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A NEW APPROACH TO ψ-CONTINUITY

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ABSTRACT. Let $\mathcal{T}_\psi$ be a ψ-density topology for a fixed function ψ. For any topological space X with the topology τ we will consider the family $C(X, \mathbb{R}_\psi)$ of all continuous functions f from (X, τ) into $(\mathbb{R}, \mathcal{T}_\psi)$. The main aim of this paper is to investigate when $C(X, \mathbb{R}_\psi)$ is a ring. This article is based on the results achieved by M. Knox [*A characterization of rings of density continuous functions*, Real Anal. Exchange **31** (2005), 165–177].

We will use the following notations: $\mathbb{R}$ will denote the set of real numbers, S – the σ-algebra of Lebesgue measurable subsets on $\mathbb{R}$, m – the Lebesgue measure on $\mathbb{R}$. For $A \subset \mathbb{R}$ let A' be its complement. Let $\mathcal{C}$ be a family of nondecreasing continuous functions $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lim_{t\to 0^+} \psi(t) = 0$.

Fix $\psi \in \mathcal{C}$. We say that $x \in \mathbb{R}$ is *a ψ-density point* of a measurable set $A \subset \mathbb{R}$ if

$$\lim_{h\to 0^+} \frac{m\big(A' \cap (x-h, x+h)\big)}{2h\psi(2h)} = 0.$$

For any $A \in S$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of A}\}.$$

From [TW-B, Theorem 1.4] we obtain that the family $\mathcal{T}_\psi = \big\{A \in S\colon A \subset \Phi_\psi(A)\big\}$ is a topology stronger than the natural topology $\mathcal{T}_e$ and weaker than the density topology $\mathcal{T}_d$. We will say that a set is $\mathcal{T}_\psi$-open ($\mathcal{T}_\psi$-closed) when it is open (closed) in topology $\mathcal{T}_\psi$. Some properties of the topology $\mathcal{T}_\psi$ are very similar to those of density topology. $\mathcal{T}_\psi$ is invariant under translation, but it does not need be invariant under multiplication.

Theorem 1 ([F]). *Let $\psi \in \mathcal{C}$. $\mathcal{T}_\psi$ is invariant under multiplication if and only if*

$$\limsup_{x\to 0^+} \frac{\psi(\alpha x)}{\psi(x)} < \infty \tag{$\Delta 2$}$$

for any $\alpha \in \mathbb{R}_+$.

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To shorten the notation, we will use the following abbreviations: $\mathbb{R}_e = (\mathbb{R}, \mathcal{T}_e)$, $\mathbb{R}_d = (\mathbb{R}, \mathcal{T}_d)$, $\mathbb{R}_\psi = (\mathbb{R}, \mathcal{T}_\psi)$, $\mathbb{R}_{discr} = (\mathbb{R}, \mathcal{T}_{discr})$, where $\mathcal{T}_{discr}$ stands for discrete topology on the real line. Let $C(X, Y)$ be the family of all continuous functions $f\colon X \to Y$, where X, Y are topological spaces. If $Y = \mathbb{R}_e$, then we write $C(X)$ instead of $C(X, \mathbb{R}_e)$.

In the articles [FT1] and [FT2] there were discussed some properties of families $C(X, Y)$, where $X, Y \in \{\mathbb{R}_e, \mathbb{R}_d, \mathbb{R}_\psi\}$. It was shown that the family $C(\mathbb{R}_\psi, \mathbb{R}_e)$ consists of constant functions only. It was also proved that the family $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ (the family of so-called ψ-continuous functions) is not closed under pointwise addition: there exists a ψ-continuous function f such that $f+x$ is not ψ-continuous. Hence, $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a ring for any function $\psi \in \mathcal{C}$. If the condition $(\Delta 2)$ is not fulfilled for a certain $\alpha > 1$, then even linear function $f(x) = \alpha x$ is not ψ-continuous. In this paper, we will concentrate on the family $C(X, \mathbb{R}_\psi)$, where $X = (X, \tau)$ is an arbitrary topological space. If $f \in C(X, \mathbb{R}_\psi)$, then f is said to be $\tau\psi$-continuous. There a natural question arose—for which topological space X the family $C(X, \mathbb{R}_\psi)$ can be a ring? We will also discuss the problem if it is a lattice.

Let $\mathbb{R}^X$ denote the space of all real-valued functions on X. With the operations of pointwise addition and multiplication, it is a ring. We can define the partial order on $\mathbb{R}^X$ by $f \leq g$ if

$$\forall\, x \in X, \qquad f(x) \leq g(x).$$

For any two elements of $\mathbb{R}^X$, there exist infimum and supremum of them. We will denote them by $f \wedge g$ and $f \vee g$, respectively, and for any $x \in X$ we have

$$(f \wedge g)(x) = f(x) \wedge g(x),$$
$$(f \vee g)(x) = f(x) \vee g(x).$$

Then $\mathbb{R}^X$ is a lattice. The space $C(X)$ is a subring and a sublattice of $\mathbb{R}^X$ for any space X.

Definition 2. Let $f \in \mathbb{R}^X$. The set

(a) $Z(f) = \big\{x \in X\colon f(x) = 0\big\}$ is called a zeroset of the function f,

(b) $Coz(f) = \big\{x \in X\colon f(x) \neq 0\big\}$ is called a cozeroset of the function f.

For each function $f \in C(X)$ the set $Z(f)$ is closed and $Coz(f)$ is open in the space X.

If Z is a subset of X such that it is a zeroset for a certain function $f \in C(X)$, than we call it a zeroset of a space X. The collection of such sets is denoted by $Z(X)$. It is closed under countable intersection. If $Z \subset X$ satisfies $Z = Z(f)$ for a certain function $f \in C(X, \mathbb{R}_\psi)$, then Z will be called a ψ-zeroset of a space X and the family of such sets will be denoted by $Z_\psi(X)$. So, $Z_\psi(X) = \big\{Z(f)\colon f \in C(X, \mathbb{R}_\psi)\big\}$. The families $Coz(X)$ and $Coz_\psi(X)$ are defined analogously.

EXAMPLE 3. If $X = \mathbb{R}_e$, then $C(X, \mathbb{R}_\psi) = Const$ ([FT1]) and $Z_\psi(X) = \{\emptyset, \mathbb{R}\}$.

Now, we will discuss some properties of the family $C(X, \mathbb{R}_\psi)$ and $\tau\psi$-continuous functions. Since $\mathcal{T}_e \subset \mathcal{T}_\psi \subset \mathcal{T}_d$, we obtain $C(X, \mathbb{R}_d) \subset C(X, \mathbb{R}_\psi) \subset C(X)$.

LEMMA 4. *Let X, Y be topological spaces and suppose that $X = A \cup B$, where A, B are closed. We assume, that $f\colon A \to Y$ and $g\colon B \to Y$ are continuous. If $f(x) = g(x)$ for any $x \in A \cap B$, then a function $h\colon X \to Y$ defined by the formula*

$$h(x) = \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B \end{cases}$$

is also continuous.

PROPOSITION 5. *For any topological space, X the space $C(X, \mathbb{R}_\psi)$ is a sublattice of $C(X)$.*

Proof. Let $f, g \in C(X, \mathbb{R}_\psi)$. From inclusion $C(X, \mathbb{R}_\psi) \subset C(X)$ we obtain that f, g are continuous functions from X to $\mathbb{R}_e$, hence the sets

$$A = \big\{x \in X\colon f(x) \geq g(x)\big\} \quad \text{and} \quad B = \big\{x \in X\colon f(x) \leq g(x)\big\}$$

are closed in X and $A \cup B = X$. Let us define the function

$$f \vee g = \begin{cases} f & \text{on } A, \\ g & \text{on } B. \end{cases}$$

As f is $\tau\psi$-continuous on A, and g is $\tau\psi$-continuous on B, so $f \vee g$ is $\tau\psi$-continuous on both sets A and B. Moreover, $f = g$ on $A \cap B$, so from Lemma 4 we have that $f \vee g \in C(X, \mathbb{R}_\psi)$. Analogously, we show that the function

$$f \wedge g = \begin{cases} f & \text{on } B, \\ g & \text{on } A \end{cases}$$

is $\tau\psi$-continuous on X. We obtain that for any $f, g \in C(X, \mathbb{R}_\psi)$ the functions $f \vee g$ and $f \wedge g$ are the elements of the space $C(X, \mathbb{R}_\psi)$, so $C(X, \mathbb{R}_\psi)$ is a sublattice of $C(X)$. $\square$

For any $f \in C(X, \mathbb{R}_\psi)$ we put

$$f^+ = f \vee \mathbf{0}, \quad f^- = -f \vee \mathbf{0}, \quad |f| = f^+ \vee f^-,$$

where $\mathbf{0}$ denotes the constant function equal to zero. From Proposition 5 we obtain the following.

COROLLARY 6. *The functions f^+, f^-, $|f|\colon X \to \mathbb{R}_\psi$ are $\tau\psi$-continuous for any $f \in C(X, \mathbb{R}_\psi)$.*

THEOREM 7. *For any topological space X, the family $Z_\psi(X)$ is a sublattice of $Z(X)$ (partially ordered by inclusion).*

P r o o f. It is sufficient to show that the union and intersection of two ψ-zerosets of a space X is also a ψ-zeroset of this space. Let $f, g \in C(X, \mathbb{R}_\psi)$. Then, from the above corollary, their absolute values $|f|, |g|$ are also $\tau\psi$-continuous on X. Hence,

$$|f| \vee |g|,\ |f| \wedge |g| \in C(X, \mathbb{R}_\psi).$$

Notice that sets

$$Z(f) \cap Z(g) = Z\big(|f| \vee |g|\big), \qquad Z(f) \cup Z(g) = Z\big(|f| \wedge |g|\big)$$

are elements of $Z_\psi(X)$. □

Corollary 8. *Let $f \in C(X, \mathbb{R}_\psi)$, $a, b \in \mathbb{R}$, $a \leq b$. Then,*

$$\big\{x \in X\colon f(x) \leq a\big\},\ \big\{x \in X\colon f(x) \geq a\big\}\ \in\ Z_\psi(X),$$
$$\big\{x \in X\colon f(x) < a\big\},\ \big\{x \in X\colon f(x) > a\big\}\ \in\ Coz_\psi(X).$$

Moreover,

$$f^{-1}\big([a, b]\big) \in Z_\psi(X), \quad f^{-1}\big((a, b)\big) \in Coz_\psi(X).$$

P r o o f. Let $\mathbf{a}$ denotes the costant function on X equal to a real number a. From Proposition 5, we have that the functions $f \vee \mathbf{a}$ and $f \wedge \mathbf{a}$ are $\tau\psi$--continuous on X. The topology $\mathcal{T}_\psi$ is invariant under translation, hence the functions $f \vee \mathbf{a} - \mathbf{a}$ and $f \wedge \mathbf{a} - \mathbf{a}$ are also $\tau\psi$-continuous on X. We obtain

$$\big\{x \in X\colon f(x) \leq a\big\} = Z(f \vee \mathbf{a} - \mathbf{a}),$$

hence

$$\big\{x \in X\colon f(x) \leq a\big\} \in Z_\psi(X).$$

The set $\big\{x \in X\colon f(x) > a\big\}$ is a complement of $\big\{x \in X\colon f(x) \leq a\big\}$, so it is the ψ-cozeroset of the space X. In the same way, we prove the next conditions. Let us consider the set

$$f^{-1}\big([a, b]\big) = \big\{x \in X\colon f(x) \geq a\big\} \cap \big\{x \in X\colon f(x) \leq b\big\}.$$

As $Z_\psi(X)$ is a sublattice of $Z(X)$, then we obtain $f^{-1}\big([a, b]\big) \in Z_\psi(X)$. □

Theorem 9. *Let $f, g \in C(X, \mathbb{R}_\psi)$ and*

$$Coz(f) \cap Coz(g) = \emptyset. \tag{1}$$

Then $f + g \in C(X, \mathbb{R}_\psi)$.

P r o o f. Let x be an arbitrary point of X. Let us consider two cases:

(1) Let $x \in Coz(f + g)$.
From (1) we obtain that $x \in Coz(f)$ or $x \in Coz(g)$. Let us suppose that $x \in Coz(f)$. Then, $f(x) \neq 0$ and $g(x) = 0$. Hence, there exists the neighbourhood U of x included in $Coz(f)$ such that $f + g = f$ on U. From this, we have that $f + g$ is $\tau\psi$-continuous at point x.

(2) Let $x \in Z(f+g)$.
From (1) we obtain that $Z(f+g) = Z(f) \cap Z(g)$. Let V be a $\mathcal{T}_\psi$-open neighbourhood of 0. We will show that there exists a set $U \in \tau$ such that $f(U) \subset V$. From the assumption $f, g \in C(X, \mathbb{R}_\psi)$, there exist the open neighbourhoods of x: $U_1, U_2 \in \tau$ such that $f(U_1) \subset V$ and $g(U_2) \subset V$. We will show $(f+g)(U_1 \cap U_2) \subset V$. Let $y \in U_1 \cap U_2$. If $y \in Coz(f)$, then from (1) we have $y \in Z(g)$, so $(f+g)(y) = f(y) \in V$. If $y \in Coz(g)$, then $(f+g)(y) = g(y) \in V$. If $y \in Z(f) \cap Z(g)$, then $(f+g)(y) = 0 \in V$. Hence, for any $y \in U_1 \cap U_2$ we obtain $(f+g)(y) \in V$, so $(f+g)(U_1 \cap U_2) \subset V$ and, consequently, $f+g \in C(X, \mathbb{R}_\psi)$.

□

Lemma 10. *Let $f \in C(X, \mathbb{R}_\psi)$. Then $\frac{1}{f}$ is $\tau\psi$-continuous on $Coz(f)$.*

As it was mentioned earlier, the space $C(\mathbb{R}_\psi, \mathbb{R}_\psi)$ is not a group under addition. Is it possible to determine when $C(X, \mathbb{R}_\psi)$ is a group for any topological space X? The next lemmas and propositions will help us to answer this question.

Fix a function $\psi \in \mathcal{C}$. Without loss of generality, we can assume that for any natural number n

$$\psi\left(\frac{1}{2^n}\right) \leq 1 \quad \text{and} \quad \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) < \frac{1}{2^n}.$$

Lemma 11. *Let*

$$V = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right). \tag{2}$$

Then 0 is a ψ-density point of V.

Proof. Notice that $V' = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi(\frac{1}{2^n})\right]$. Let $h > 0$. Then there exists a natural number n for which $h \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$. We estimate the expression

$$\frac{m\big(V' \cap [0,h]\big)}{2h\psi(2h)} \leq \frac{m\Big(V' \cap \big[0, \frac{1}{2^n}\big]\Big)}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\big(2 \cdot \frac{1}{2^{n+1}}\big)} \leq$$

$$\frac{\sum\limits_{k=n+1}^{\infty} \frac{1}{4^k}\psi\big(\frac{1}{2^k}\big)}{\frac{1}{2^n} \cdot \psi\big(\frac{1}{2^n}\big)} \leq \frac{\psi\big(\frac{1}{2^n}\big) \sum\limits_{k=n+1}^{\infty} \frac{1}{4^k}}{\frac{1}{2^n}\psi\big(\frac{1}{2^n}\big)} < \frac{1}{2^n}.$$

Therefore, 0 is a ψ-density point of the set V. □

Lemma 12. *Assume that W is an arbitrary $\mathcal{T}_\psi$-open neighbourhood of 0. Then the set*

$$W_0 = \big((-\infty, 0] \cap W\big) \cup \bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) \right). \tag{3}$$

is also a $\mathcal{T}_\psi$-open neighbourhood of 0.

Proof. Assume that V is the set given by formula (2). Let W be an arbitrary $\mathcal{T}_\psi$-open neighbourhood of 0. Then the set

$$V \cap W = \big((-\infty, 0] \cap W\big) \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right) \cap W$$

is also $\mathcal{T}_\psi$-open neighbourhood of 0. We will show that 0 is a right-hand ψ-density point of the set

$$\bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right), \frac{1}{2^n} \right) \cap W - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right) \right),$$

hence 0 is a ψ-dispersion point of W_0'. Let $\varepsilon > 0$. From the assumption, there exists a positive number δ such that for any $x \in (0, \delta)$ we have

$$\frac{m\big(W' \cap [0, x]\big)}{2x \cdot \psi(2x)} < \frac{\varepsilon}{3}. \tag{4}$$

Let N stands for a natural number for which $\frac{1}{2^N} \le \min\big(\delta, \frac{\varepsilon}{3}\big)$ and $h \in \big(0, \frac{1}{2^N}\big)$. There is $n \ge N$ such that $h \in \big[\frac{1}{2^{n+1}}, \frac{1}{2^n}\big)$. Let us notice that for any $k \ge n+1$ we have

$$\begin{aligned} m\left(W_0 \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) &= m\left(\left[\frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \cdot \psi\left(\frac{1}{2^k}\right), \frac{1}{2^k}\right) \cap W \right. \\ &\qquad \left. - \frac{1}{4^{k+1}} \cdot \psi\left(\frac{1}{2^k}\right)\right) \\ &= m\left(\left[\frac{1}{2^{k+1}} + \frac{1}{4^{k+1}} \cdot \psi\left(\frac{1}{2^k}\right), \frac{1}{2^k}\right) \cap W\right) \\ &= m\left(\big(V \cap W\big) \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right). \end{aligned}$$

Hence,

$$m\left(W_0' \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) = m\left(\left(V' \cup W'\right) \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right)$$
$$\leq m\left(V' \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right) + m\left(W' \cap \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)\right). \quad (5)$$

Moreover,

$$m\left(W_0' \cap \left[\frac{1}{2^{n+1}}, h\right]\right)$$
$$\leq m\left(V' \cap \left[\frac{1}{2^{n+1}}, h\right]\right) + m\left(W' \cap \left[\frac{1}{2^{n+1}}, h + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right]\right)$$
$$\leq m\left(V' \cap \left[\frac{1}{2^{n+1}}, h\right]\right) + m\left(W' \cap \left[\frac{1}{2^{n+1}}, h\right]\right) + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right). \quad (6)$$

From (5) and (6) we obtain

$$m\big(W_0' \cap [0,h]\big) \leq m\big(V' \cap [0,h]\big) + m\big(W' \cap [0,h]\big) + \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right).$$

From the above and (4), we have inequalities

$$\frac{m\big(W_0' \cap [0,h]\big)}{2h \cdot \psi(2h)} \leq \frac{m\big(W' \cap [0,h]\big)}{2h \cdot \psi(2h)} + \frac{m\big(V' \cap [0, \frac{1}{2^n}]\big)}{2 \cdot \frac{1}{2^{n+1}} \cdot \psi\big(\frac{2}{2^{n+1}}\big)} + \frac{\frac{1}{4^{n+1}} \cdot \psi\big(\frac{1}{2^n}\big)}{\frac{1}{2^n} \cdot \psi\big(\frac{1}{2^n}\big)}$$
$$< \frac{\varepsilon}{3} + \frac{\sum\limits_{k=n}^{\infty} \frac{1}{4^{k+1}} \psi\big(\frac{1}{2^k}\big)}{\frac{1}{2^n} \psi\big(\frac{1}{2^n}\big)} + \frac{1}{2^n} < \varepsilon$$

and we obtain, that 0 is a ψ-dispersion point of W_0'. □

Now, we construct the functions which enable us to solve the main problem of this paper.

Lemma 13. *Let $\psi \in \mathcal{C}$ fulfil the condition $(\Delta 2)$.*
Suppose that $f \in C(X, \mathbb{R}_\psi)$ is a nonnegative function and its zeroset is not open. Let g_n be a linear mapping from $\left[\frac{1}{2^{n+1}} - \frac{1}{4^{n+2}} \cdot \psi\big(\frac{1}{2^{n+1}}\big), \frac{1}{2^{n+1}}\right]$ onto $\left[\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \psi\big(\frac{1}{2^n}\big)\right]$, $n \in \mathbb{N}$.

If $g\colon X \to \mathbb{R}_\psi$ *is defined by the following formula*

$$g(x) = \begin{cases} \frac{1}{16}\cdot\psi\big(\frac{1}{2}\big) + f(x) & \textit{for } x \in f^{-1}\Big(\Big[\frac{1}{2} - \frac{1}{16}\cdot\psi\big(\frac{1}{2}\big), \infty\Big)\Big), \\ \frac{1}{4^{n+1}}\cdot\psi\big(\frac{1}{2^n}\big) + f(x) & \textit{for } x \in f^{-1}\Big(\Big[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}}\cdot\psi\big(\frac{1}{2^n}\big)\Big]\Big), \\ & \quad n \geq 2 \\ (g_n \circ f)(x) & \textit{for } x \in f^{-1}\Big(\Big[\frac{1}{2^{n+1}} - \frac{1}{4^{n+2}}\cdot\psi\big(\frac{1}{2^{n+1}}\big), \frac{1}{2^{n+1}}\Big]\Big), \\ & \quad n \geq 2 \\ 0 & \textit{for } x \in Z(f), \end{cases} \tag{7}$$

then g *is* $\tau\psi$*-continuous on* X.

P r o o f. If $f \in C(X, \mathbb{R}_\psi)$ and condition $(\Delta 2)$ is fulfilled, then g_n is ψ-continuous, hence $g_n \circ f$ is $\tau\psi$-continuous on X for any $n \in \mathbb{N}$. From (7) we have that g is $\tau\psi$-continuous on $X \setminus Z(f)$ and $\operatorname{Int} Z(f)$. We will show that g is $\tau\psi$-continuous at each point of the set $Z(f) \setminus \operatorname{Int} Z(f)$. Let W be any $\mathcal{T}_\psi$-open neighbourhood of 0. Then, the set W_0 defined in Lemma 12, is the $\mathcal{T}_\psi$-open neighbourhood of 0. Let $a \in Z(f) \setminus \operatorname{Int} Z(f)$. Function f is $\tau\psi$-continuous, so there exists a neighbourhood $U \in \tau$ of a such that $f(U) \subset W_0$. We will prove that $g(U) \subset W$. Let $x \in U$.

(a) If $x \in Z(f)$, then $g(x) = 0$. So, $g(x) \in W$.

(b) If $x \notin Z(f)$, then $f(x) \in W_0$ and there exists $N \in \mathbb{N}$ such that

$$f(x) \in \left(\frac{1}{2^{N+1}} + \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right), \frac{1}{2^N}\right) \cap W - \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right).$$

Hence,

$$f(x) + \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right) \in \left(\frac{1}{2^{N+1}} + \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right), \frac{1}{2^N}\right) \cap W \subset W. \tag{8}$$

On the other hand,

$$\begin{aligned} f(x) &\in \left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right)\right) \cap \left(W - \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right)\right) \\ &\subset \left(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right)\right). \end{aligned}$$

Therefore, on the set $f^{-1}\big(\frac{1}{2^{N+1}}, \frac{1}{2^N} + \frac{1}{4^{N+1}}\cdot\psi\big(\frac{1}{2^N}\big)\big)$ we have

$$g(x) = f(x) + \frac{1}{4^{N+1}}\cdot\psi\left(\frac{1}{2^N}\right).$$

From this and (8), we get $g(x) \in W$.

We obtain that $g(U) \subset W$. Hence the function g is $\tau\psi$-continuous at a and $g \in C(X, \mathbb{R}_\psi)$. $\square$

Proposition 14. *Suppose that all assumptions of Proposition* 13 *are fulfilled and g is the function defined by the formula* (7)*. Then $g - f\colon X \to \mathbb{R}_\psi$ is not $\tau\psi$-continuous on the set $Z(f) \setminus \mathit{Int}\, Z(f)$.*

Proof. Let $h = g - f$. Then

$$h(x) = \begin{cases} \frac{1}{16} \cdot \psi\big(\frac{1}{2}\big) & \text{for } x \in f^{-1}\Big(\Big[\frac{1}{2} - \frac{1}{16} \cdot \psi\big(\frac{1}{2}\big), \infty\Big)\Big), \\ \frac{1}{4^{n+1}} \cdot \psi\big(\frac{1}{2^n}\big) & \text{for } x \in f^{-1}\Big(\Big[\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\big(\frac{1}{2^n}\big)\Big]\Big), \\ & \quad n \geq 2 \\ (g_n \circ f - f)(x) & \text{for } x \in f^{-1}\Big(\Big[\frac{1}{2^{n+1}} - \frac{1}{4^{n+2}} \cdot \psi\big(\frac{1}{2^{n+1}}\big), \frac{1}{2^{n+1}}\Big]\Big), \\ & \quad n \geq 2 \\ 0 & \text{for } x \in Z(f). \end{cases} \tag{9}$$

We will show that h is not $\tau\psi$-continuous on the set $Z(f) \setminus \mathrm{Int}\, Z(f)$. Let U be a $\mathcal{T}_\psi$-open set defined as follows:

$$U = (-\infty, 0] \cup \bigcup_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right).$$

Put

$$C = h^{-1}\left(\left\{\frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right\}_{n=1}^{\infty}\right).$$

Let $a \in Z(f) \setminus \mathrm{Int}\, Z(f)$. Suppose that C is closed. Then, the set

$$\begin{aligned} A &= (X \setminus C) \cap f^{-1}(U) \\ &= (X \setminus C) \cap \left(Z(f) \cup \bigcup_{n=n_0}^{\infty} f^{-1}\left(\frac{1}{2^{n+1}}, \frac{1}{2^n} - \frac{1}{4^{n+1}} \cdot \psi\left(\frac{1}{2^n}\right)\right)\right) \end{aligned}$$

is the open neighbourhood of the point a and A is the subset of $Z(f)$. We obtain that $a \in A$ and $a \notin \mathrm{Int}\, Z(f)$, hence,

$$A \cap Coz(f) \neq \emptyset.$$

We have a contradiction.

From the above, C is not closed. The set $Y = \big\{\frac{1}{4^{n+1}} \cdot \psi\big(\frac{1}{2^n}\big)\big\}_{n=1}^{\infty}$ is $\mathcal{T}_\psi$-closed (as a set of measure 0), but its preimage is not closed in X. Hence, the function h is not $\tau\psi$-continuous at a. □

The next theorem presents the main result of this paper.

Theorem 15. *Let $\psi \in \mathcal{C}$ fulfil the condition* ($\Delta 2$)*. The following conditions are equivalent:*

(1) $C(X, \mathbb{R}_\psi) = C(X, \mathbb{R}_{discr})$.

(2) $C(X, \mathbb{R}_\psi)$ *is a ring.*

(3) $C(X, \mathbb{R}_\psi)$ *is closed under multiplication.*

(4) *$C(X,\mathbb{R}_\psi)$ is a group.*

(5) *$Z(f)$ is open for any function $f \in C(X,\mathbb{R}_\psi)$.*

Proof. Implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow from the definition of a ring. Implication (1) $\Rightarrow$ (2) is obvious, because $C(X,\mathbb{R}_{discr})$ is a ring.

Let us consider the implication (3) $\Rightarrow$ (4). We will show that $C(X,\mathbb{R}_\psi)$ is closed under addition. Let $f,g \in C(X,\mathbb{R}_\psi)$. If $f_1(x)=e^x$, $g_1(x)=\ln x$ and ($\Delta 2$) is fulfilled, then the functions f_1,g_1 are ψ-continuous [FT2, Theorem 11]. Then compositions $(f_1\circ f)(x)=e^{f(x)}$ and $(g_1\circ g)(x)=e^{g(x)}$ are $\tau\psi$-continuous. From the assumption, $C(X,\mathbb{R}_\psi)$ is closed under multiplication, so

$$\begin{aligned} e^{f(x)}\cdot e^{g(x)} = e^{f(x)+g(x)} &\in C(X,\mathbb{R}_\psi),\\ \ln e^{f(x)+g(x)} &\in C(X,\mathbb{R}_\psi),\\ f(x)+g(x) &\in C(X,\mathbb{R}_\psi). \end{aligned}$$

We obtain that $C(X,\mathbb{R}_\psi)$ is closed under addition, so it is a group.

Next, we show the implication (1) $\Rightarrow$ (5). If $C(X,\mathbb{R}_\psi)=C(X,\mathbb{R}_{discr})$, then for any function $f\in C(X,\mathbb{R}_\psi)$ preimage of any set open in $\mathbb{R}_{discr}$ is open in X. So, the set $f^{-1}\big(\{0\}\big)=Z(f)$ is open in X.

We will prove the implication (5) $\Rightarrow$ (1) now. Observe that $f\in C(X,\mathbb{R}_{discr})$ if and only if $f^{-1}\big(\{y\}\big)\in\tau$ for any $y\in\mathbb{R}$, that means, f is locally constant. We will show that any function $f\in C(X,\mathbb{R}_\psi)$ with open zeroset belongs to the family $C(X,\mathbb{R}_{discr})$. Take an arbitrary function $f\in C(X,\mathbb{R}_\psi)$ and a point $y\in\mathbb{R}$. If there is no x_0 such that $y=f(x_0)$, then $f^{-1}\big(\{y\}\big)=\emptyset\in\tau$. Consider the case when $y=f(x_0)$ for a certain $x_0\in X$. Put $g(x)=f(x)-f(x_0)$, $x\in X$. Then $g\in C(X,\mathbb{R}_\psi)$ and the set $Z(g)=\big\{x\in X\colon f(x)=f(x_0)\big\}$ is open from (5) and not empty as $x_0\in Z(g)$. Hence, we obtain that $f^{-1}\big(\{y\}\big)=Z(g)\in\tau$ and $f\in C(X,\mathbb{R}_{discr})$. As $\mathbb{R}_\psi\subset\mathbb{R}_{discr}$, so $C(X,\mathbb{R}_\psi)\supset C(X,\mathbb{R}_{discr})$ and we have the equality $C(X,\mathbb{R}_\psi)=C(X,\mathbb{R}_{discr})$.

Let us consider the implication (4) $\Rightarrow$ (1). Suppose that $C(X,\mathbb{R}_\psi)\neq C(X,\mathbb{R}_{discr})$. Then there exists a function $f\in C(X,\mathbb{R}_\psi)$ which zeroset is not open in X. From Proposition 14, there is a function $g\in C(X,\mathbb{R}_\psi)$ such that $g-f$ is not $\tau\psi$-continuous on $Z(f)\setminus \operatorname{Int} Z(f)$. Hence, $C(X,\mathbb{R}_\psi)$ is not a group.

The proof of the last implication completes the proof of theorem. □

Corollary 16. *The identity function $f(x)=x$ is ψ-continuous. Its zeroset is not open in $\mathcal{T}_\psi$, because it is a set of measure zero. Hence, $C(\mathbb{R}_\psi,\mathbb{R}_\psi)$ is not a group, so, it is not a ring. Analogously, we can show that $C(\mathbb{R}_d,\mathbb{R}_d)$ and $C(\mathbb{R}_d,\mathbb{R}_\psi)$ are not groups.*

Similar results for the family $C(X,\mathbb{R}_d)$ have been obtained by Michelle Knox in [K].

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