, on the set \( f^{-1}\left(\frac{1}{2n+1} + \frac{1}{4n+1} \cdot \psi\left(\frac{1}{2^n}\right)\right) \) we have

\[
g(x) = f(x) + \frac{1}{4^{N+1}} \cdot \psi\left(\frac{1}{2^N}\right).
\]

From this and (5), we get \( g(x) \in W \).

We obtain that \( g(U) \subset W \). Hence the function \( g \) is \( \tau\psi \)-continuous at \( a \) and \( g \in C(X, \mathbb{R}_\psi) \). \( \square \)
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**Proposition 14.** Suppose that all assumptions of Proposition 13 are fulfilled and $g$ is the function defined by the formula (7). Then $g - f : X \to \mathbb{R}_\psi$ is not $\tau\psi$-continuous on the set $Z(f) \setminus \text{Int} Z(f)$.

**Proof.** Let $h = g - f$. Then

$$h(x) = \begin{cases} 
\frac{1}{16} \cdot \psi\left(\frac{1}{2^n}\right) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2} - \frac{1}{16} \cdot \psi\left(\frac{1}{2}\right), \infty\right)\right), \\
\frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^{n+1}}\right)\right]\right), \\
(g_n \circ f - f)(x) & \text{for } x \in f^{-1}\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^{n+1}}\right), \frac{1}{2^n}\right]\right), \\
0 & \text{for } x \in Z(f).
\end{cases} \quad (9)$$

We will show that $h$ is not $\tau\psi$-continuous on the set $Z(f) \setminus \text{Int} Z(f)$. Let $U$ be a $\mathcal{T}_\psi$-open set defined as follows:

$$U = (-\infty, 0] \cup \bigcup_{n=1}^\infty \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} \cdot \psi\left(\frac{1}{2^n}\right)\right).$$

Put

$$C = h^{-1}\left(\left\{ \frac{1}{4^n+1} \cdot \psi\left(\frac{1}{2^n}\right) \right\}_{n=1}^\infty\right).$$

Let $a \in Z(f) \setminus \text{Int} Z(f)$. Suppose that $C$ is closed. Then, the set

$$A = (X \setminus C) \cap f^{-1}(U)$$

$$= (X \setminus C) \cap \left( Z(f) \cup \bigcup_{n=n_0}^\infty f^{-1}\left(\left[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right]\right) \right)$$

is the open neighbourhood of the point $a$ and $A$ is the subset of $Z(f)$. We obtain that $a \in A$ and $a \notin \text{Int} Z(f)$, hence,

$$A \cap \text{Coz}(f) \neq \emptyset.$$  

We have a contradiction.

From the above, $C$ is not closed. The set $Y = \left\{ \frac{1}{4^n+1} \cdot \psi\left(\frac{1}{2^n}\right) \right\}_{n=1}^\infty$ is $\mathcal{T}_\psi$-closed (as a set of measure 0), but its preimage is not closed in $X$. Hence, the function $h$ is not $\tau\psi$-continuous at $a$. \[\Box\]

The next theorem presents the main result of this paper.

**Theorem 15.** Let $\psi \in C$ fulfil the condition (Δ2). The following conditions are equivalent:

1. $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{\text{discr}})$.
2. $C(X, \mathbb{R}_\psi)$ is a ring.
3. $C(X, \mathbb{R}_\psi)$ is closed under multiplication.

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(4) $C(X, \mathbb{R}_\psi)$ is a group.

(5) $Z(f)$ is open for any function $f \in C(X, \mathbb{R}_\psi)$.

**Proof.** Implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow from the definition of a ring. Implication (1) $\Rightarrow$ (2) is obvious, because $C(X, \mathbb{R}_{\text{discr}})$ is a ring.

Let us consider the implication (3) $\Rightarrow$ (4). We will show that $C(X, \mathbb{R}_\psi)$ is closed under addition. Let $f, g \in C(X, \mathbb{R}_\psi)$. If $f_1(x) = e^x$, $g_1(x) = \ln x$ and $\Delta_2$ is fulfilled, then the functions $f_1, g_1$ are $\psi$-continuous [FT2, Theorem 11]. Then compositions $(f_1 \circ f)(x) = e^{f(x)}$ and $(g_1 \circ g)(x) = e^{g(x)}$ are $\tau\psi$-continuous. From the assumption, $C(X, \mathbb{R}_\psi)$ is closed under multiplication, so

$$e^{f(x)} \cdot e^{g(x)} = e^{f(x)+g(x)} \in C(X, \mathbb{R}_\psi),$$

$$\ln e^{f(x)+g(x)} \in C(X, \mathbb{R}_\psi),$$

$$f(x) + g(x) \in C(X, \mathbb{R}_\psi).$$

We obtain that $C(X, \mathbb{R}_\psi)$ is closed under addition, so it is a group.

Next, we show the implication (1) $\Rightarrow$ (5). If $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{\text{discr}})$, then for any function $f \in C(X, \mathbb{R}_\psi)$ preimage of any set open in $\mathbb{R}_{\text{discr}}$ is open in $X$.

So, the set $f^{-1}(\{0\}) = Z(f)$ is open in $X$.

We will prove the implication (5) $\Rightarrow$ (1) now. Observe that $f \in C(X, \mathbb{R}_{\text{discr}})$ if and only if $f^{-1}(\{y\}) \in \tau$ for any $y \in \mathbb{R}$, that means, $f$ is locally constant. We will show that any function $f \in C(X, \mathbb{R}_\psi)$ with open zeroset belongs to the family $C(X, \mathbb{R}_{\text{discr}})$. Take an arbitrary function $f \in C(X, \mathbb{R}_\psi)$ and a point $y \in \mathbb{R}$. If there is no $x_0$ such that $y = f(x_0)$, then $f^{-1}(\{y\}) = \emptyset \in \tau$. Consider the case when $y = f(x_0)$ for a certain $x_0 \in X$. Put $g(x) = f(x) - f(x_0)$, $x \in X$. Then $g \in C(X, \mathbb{R}_\psi)$ and the set $Z(g) = \{x \in X : f(x) = f(x_0)\}$ is open from (5) and not empty as $x_0 \in Z(g)$. Hence, we obtain that $f^{-1}(\{y\}) = Z(g) \in \tau$ and $f \in C(X, \mathbb{R}_{\text{discr}})$. As $\mathbb{R}_\psi \subset \mathbb{R}_{\text{discr}}$, so $C(X, \mathbb{R}_\psi) \supset C(X, \mathbb{R}_{\text{discr}})$ and we have the equality $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{\text{discr}})$.

Let us consider the implication (4) $\Rightarrow$ (1). Suppose that $C(X, \mathbb{R}_\psi) \neq C(X, \mathbb{R}_{\text{discr}})$. Then there exists a function $f \in C(X, \mathbb{R}_\psi)$ which zeroset is not open in $X$. From Proposition 14 there is a function $g \in C(X, \mathbb{R}_\psi)$ such that $g - f$ is not $\tau\psi$-continuous on $Z(f) \setminus \text{Int} Z(f)$. Hence, $C(X, \mathbb{R}_\psi)$ is not a group.

The proof of the last implication completes the proof of theorem. \[\Box\]

**Corollary 16.** The identity function $f(x) = x$ is $\psi$-continuous. Its zeroset is not open in $\mathcal{T}_\psi$, because it is a set of measure zero. Hence, $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a group, so, it is not a ring. Analogously, we can show that $C(\mathbb{R}_d, \mathbb{R}_d)$ and $C(\mathbb{R}_d, \mathbb{R}_\psi)$ are not groups.

Similar results for the family $C(X, \mathbb{R}_d)$ have been obtained by Michelle Knox in [K].
A NEW APPROACH TO $\psi$-CONTINUITY

REFERENCES


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