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A NOTE ON MEASURE EXTENSION PROBLEM OF ℓ-GROUP-VALUED MEASURES

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ABSTRACT. For an Abelian, Archimedean Dedekind complete lattice-ordered, weakly σ-distributive group G and an algebra $\mathcal{F}$ of subsets of a set X, every positive measure $\mu\colon \mathcal{F} \to G$ has a unique countably additive (in order convergence) extension to σ-algebra generated by $\mathcal{F}$.

1. Introduction and notations

In this note, R is the set of real numbers and $\bar{R} = [-\infty, \infty]$, G is an Abelian, Archimedean Dedekind complete lattice-ordered, weakly σ-distributive group, $\mathcal{F}$ is an algebra of subsets of a set X and $\mu\colon \mathcal{F} \to G$ a positive measure (this means for a disjoint sequence $\{A_n\} \subset \mathcal{F}$ with $\cup A_n \in \mathcal{F}$, $\mu(A) = \sum \mu(A_n)$, considering limit in order convergence) and $\mathcal{A}$ the σ-algebra generated by $\mathcal{F}$.

In [6], using Carathéodory measurability, the author has given a very interesting sophisticated proof of the unique extension, in order convergence, of the measure to $\mathcal{A}$ (in [4, 5] the author has given other proofs of this result). In this note we prove that this result can also be easily obtained from the representation theorem proved in [8] (see also [3]).

2. Main result

Theorem 1 ([6, Theorem 5.1]). *Let $\mu\colon \mathcal{F} \to G$ be a positive measure. Then there is a unique extension $\mu\colon \mathcal{A} \to G$ which is positive and countably additive in order convergence.*

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P r o o f. For a compact Hausdorff Stonian (extremely disconnected) space Y, $\mathcal{E}(Y)$ will denote the space of all continuous functions $f\colon Y \to \bar{R}$ such that f is finite-valued almost everywhere (this means the set where it is $\pm\infty$ is nowhere dense). □

As in [8], we can assume, that $\mathcal{F}$ separates the points of X. As in [8, p. 70], there is a totally disconnected compact Hausdorff space $\tilde{X}$, the Boolean space for $\mathcal{F}$, in which X is dense. If $\tilde{\mathcal{F}}$ and $\mathcal{B}$ are the classes of all clopen subsets and all Baire subsets of $\tilde{X}$, then the mapping $\rho\colon B \to B \cap X$, when restricted to $\tilde{\mathcal{F}}$, is an isomorphism onto $\mathcal{F}$ and when restricted to $\mathcal{B}$ is a σ-homomorphism of $\mathcal{B}$ onto a class which contains $\mathcal{A}$ with kernel being the class of Baire sets disjoint from X [8]. The measure μ gives a finitely additive positive mapping

$$\tilde{\mu}\colon \tilde{\mathcal{F}} \to G, \qquad \tilde{\mu}(A) = \mu(A \cap X).$$

By [2, Theorem 4], there is a compact Hausdorff Stonian space Y such that G is lattice group isomorphic to a sub-lattice group of $\mathcal{E}(Y)$, preserving arbitrary sup and inf; we denote this isomorphism by i. Put $i \circ \mu(X) = e_0$ and let Y_0 be an open dense subset of Y where e_0 is finite-valued. We denote by $C(Y_0)$ the space of all real-valued continuous functions on the locally compact space Y_0. The mapping $C(Y_0) \to \mathcal{E}(Y)$, $f \to \tilde{f}$ (continuous extension) is a lattice isomorphism preserving arbitrary sup and inf. Put $e = (e_0)_{|Y_0}$. The finitely additive positive measure

$$\nu\colon \tilde{\mathcal{F}} \to C(Y_0)$$

is defined as

$$\nu(A) = \big(i \circ \tilde{\mu}(A)\big)_{|Y_0}.$$

Putting

$$I = \big\{f \in C(Y_0) : |f| \le e\big\},$$

we get a Dedekind complete vector sub-lattice E of $C(Y_0)$, $E = \cup_{n=1}^{\infty} nI$ with a strong unit e and a positive finitely additive measure $\nu\colon \tilde{\mathcal{F}} \to E$. Defining the norm $\|.\|$ on E, $\|f\| = \inf\{\lambda > 0 : |f| \le \lambda e\}$, E becomes a Dedekind complete M-space [1] and so there is an extremely disconnected compact Hausdorff space S such that E is linear lattice isomorphic to $C(S)$, preserving arbitrary sup and inf. Thus we get a finitely additive positive measure $\nu\colon \tilde{\mathcal{F}} \to C(S)$ which extends to linear positive mapping

$$\nu : \mathcal{S}(\tilde{\mathcal{F}}) \to C(S),\ \mathcal{S}(\tilde{\mathcal{F}})$$

being the space of all real-valued $\tilde{\mathcal{F}}$-simple functions on X; with sup norm topology on $\mathcal{S}(\tilde{\mathcal{F}})$, this mapping is easily verified to be continuous. Since X is totally disconnected, by Stone-Weierstrass theorem, $\mathcal{S}(\tilde{\mathcal{F}})$ is norm dense in $C(\tilde{X})$. Thus, we get a positive continuous mapping

$$\nu\colon C(\tilde{X}) \to C(S).$$

By [7], we get a quasi-regular Baire measure on $\tilde{X}$ which extends ν. Since G is weakly σ-distributive, the measure $\nu\colon \mathcal{B} \to G$ is regular. Now, for any compact G_δ-set $C \subset \tilde{X} \setminus X$, there is a decreasing sequence $\{F_n\} \subset \tilde{\mathcal{F}}$ such that $F_n \downarrow \chi_C$; since μ is a measure, we get $\nu(C) = 0$. By regularity, $\nu(B) = 0$ for any Baire subset of $\tilde{X}$ disjoint from X. Now, for any $A \in \mathcal{A}$ we can define $\mu(A) = \nu(B)$ for any Baire subset B of $\tilde{X}$ with $B \cap X = A$. The rest is easily verified.

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