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BOUNDEDNESS AND OSCILLATION OF THIRD ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The relationship between boundedness and oscillation of solutions of the third order neutral differential equations are presented.

1. Introduction

In this paper we consider third order neutral differential equations of the form

$$\left(r_2(t)\left(r_1(t)\left(x(t) - p\,x\big(\tau(t)\big)'\right)'\right)' + q(t)\,f\left(x\big(\sigma(t)\big)\right)\right), \qquad t \ge t_0 \qquad (1)$$

and the following conditions are assumed to hold:

(H_1) $0 < p < 1$;

(H_2) $\tau \in C\big([t_0,\infty), R\big)$, $\tau(t) < t$, τ is strictly increasing, $\lim_{t\to\infty} \tau(t) = \infty$ and we define

$$\tau^0(t) = t, \quad \tau^i(t) = \tau(\tau^{i-1}(t)), \qquad i = 1, 2, \dots ;$$

(H_3) q, $r_j \in C\big([t_0,\infty), (0,\infty)\big)$, and

$$\int_0^\infty r_j(t)\,\mathrm{d}t = \infty, \qquad j = 1, 2;$$

(H_4) $\sigma \in C^1\big([t_0,\infty), R\big)$, $\sigma'(t) > 0$, $\sigma(t) < t$ and $\lim_{t\to\infty} \sigma(t) = \infty$;

(H_5) $f \in C(R,R)$, f is nondecreasing, $u\,f(u) > 0$ for $u \neq 0$ and

$$\liminf_{u\to 0} \frac{f(u)}{u} > 0.$$

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For the sake of convenience we introduce the following notation:

$$\begin{aligned} z(t) &= x(t) - p\, x\big(\tau(t)\big), \qquad (2)\\ L_0\, z(t) &= z(t),\\ L_i\, z(t) &= r_i(t)\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}\, z(t), \qquad i = 1, 2,\\ L_3\, z(t) &= \frac{\mathrm{d}}{\mathrm{d}t}L_2\, z(t). \end{aligned}$$

Asymptotic properties of solutions of differential equations (ordinary, with deviating argument) of the third order have been subject of intensive studying in the literature. This problem for neutral differential equations has received considerable attention in recent years (see the references cited, for example, in [6]. Boundedness and oscillation are, generally speaking, independent properties. Nevertheless, there exists a precise relation between them. In this paper we give sufficient conditions for the bounded solution of the equation (1) to be oscillatory.

By a *solution* of the equation (1) we mean a continuous function $x(t)$ satisfying (1) on $[t_x, \infty)$ such that $L_i\, z(t)$, $0 \le i \le 3$ exist and are continuous on $[t_x, \infty)$. A nontrivial solution of (1) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*.

2. Properties of nonoscillatory solutions

We begin with analyzing of the asymptotic behavior of possible non-oscillatory solutions of the equation (1). Let x be a non-oscillatory solution of (1) on $[t_0, \infty)$. From the equation (1) it follows that the function z has to be eventually of constant sign, so either

$$x(t)\, z(t) > 0, \qquad (3)$$

or

$$x(t)\, z(t) < 0, \qquad (4)$$

for all sufficiently large t. Denote by N^+ [or N^-] the set of all non-oscillatory solutions x of the equation (1) such that (3) [or (4)] is satisfied.

The following lemmas will be useful in the proofs of main results.

Lemma 2.1. *Let* (H_1), (H_2) *hold and x be continuous nonoscillatory solution of the functional inequality* (3). *Then*

$$\lim_{t\to\infty} x(t) = 0, \qquad \lim_{t\to\infty} z(t) = 0.$$

Lemma 2.2. *Let* (H_1), (H_2) *hold and* x *be continuous nonoscillatory solution of the functional inequality (4). If*

$$\lim_{t\to\infty} z(t) = 0, \quad \textit{then} \quad \lim_{t\to\infty} x(t) = 0.$$

These lemmas are modifications of the Lemma 2.1 in the paper [7] and the Lemma 2 in the paper [8].

3. Main results

Theorem 3.1. *Let the conditions* (H_1)–(H_5) *hold. If*

$$(\mathrm{H}_6) \qquad \limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_{u}^{\sigma(t)} \frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s \;>\; 0,$$

then every bounded solution x *of (1) is either oscillatory or tends to zero as* $t \to \infty$.

P r o o f. Let x be a bounded non-oscillatory solution of (1) on the interval $[t_0, \infty)$. Without loss of generality we may assume that $x(t) > 0$, $x\big(\sigma(t)\big) > 0$ for $t \ge t_1 \ge t_0$. Then z is bounded and nonoscillatory, too. The equation (1) implies that $L_2\, z$ is a decreasing function and there are two possibilities:

I. $L_2\, z(t) < 0$, $t \ge t_2 \ge t_1$;

II. $L_2\, z(t) > 0$, $t \ge t_2 \ge t_1$.

Let **I.** hold. Then there exist a $t_3 \ge t_2$ and a constant $A < 0$ such that

$$L_2\, z(t) < A, \quad \text{resp.} \quad \frac{\mathrm{d}}{\mathrm{d}t} L_1\, z(t) < \frac{A}{r_2(t)}, \qquad \text{for} \quad t \ge t_3.$$

Integrating the last inequality and using (H_3) we get $L_1\, z(t) \to -\infty$ as $t \to \infty$ and then there exists a $t_4 \ge t_3$ and a constant $B < 0$ such that

$$L_1\, z(t) \le B, \quad \text{resp.} \quad \frac{\mathrm{d}}{\mathrm{d}t} L_0\, z(t) < \frac{B}{r_1(t)}, \qquad \text{for} \quad t \ge t_4.$$

Again integrating this inequality from t_4 to t and using (H_3) we get $z(t) \to -\infty$ as $t \to \infty$. This yields that $x \in N^-$ and this contradicts the Lemma 2.1.

Now we assume that **II.** holds, i.e., $L_2\, z(t) > 0$, $t \ge t_2$. Then $L_1\, z$ is the increasing function and we have two possibilities for $L_1\, z$:

(a) $L_1\, z(t) > 0$ for $t \ge t_3 \ge t_2$;

(b) $L_1\, z(t) < 0$ for $t \ge t_3 \ge t_2$.

In the case **(a)** there exist a $t_4 \geq t_3$ and a constant $C > 0$ such that

$$L_1\, z(t) \geq C > 0, \quad \text{resp.} \quad \frac{\mathrm{d}}{\mathrm{d}t} L_0\, z(t) > \frac{C}{r_1(t)}, \qquad \text{for} \quad t \geq t_4.$$

The last inequality and (H_3) imply $\lim_{t\to\infty} z(t) = \infty$. This means that $z(t) > 0$ and according to (2) we get $x(t) \geq z(t)$ for all sufficiently large t. Then $\lim_{t\to\infty} x(t) = \infty$, which contradicts the boundedness of x.

Let the case **(b)** hold. Then z is decreasing and there exists $\lim_{t\to\infty} z(t) = \alpha$. If $z(t) < 0$ for $t \geq t_4 \geq t_3$, then $\alpha < 0$, which contradicts the Lemma 2.1. Therefore $z(t) > 0$ for $t \geq t_4 \geq t_3$ and $\alpha \geq 0$. We shall prove that $\alpha = 0$.

Let $\alpha > 0$. Integration of the identity $L_1\, z(t) = L_1\, z(t)$ from $\sigma(s)$ to $\sigma(t)$, $t > s > T \geq t_4$ leads to

$$z\big(\sigma(t)\big) - z\big(\sigma(s)\big) = \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} L_1\, z(u)\, \mathrm{d}u\,. \tag{5}$$

Integrating the identity $L_2\, z(t) = L_2\, z(t)$ from u to $\sigma(t)$, $T < s < u < t$ we obtain

$$-L_1\, z(u) \geq L_1\, z\big(\sigma(t)\big) - L_1\, z(u) = \int_u^{\sigma(t)} \frac{1}{r_2(v)} L_2\, z(v)\, \mathrm{d}v\,,$$

for $t \geq u \geq T$. We know that $L_2\, z$ is decreasing and so the last inequality implies

$$L_1\, z(u) \leq -L_2\, z\big(\sigma(t)\big) \int_u^{\sigma(t)} \frac{1}{r_2(v)}\, \mathrm{d}v\,. \tag{6}$$

Combining (5), (6) we get

$$z\big(\sigma(t)\big) - z\big(\sigma(s)\big) \leq -L_2\, z\big(\sigma(t)\big) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_u^{\sigma(t)} \frac{1}{r_2(v)}\, \mathrm{d}v\, \mathrm{d}u\,, \qquad t > s \geq T, \tag{7}$$

for $t \geq s \geq T$.

Let us define a function

$$F(s,t) = \Big(L_2\, z(s) - L_2\, \big(\sigma(t)\big)\Big) \int_s^t \frac{z'\big(\sigma(u)\big)\sigma'(u)}{f\Big(z\big(\sigma(u)\big)\Big)}\, \mathrm{d}u\,, \qquad t \geq s \geq T.$$

Hence $F(t,t) = 0 = F\big(\sigma(t), t\big)$. Deriving $F(s,t)$ partially with respect to s, using the equation (1), (H_4), monotonicity of z, f, positivity of $L_2\, z$ we get

$$F_s'(s,t) \geq -q(s) f\Big(x\big(\sigma(s)\big)\Big) \int_s^t \frac{z'\big(\sigma(u)\big)\sigma'(u)}{f\Big(z\big(\sigma(u)\big)\Big)}\, \mathrm{d}u + L_2\, z\big(\sigma(t)\big) \frac{z'\big(\sigma(s)\big)\sigma'(s)}{f\Big(z\big(\sigma(s)\big)\Big)}$$

for $t \geq s \geq T$. Taking into account the monotonicity of z, f, σ, the last inequality may be rewritten to the form

$$F'_s(s,t) \geq -g(s)\frac{f\Big(x\big(\sigma(s)\big)\Big)}{f\Big(z\big(\sigma(s)\big)\Big)}\Big[z\big(\sigma(t)\big) - z\big(\sigma(s)\big)\Big] + L_2\, z\big(\sigma(t)\big)\frac{z'\big(\sigma(s)\big)\sigma'(s)}{f\Big(z\big(\sigma(s)\big)\Big)}$$

for $t \geq s \geq T$. In view of $x \in N^+$ from (2) we have that $x\big(\sigma(s)\big) \geq z\big(\sigma(s)\big)$ and then

$$\frac{f\Big(x\big(\sigma(s)\big)\Big)}{f\Big(z\big(\sigma(s)\big)\Big)} \geq 1$$

for all sufficiently large s. Therefore

$$F'_s(s,t) \geq -q(s)\Big[z\big(\sigma(t)\big) - z\big(\sigma(s)\big)\Big] + L_2\, z\big(\sigma(t)\big)\frac{z'\big(\sigma(s)\big)\sigma'(s)}{f\Big(z\big(\sigma(s)\big)\Big)} \tag{8}$$

for $t \geq s \geq T$. Combining (7), (8) and then integrating from $\sigma(t)$ to t with respect to s we obtain

$$0 \geq L_2\, z\big(\sigma(t)\big)\left[\int_{\sigma(t)}^{t} q(s)\int_{\sigma(s)}^{\sigma(t)}\frac{1}{r_1(u)}\int_{u}^{\sigma(t)}\frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s + \int_{z(\sigma(\sigma(t)))}^{z(\sigma(t))}\frac{\mathrm{d}w}{f(w)}\right] \tag{9}$$

for $t \geq s \geq T$. Because $\lim_{t\to\infty} z(t) = a > 0$, we get

$$\lim_{t\to\infty}\int_{z(\sigma(\sigma(t)))}^{z(\sigma(t))}\frac{\mathrm{d}w}{f(w)} = 0.$$

Then in view of the sign property of $L_2\, z(t)$ we obtain from (9) that

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\int_{\sigma(s)}^{\sigma(t)}\frac{1}{r_1(u)}\int_{u}^{\sigma(t)}\frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s \leq 0,$$

which contradicts the condition (H_6). Therefore $\lim_{t\to\infty} z(t) = 0$ and according to the Lemma 2.2 we have that $\lim_{t\to\infty} x(t) = 0$. □

Theorem 3.2. *Let* (H_1)–(H_5) *hold. Let there exist an integer* $n \geq 0$ *such that*

$$(\mathrm{H}_7)\qquad \limsup_{t\to\infty}\int_{\sigma(t)}^{t} q(s)\int_{\sigma(s)}^{\sigma(t)}\frac{1}{r_1(u)}\int_{u}^{\sigma(t)}\frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s > \frac{1-p}{1-p^{n+1}}\limsup_{u\to 0}\frac{u}{f(u)}.$$

Then every bounded solution x of equation (1) is oscillatory.

P r o o f. Let x be a bounded non-oscillatory solution of (1) on the interval $[t_0, \infty)$. Without loss of generality we may assume that $x(t) > 0$, $x\big(\sigma(t)\big) > 0$ for $t \ge t_1 \ge t_0$. We can proceed exactly as in the proof of the Theorem 3.1. In the case **II. (b)** we use the fact that (H_7) implies (H_6) and so we get a non-oscillatory solution with the properties $x(t) > 0, z(t) = L_0\, z(t) > 0,\ L_1\, z(t) < 0,$ $L_2\, z(t) > 0,\ L_3\, z(t) < 0,\ t \ge t_2$ ($t_2 \ge t_1$ is sufficiently large) and $\lim_{t\to\infty} z(t) = 0$, $\lim_{t\to\infty} x(t) = 0$. Using (2) we get

$$x(t) = z(t) + p\, x\big(\tau(t)\big) = z(t) + p\, z\big(\tau(t)\big) + p^2 x\big(\tau^2(t)\big)$$

for all sufficiently large t. Repeating this procedure and using (H_2) and the monotonicity of z we obtain that there exists an integer $n \ge 0$ such that $\tau^{n+1}\big(\sigma(t)\big) \ge t_2$ and

$$x\big(\sigma(t)\big) = \sum_{j=0}^{n} p^j z\Big(\tau^j\big(\sigma(t)\big)\Big) + p^{n+1} x\Big(\tau^{n+1}\big(\sigma(t)\big)\Big) \ge K z\big(\sigma(t)\big),$$

where $K = \sum_{j=0}^{n} p^j > 0,\ t \ge T \ge t_2$. Hence $f\big(x(\sigma(t))\big) \ge f\big(Kz(\sigma(t))\big)$, integrating the equation (1) from $\sigma(t)$ to t we have

$$0 < L_2\, z(t) \le L_2\, z\big(\sigma(t)\big) - MK \int_{\sigma(t)}^{t} q(s) z\big(\sigma(s)\big)\, \mathrm{d}s\,, \tag{10}$$

where $t \ge T_1 \ge T,\ M_3 = \inf\Big\{\frac{f(u)}{u};\ 0 \le |z| \le \big|K\, z\big(\sigma(T)\big)\big|\Big\}$. The inequality (7) yields

$$-\, z\big(\sigma(s)\big) \le -L_2\, z\big(\sigma(t)\big) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_{u)}^{\sigma(t)} \frac{1}{r_2(v)}\, \mathrm{d}v\, \mathrm{d}u\,, \tag{11}$$

where $t \ge s \ge T_1$. Combining (10) and (11) we obtain

$$0 < L_2\, z\big(\sigma(t)\big) - L_2\, z\big(\sigma(t)\big) MK \int_{\sigma(t)}^{t} q(s) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_{u}^{\sigma(t)} \frac{1}{r_2(v)}\, \mathrm{d}v\, \mathrm{d}u\, \mathrm{d}s\,,$$

Because $L_2\, z\big(\sigma(t)\big) > 0$ we have

$$1 > MK \int_{\sigma(t)}^{t} q(s) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_{u}^{\sigma(t)} \frac{1}{r_2(v)}\, \mathrm{d}v\, \mathrm{d}u\, \mathrm{d}s\,,$$

for $t \ge s \ge T_1$ and this contradicts the condition (H_7). □

Theorem 3.3. *Let* (H_1)–(H_6) *hold and in addition*

$$(\mathrm{H}_8) \qquad \limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_{u}^{\sigma(t)} \frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s > (1-p) \limsup_{u\to 0} \frac{u}{f(u)}.$$

Then the conclusion of the Theorem 3.2 holds.

Proof. Denote

$$a = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s) \int_{\sigma(s)}^{\sigma(t)} \frac{1}{r_1(u)} \int_{u}^{\sigma(t)} \frac{1}{r_2(v)}\,\mathrm{d}v\,\mathrm{d}u\,\mathrm{d}s.$$

Let the integer n be chosen such that

$$a > \frac{1-p}{1-p^{n+1}} \limsup_{u\to 0} \frac{u}{f(u)}.$$

Then the assertion of this theorem follows from the Theorem 3.2. □

Example 1. Let us consider the differential equation

$$\left(\sqrt{t}\left(\sqrt{t}\left(x(t) - \frac{1}{2}x\left(\left(\sqrt{t}-\pi\right)^2\right)\right)'\right)'\right)' + \frac{3}{8\sqrt{t}}\,x\left(\left(\sqrt{t}-\frac{3}{2}\pi\right)^2\right) = 0, \qquad t > \frac{27}{16}\pi^2.$$

All conditions of Theorem 3.3 are satisfied and all bounded solutions of the above equation are oscillatory. For instance, $x(t) = \sin\sqrt{t}$ is such solution.

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